PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 6, Pages 1745–1754
S 0002-9939(00)05754-3
Article electronically published on October 31, 2000

IMAGINARY POWERS OF LAPLACE OPERATORS

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(Communicated by Christopher D. Sogge)

ABSTRACT. We show that if L is a second-order uniformly elliptic operator in divergence form on $\mathbf{R}^d$, then $C_1(1+|\alpha|)^{d/2} \le \|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2}$. We also prove that the upper bounds remain true for any operator with the finite speed propagation property.

1. Introduction

Assume that $a_{ij} \in C^\infty(\mathbf{R}^d)$, $a_{ij} = a_{ji}$ for $1 \le i,j \le d$ and that $\kappa I \le (a_{ij}) \le \tau I$ for some positive constants κ and τ. We define a positive self-adjoint operator L on $L^2(\mathbf{R}^d)$ by the formula

$$L = -\sum \partial_i a_{ij} \partial_j. \tag{1}$$

We refer readers to [8] for the precise definition and basic properties of L. In particular, L admits a spectral resolution $E(t)$ and we can define the operator $L^{i\alpha}$ by the formula

$$L^{i\alpha} = \int_0^\infty t^{i\alpha} dE(t).$$

By spectral theory $\|L^{i\alpha}\|_{L^2\to L^2} = 1$. It is well known that $L^{i\alpha}$ falls within the scope of classical Calderón-Zygmund theory (as described in [3] or [22]) and so it extends to a bounded operator on $L^p, 1 < p < \infty$, and is also weak type (1,1). The main aim of this paper is to obtain the sharp estimate for the weak type $(1,1)$ norm of $L^{i\alpha}$ in terms of α.

The study of imaginary powers of operators is an important part of the theory of operators of type ω with H^∞ functional calculus (see e.g., [6], [9] and [17]). What is perhaps more interesting and relevant from the point of view of this paper is that the weak type $(1,1)$ norm of imaginary powers of self-adjoint operators can play a central role in the theory of spectral multipliers. See [5] and [15]. Imaginary powers of Laplace operators on compact Lie groups were also investigated in [20]. Theorem 2 below applied to Laplace operators on compact Lie groups gives the sharp endpoint result of Theorem 3 in [20], pp. 58. See also Corollary 4 of [20], pp. 121.

Received by the editors June 22, 1999 and, in revised form, September 27, 1999.
2000 *Mathematics Subject Classification.* Primary 42B15; Secondary 35P99.
Key words and phrases. Spectral multiplier, imaginary powers.
The research for this paper was supported by the Australian National University, the University of New South Wales, the University of Wroclaw, the Australian Research Council and the Polish Research Council KBN. We thank these institutions for their contributions.

However, the starting point for this paper is the following observation from [2]. If we denote the weak type (1,1) norm of an operator T on a measure space (X,μ) by $\|T\|_{L^1\to L^{1,\infty}} = \sup \lambda\ \mu(\{x\in X:\ |Tf(x)|>\lambda\})$ where the supremum is taken over $\lambda>0$ and functions f with $L^1(X)$ norm less than one, then for the standard Laplace operator on $\mathbf{R}^d$,

$$C_1(1+|\alpha|)^{d/2} \le \|(-\Delta_d)^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2}\log(1+|\alpha|). \tag{2}$$

The classical Hörmander multiplier theorem (see [13]) states that a multiplier operator T_m on $\mathbf{R}^d$ with multiplier m satisfies

$$\|T_m\|_{L^1\to L^{1,\infty}} \ \le\ C_s \sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{H_s} \ \le\ A \tag{3}$$

for any $s>d/2$ and any $\eta\in C_c^\infty(\mathbf{R}_+)$ not identically zero. Here H_s is the Sobolev space of order s on $\mathbf{R}^d$. Since the Sobolev norm in (3) behaves like $(1+|\alpha|)^s$ for the multiplier $m(x)=|x|^{i\alpha}$ of $(-\Delta)^{i\alpha}$, (2) shows that the exponent $d/2$ in Hörmander's theorem is sharp. Furthermore, if (3) is satisfied with $A<\infty$, then the distribution $K=\hat{m}$ agrees with a locally integrable function away from the origin which satisfies

