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ON THE LYUBEZNIK NUMBERS OF A LOCAL RING

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ABSTRACT. We collect some information about the invariants $\lambda_{p,i}(A)$ of a commutative local ring A containing a field introduced by G. Lyubeznik in 1993 (*Finiteness properties of local cohomology modules*, Invent. Math. **113**, 41–55). We