$$I(B) \ =\ \sup_{y\neq 0} \int_{|x|\ge B|y|} |K(x-y)-K(x)|\,dx \ \le\ A \tag{4}$$

for $B\ge 2$ and Hörmander's theorem actually shows that the weak type (1,1) norm of T_m is bounded by $I(B)+\|m\|_{L^\infty}^2+B^d$. One can easily compute that for the convolution kernel K of $(-\Delta)^{i\alpha}$, the integral $I(B)$ is bounded above and below by $(1+|\alpha|)^{d/2}\log(1+|\alpha|/B)$. Hence Hörmander's theorem gives the upper bound in (2). The lower bound is a simple consequence of the explicit formula for the kernel K of $(-\Delta)^{i\alpha}$. See for example, [21] pp. 51-52.

The main observation of this paper is to note that there is a slight improvement of the bound $I(B)+\|m\|_{L^\infty}^2+B^d$ to $I(B)+(\|m\|_{L^\infty}^2 B^d)^{1/2}$. This can be achieved either by using C. Fefferman's ideas in [11] of exploiting more information of L^2 bounds or by varying the level of the Calderón-Zygmund decomposition and optimising. Hence we will be able to remove the *log* term in (2). We will show that this more precise estimate holds for a general class of operators.

Theorem 1. *Suppose that L is defined by* (1). *Then*

$$C_1(1+|\alpha|)^{d/2} \le \|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2} \tag{5}$$

for all $\alpha\in\mathbf{R}$.

Proof of the lower bound. We begin with some known estimates for the kernel $p_t(x,y)$ of the heat operator e^{-tL} associated to L. First, this kernel satisfies Gaussian bounds

$$C_1\,\frac{1}{t^{d/2}}\,\mathrm{e}^{-b_1\rho^2(x,y)/t} \ \le\ p_t(x,y) \ \le\ C_2\,\frac{1}{t^{d/2}}\,\mathrm{e}^{-b_2\rho^2(x,y)/t} \tag{6}$$

(see [8]) for some positive constants C_1, C_2, b_1 and b_2 and where $\rho(x,y)$ denotes the geodesic distance between x and y given by the Riemannian metric $(a_{i,j})$. In this setting of uniform ellipticity, $\kappa|x-y|\le\rho(x,y)\le\tau|x-y|$. Secondly, from the construction of a parametrix for the heat equation with respect to L (either via Hadamard's construction, see §17.4 of [14], or using pseudodifferential operator techniques, see chapter 7, §13 of [23]), we have for each $y \in \mathbf{R}^d$ a ball $B(y,r)$ such that for $x \in B(y,r)$ and $0 < t < 1$,

$$|p_t(x,y) - (\det a_{ij}(y))^{-1/2}(4\pi t)^{-d/2}\mathrm{e}^{-\rho^2(x,y)/4t}| \leq Ct^{1/2}t^{-d/2}. \tag{7}$$

Here we are using the fact that p_t is symmetric, $p_t(x,y) = p_t(y,x)$. From (6) and (7), we have for $x \in B(y,r)$ the bound

$$|p_t(x,y) - (\det a_{ij}(y))^{-1/2}(4\pi t)^{-d/2}\mathrm{e}^{-\rho^2(x,y)/4t}| \leq Ct^{1/4}t^{-d/2}\exp(-b'\rho(x,y)^2/t)$$

which translates into a bound for the kernel $K_{L^{i\alpha}}$ of $L^{i\alpha}$ since the functional calculus for L gives us the relationship

$$L^{i\alpha} = \Gamma(-i\alpha)^{-1}\int_0^\infty t^{-i\alpha-1}e^{-tL}dt$$

for $\alpha \neq 0$. Thus for $x \in B(y,r)$,

$$\begin{aligned} &|K_{L^{i\alpha}}(x,y) - (\det a_{ij}(y))^{-1/2}4^{i\alpha}\pi^{-d/2}\gamma(\alpha)\rho(x,y)^{-d-i2\alpha}| \\ &\qquad \leq C|\Gamma(-i\alpha)|^{-1}\rho(x,y)^{-d+1/2} \end{aligned} \tag{8}$$

where $\gamma(\alpha) = \Gamma(i\alpha + d/2)/\Gamma(-i\alpha)$. Using (8) with $y = 0$ we obtain for λ large enough

$$\begin{aligned} &\mu(\{|K_{L^{i\alpha}}(x,0)| \geq \lambda\}) \\ &\quad \geq \mu(\{C_1|\gamma(\alpha)|\rho^{-d}(x,0) \geq 2\lambda\}) - \mu(\{C_2|\Gamma(-i\alpha)|\rho^{-d+\frac{1}{2}}(x,0) \geq \lambda\}) \\ &\quad = \mu(B(0,(2C_1|\gamma(\alpha)|/\lambda)^{1/d})) - \mu(B(0,(C_2|\Gamma(-i\alpha)|/\lambda)^{1/(d-1/2)})) \\ &\quad \geq C'|\gamma(\alpha)|/\lambda. \end{aligned}$$

Here μ is Lebesgue measure and the sets above have the further restriction that $x \in B(0,r)$. Since $K_{L^{i\alpha}}$ is smooth away from the diagonal, we see that $L^{i\alpha}\phi_\delta(x)$ tends to $K_{L^{i\alpha}}(x,0)$ as $\delta \to 0$ for any $x \neq 0$ and any approximation of the identity $\{\phi_\delta\}$. Hence the above estimate shows that the weak type (1,1) norm of $L^{i\alpha}$ is bounded below by $|\gamma(\alpha)| = |\Gamma(i\alpha + d/2)/\Gamma(-i\alpha)| \sim (1+|\alpha|)^{\frac{d}{2}}$ (see [10]).

The upper bound in Theorem 1 holds in a much more general setting which we describe now. Assume that (X,μ,ρ) is a space with measure μ and metric ρ. If $\|P\|_{L^2\to L^\infty} < \infty$, then we can define the kernel K_P of the operator P by the formula

$$\langle P(\psi),\phi\rangle = \int P(\psi)\overline{\phi}d\mu = \int K_P(x,y)\psi(x)\overline{\phi(y)}d\mu(x)d\mu(y).$$

Note that $\sup_x \|K_P(x,\cdot)\|_{L^2} = \|P\|_{L^2\to L^\infty}$. Next, we say that

$$\operatorname{supp} K_P \subset \{(x,y) \in X^2 : \rho(x,y) \leq r\} \tag{9}$$

if $\langle P(\psi),\phi\rangle = 0$ for every $\phi,\psi \in L^2$ and every $r_1 + r_2 + r < \rho(x',y')$ such that $\psi(x) = 0$ for $\rho(x,x') > r_1$ and $\phi(x) = 0$ for $\rho(x,y') > r_2$. This definition (9) makes sense even if $\|P\|_{L^2\to L^\infty} = \infty$. Now if L is a self-adjoint positive definite operator acting on $L^2(\mu)$, then we say that it satisfies the finite speed propagation property of the corresponding wave equation if

$$\operatorname{supp} K_{C_t(\sqrt{L})} \subset \{(x,y) \in X^2 : \rho(x,y) \leq t\}, \tag{10}$$

where $C_t(\sqrt{L}) = \int \cos(t\sqrt{\lambda})\,dE(\lambda)$.

Theorem 2. *Suppose that L satisfies* (10). *Next assume that*

(11) $\|\exp(-tL)\|^2_{L^2\to L^\infty} \le C_1 V_{d,D}(t^{1/2})^{-1} \le C\mu(B(x,t^{1/2}))^{-1} \le C_2 V_{d,D}(t^{1/2})^{-1}$

for all $t>0$ and $x\in X$, where $B(x,t)$ is a ball with radius t centred at x and

$$V_{d,D}(t) = \begin{cases} t^d & \text{for } t\le 1,\\ t^D & \text{for } t>1,\end{cases}$$

for $d,D\ge 0$. Then

$$\|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{\max(d,D)/2}$$

for all $\alpha\in\mathbf{R}$.

We remark that (10) and (11) are equivalent to having Gaussian upper bounds on the heat kernel and the associated volume growth on balls. See [18]. Furthermore, the upper bound in Theorem 1 follows from Theorem 2. Indeed, if $X=\mathbf{R}^d$, $\rho(x,y)=\tau|x-y|$ and μ is Lebesgue measure, then it is well known (see e.g. [8] and [19]) that (11) and (10) hold. We are going to prove Theorem 2 only in the case $d=D$. The argument for the other cases is similar.

2. Preliminaries

The following lemma is a very simple but useful consequence of (10).

Lemma 1. *Assume that L satisfies* (10) *and that $\hat{F}$ is a Fourier transform of an even bounded Borel function F with* supp $\hat{F}\subset[-r,r]$. *Then*

$$\text{supp } K_{F(\sqrt{L})} \subset \{(x,y)\in X^2 : \rho(x,y)\le r\}.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{F}(t)C_t(\sqrt{L})\,dt.$$

But since supp $\hat{F}\subset[-r,r]$,

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-r}^{r} \hat{F}(t)C_t(\sqrt{L})\,dt$$

and Lemma 1 follows from (10).

Lemma 2. *Let $\phi\in C_c^\infty(\mathbf{R})$ be even, $\phi\ge 0$, $\|\phi\|_{L^1}=1$, $\mathrm{supp}(\phi)\subset[-1,1]$, and set $\phi_r(x)=1/r\,\phi(x/r)$ for $r>0$. Let Φ denote the Fourier transform of ϕ and Φ^r denote the Fourier transform of ϕ_r. If* (11) *and* (10) *hold, then the kernel $K_{\Phi^r(\sqrt{L})}$ of the self-adjoint operator $\Phi^r(\sqrt{L})$ satisfies*

$$\text{supp } K_{\Phi^r(\sqrt{L})} \subset \{(x,y)\in X^2;\rho(x,y)\le r\} \tag{12}$$

and

$$|K_{\Phi^r(\sqrt{L})}(x,y)| \le C\,r^{-d} \tag{13}$$

for all $r>0$ and $x,y\in X$.

Proof. (12) follows from Lemma 1. For any $m \in \mathbf{N}$ and $r > 0$, we have the relationship

$$(I + rL)^{-m} \;=\; \frac{1}{m!} \int_0^\infty \mathrm{e}^{-rtL} \mathrm{e}^{-t} t^{m-1}\, dt$$

and so when $m > d/4$, (11) implies

$$\begin{aligned} \|(I + rL)^{-m}\|_{L^2 \to L^\infty} &\le \frac{1}{m!} \int_0^\infty \|\exp(-rtL)\|_{L^2 \to L^\infty} e^{-t} t^{m-1} dt \\ &\le C_1\, r^{-d/4} \end{aligned} \tag{14}$$

for all $r > 0$. Now $\|(I + r^2L)^{-m}\|_{L^1 \to L^2} = \|(I + r^2L)^{-m}\|_{L^2 \to L^\infty}$ and so

$$\|\Phi^r(\sqrt{L})\|_{L^1 \to L^\infty} \;\le\; \|(I + r^2L)^{2m}\Phi^r(\sqrt{L})\|_{L^2 \to L^2}\, \|(I + r^2L)^{-m}\|^2_{L^2 \to L^\infty}.$$

The L^2 operator norm of the first term is equal to the L^∞ norm of the function $(1 + r^2|t|)^{2m}\Phi(r\sqrt{|t|})$ which is uniformly bounded in $r > 0$ and so (13) follows by (14).

Next we recall the Calderón-Zygmund decomposition in the general setting of spaces of homogeneous type (see e.g. [3] or [22]).

Lemma 3. *There exists C such that, given $f \in L^1(X, \mu)$ and $\lambda > 0$, one can decompose f as*

$$f = g + b = g + \sum b_i$$

so that

1. $|g(x)| \le C\lambda$, *a.e.* x *and* $\|g\|_{L^1} \le C\|f\|_{L^1}$.
2. *There exists a sequence of balls $B_i = B(x_i, r_i)$ such that the support of each b_i is contained in B_i and*
$$\int |b_i(x)| d\mu(x) \;\le\; C\lambda\mu(B_i).$$
3. $\sum \mu(B_i) \le C\,\frac{1}{\lambda} \int |f(x)|\, d\mu(x)$.
4. *There exists $k \in \mathbf{N}$ such that each point of X is contained in at most k of the balls $B(x_i, 2r_i)$.*

We are now in a position to prove Theorem 2.

3. Proof of Theorem 2

The proof follows closely the line of argument in [1] (which of course generalises to this setting). We are attempting to prove

$$\lambda\, \mu(\{x \in X \,:\, |L^{i\alpha} f(x)| \ge \lambda\}) \;\le\; C(1 + |\alpha|)^{\frac{d}{2}}\, \|f\|_{L^1}.$$

As usual we start by decomposing f into $g + \sum b_i$ at the level of λ according to Lemma 3. We will follow the idea of C. Fefferman [11] of using more information of the L^2 operator norm (in our case, $\|L^{i\alpha}\|_{L^2 \to L^2} = 1$) by smoothing out the bad functions b_i at a scale smaller than the size of its support and considering this part of the good function where L^2 estimates can be used (see also [4]). In our case for each b_i, consider $\Phi^{s_i}(\sqrt{L})b_i$ where $s_i = \theta r_i$, $\theta = (1 + |\alpha|)^{-\frac{1}{2}}$, and let

$G = g + \sum \Phi^{s_i}(\sqrt{L})b_i$ be the modified good function. Hence $f = G + B$ where $B = \sum(I - \Phi^{s_i}(\sqrt{L}))b_i$ and we write

$$\begin{aligned}\lambda\,\mu(\{|L^{i\alpha}f(x)| \geq \lambda\}) \leq\ & \lambda\,\mu(\{|L^{i\alpha}G(x)| \geq \lambda/2\}) \\ &+\ \lambda\,\mu(\{|L^{i\alpha}B(x)| \geq \lambda/2\}).\end{aligned} \tag{15}$$

The first term is less than $4/\lambda\,\|L^{i\alpha}G\|_{L^2}^2 \ \leq\ 4/\lambda\,\|G\|_{L^2}^2$. However, according to Lemma 2,

$$|\Phi^{s_i}(\sqrt{L})b_i(x)| \ \leq\ \int |K_{\Phi^{s_i}(\sqrt{L})}(x,y)b_i(y)|\,d\mu(y) \ \leq\ C\,(\theta r_i)^{-d}\|b_i\|_{L^1}\,\mathbb{1}_{B(x_i,2r_i)}$$

and therefore by Lemma 3, $|G(x)| \ \leq\ C\theta^{-d}\lambda$ for *a.e.*, x. Using Lemma 2 again which shows that the $L^p \to L^p$ operator norms of $\Phi^r(\sqrt{L})$ are uniformily bounded in $r > 0$, we also have that

$$\|G\|_{L^1} \leq \|g\|_{L^1} + C\sum \|\Phi^{s_i}(\sqrt{L})b_i\|_{L^1} \leq \|g\|_{L^1} + C\sum \|b_i\|_{L^1} \leq C\|f\|_{L^1}.$$

Therefore the first term in (15) is bounded by $(1+|\alpha|)^{\frac{d}{2}}\,\|f\|_{L^1}$.

Since $\mu(\bigcup B(x_i,\theta^{-1}r_i)) \leq C\theta^{-d}\sum\mu(B_i) \leq C(1+|\alpha|)^{\frac{d}{2}}\|f\|_{L^1}/\lambda$, then to bound the second term in (15), it suffices to show

$$\int\limits_{x\notin\cup B_i^*} |L^{i\alpha}B(x)|\,d\mu(x) \ \leq\ C(1+|\alpha|)^{\frac{d}{2}}\|f\|_{L^1}, \tag{16}$$

where $B_i^* = B(x_i,\theta^{-1}r_i)$. Let $H^\alpha(t) = |t|^{2i\alpha}$ so that

$$L^{i\alpha}B(x) = \sum H^\alpha(1-\Phi^{s_i})(\sqrt{L})b_i(x)$$

and therefore the left side of (16) is less than

$$\begin{aligned}&\sum_i \int\limits_{x\notin\cup_j B_j^*} \Big|\int K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)b_i(y)\,d\mu(y)\Big|\,d\mu(x) \\ &\qquad\leq\ \sum_i \int |b_i(y)| \int\limits_{x\notin B_i^*} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(x)\,d\mu(y).\end{aligned}$$

Since $F(L)^* = \overline{F}(L)$, we may interchange the roles of x and y, and so (16) will follow from Lemma 3 once we establish

$$\sup_{x,i} \int\limits_{\rho(x,y)\geq\theta^{-1}r_i} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y) \ \leq\ C\,(1+|\alpha|)^{\frac{d}{2}}. \tag{17}$$

We now fix $x \in X$ and i. Let $\eta \in C_c^\infty(\mathbf{R})$ be an even function supported in $\{t \in \mathbf{R} : 1 \leq |t| \leq 4\}$ such that

$$\sum_{n=-\infty}^{\infty} \eta(2^{-n}t) = 1 \ \text{ for all } \ t \neq 0.$$

We put $H_n^\alpha(t) = \eta(2^{-n}t)H^\alpha(t)$ so that

$$H^\alpha(1-\Phi^{s_i})(\sqrt{L}) = \sum_n H_n^\alpha(1-\Phi^{s_i})(\sqrt{L}).$$

Thus

$$\int_{y\notin B_i^*} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y) \;\le\; \sum_n \int_{y\notin B_i^*} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y) \tag{18}$$

and we will estimate each term in the sum on the right side in terms of n and i, uniformly in $x\in X$.

Let $k_o=[d/2]+1$ so that

$$\int_{y\notin B_i^*} (1+2^n\rho(x,y))^{-2k_o}d\mu(y)\le C\int_{\theta^{-1}r_i}^{\infty}(1+2^nr)^{-2k_0}r^{d-1}dr\le C2^{-2nk_o}(\theta^{-1}r_i)^{d-2k_o}$$

and therefore by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\int_{y\notin B_i^*}|K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y)\\ &\quad\le\; C\,2^{-nk_o}(\theta^{-1}r_i)^{\frac{d}{2}-k_o}\\ &\qquad\cdot\Big(\int_{\rho(x,y)\ge\theta^{-1}r_i}|K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2(1+2^n\rho(x,y))^{2k_o}d\mu(y)\Big)^{1/2}.\end{aligned} \tag{19}$$

We break up the integral on the right side of (19) where $2^n\rho(x,y)$ is roughly constant and consider

$$\sum_{2^j\ge 2^nr_i\theta^{-1}} 2^{2jk_o}\int_{2^{j-1-n}<\rho(x,y)\le 2^{j-n}}|K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2\,d\mu(y). \tag{20}$$

Fix a nonnegative even $\varphi\in C_c^\infty(\mathbf{R})$ such that $\varphi=1$ on $[-1/4,1/4]$ and $\varphi=0$ on $\mathbf{R}\setminus[-1/2,1/2]$. Then the Fourier transforms of $H_n^\alpha(1-\Phi^{s_i})$ and $H_n^\alpha(1-\Phi^{s_i}) * (\delta-\hat\varphi_{2^{n-j}})$ agree on $\{\xi : |\xi|\ge 2^{j-1-n}\}$ and so by Lemma 1, the kernels of $H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})$ and $H_n^\alpha(1-\Phi^{s_i})*(\delta-\hat\varphi_{2^{n-j}})(\sqrt{L})$ agree on the set $\{(x,y)\in X^2 : \rho(x,y)\ge 2^{j-1-n}\}$. Here δ denotes the Dirac mass at 0. For each j, the integrals in (20) satisfy the bound

$$\int_{2^{j-1-n}<\rho(x,y)\le 2^{j-n}}|K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2\,d\mu(y)\;\le\;\|K_{F_{n,j}^\alpha(\sqrt{L})}\|^2_{L^2\to L^\infty},$$

where we are defining $F_{n,j}^\alpha(t)=H_n^\alpha(1-\Phi^{s_i})*(\delta-\hat\varphi_{2^{n-j}})(t)$. So by (14), the right side of this inequality is bounded by $\|(I+2^{-2n}L)^mF_{n,j}^\alpha(\sqrt{L})\|^2_{L^2\to L^2}\,2^{nd}$ as long as $m>d/4$. Everything then comes down to estimating the L^∞ norm of $(1+2^{-2n}t^2)^mF_{n,j}^\alpha(t)$. We make the following claim.

Claim. For each j, n and $m>d/4$,

$$(1+2^{-2n}t^2)^m|F_{n,j}^\alpha(t)|\;\le\;C_m|\alpha|^{k_o}2^{-jk_o}\min(1,(2^nr_i\theta)^2)\,\min(1,|\alpha|2^{-j})$$

uniformly in $t\in\mathbf{R}$.

The claim shows that

$$\|K_{F_{n,j}^\alpha}(\sqrt{L})\|_{L^2\to L^\infty}\;\le\;C\,|\alpha|^{k_o}2^{-jk_o}2^{\frac{nd}{2}}\,\min(1,(2^nr_i\theta)^2)\,\min(1,|\alpha|2^{-j})$$

and hence the sum in (20) is bounded by

$$|\alpha|^{2k_o}2^{nd}\min{}^2(1,(2^n r_i\theta)^2)\sum_{2^j\geq 2^n r_i\theta^{-1}}\min{}^2(1,|\alpha|2^{-j})$$
$$\leq |\alpha|^{2k_o}2^{nd}\min{}^2(1,(2^n r_i\theta)^2)\log(2+\frac{|\alpha|}{2^n r_i\theta^{-1}}).$$

Recall that θ and α are related so that $\theta|\alpha| = |\alpha|/(1+|\alpha|)^{\frac{1}{2}} \leq \theta^{-1}$. Plugging this into (19) gives

$$\int_{y\notin B_i^*} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)d\mu(y) \leq \theta^{-d}(2^n r_i\theta)^{\frac{d}{2}-k_o}\ \min(1,(2^n r_i\theta)^2)\log(2+\frac{1}{2^n r_i\theta})$$

and this makes the sum in (18) bounded by $\theta^{-d} = (1+|\alpha|)^{\frac{d}{2}}$, proving (17) and hence Theorem 2.

Proof of the Claim. If $G_n(t) = H_n^\alpha(t)(1-\Phi^{s_i}(t))$, then $F_{n,j}^\alpha(t) = 2^{(n-j)k_o}G_n^{(k_o)} * \hat{\psi}_{2^{n-j}}(t)$ where $\psi(\xi) = \xi^{-k_o}(1-\varphi(\xi))$ (and so $\hat{\psi}$ is continuous, rapidly decreasing and has vanishing moments, $\int t^\ell\hat{\psi}(t)dt = 0,\ \ell = 0,1,2,\dots$). Hence

$$F_{n,j}^\alpha(t) = 2^{(n-j)k_o}\int_{\mathbf{R}}\big[G_n^{(k_o)}(t-s) - G_n^{(k_o)}(t)\big]\hat{\psi}_{2^{n-j}}(s)\,ds$$
$$= \ 2^{(n-j)k_o}\int_{\mathbf{R}}\big[G_n^{(k_o)}(t-2^{n-j}s) - G_n^{(k_o)}(t)\big]\hat{\psi}(s)\,ds.$$

However $G_n(t) = \eta(2^{-n}t)|t|^{2i\alpha}(1-\Phi(s_i t))$ and thereby each time we take a derivative, we gain a factor of 2^{-n}. $G_n^{(k_o)}(t)$ is thus a finite sum of terms of the form $\alpha^p 2^{-nk_o}\tilde{\eta}(2^{-n}t)|t|^{2i\alpha}\Psi(s_i t)$ where $\tilde{\eta}\in C_c^\infty(\mathbf{R})$, $\operatorname{supp}(\tilde{\eta})\subset\operatorname{supp}(\eta)$ and Ψ is a Schwartz function which is $0(t^2)$ as $t\to 0$ (note that $\Phi'(0) = \int x\phi(x)dx = 0$ since ϕ is even). The worst power p is k_o which occurs when all derivatives land on the factor $|t|^{2i\alpha}$.

Without loss of generality, let us suppose that

$$G^{(k_o)}(t) = \alpha^{k_o}2^{-nk_o}\eta(2^{-n}t)|t|^{2i\alpha}\Psi(s_i t).$$

From the above integral representation of $F_{n,j}^\alpha(t)$, we see that the main contribution to $(1+2^{-2n}t^2)^m|F_{n,j}^\alpha(t)|$ occurs when $|t|\sim 2^n$ and in this case,

$$|F_{n,j}^\alpha(t)| \leq C|\alpha|^{k_o}2^{(n-j)k_o}2^{-nk_o}\min(1,(s_i 2^n)^2) \leq C|\alpha|^{k_o}2^{-jk_o}\min(1,(2^n r_i\theta)^2).$$

However we may write

$$F_{n,j}^\alpha(t) = -2^{(n-j)k_o}2^{n-j}\int_0^1\int_{\mathbf{R}} G_n^{(k_o+1)}(t-\sigma 2^{j-n}s)s\hat{\psi}(s)\,ds\,d\sigma$$

and therefore we also have

$$|F_{n,j}^\alpha(t)| \leq C|\alpha|^{k_o+1}2^{(n-j)k_o}2^{n-j}2^{-n(k_o+1)}\min(1,(s_i 2^n)^2)$$
$$\leq C|\alpha|^{k_o}2^{-jk_o}|\alpha|2^{-j}\min(1,(2^n r_i\theta)^2),$$

establishing the claim.

Remarks. Theorem 1 holds also for Laplace-Beltrami operators on compact manifolds of dimension d. The proof is essentially the same as the proof of Theorem 1.

The hypotheses of Theorem 2 are satisfied for Laplace operators on Lie groups of polynomial growth. However, if L is a sub-Laplacian on the three dimensional Heisenberg group, then $d = 4$ but

$$C_1(1+|\alpha|)^{3/2} \le \|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_\epsilon(1+|\alpha|)^{3/2+\epsilon}.$$

(See [16]; see also [12].) The same estimates hold for a sub-Laplacian on SU(2) for which $d = 4$ and $D = 0$ (see [7]). Thus there are situations where the upper bound is better than the one given by Theorem 2 and where the lower bound in Theorem 1 is false. For general groups of polynomial growth Theorem 2 gives the best known estimates; however as the above examples show, these bounds are not always best possible.

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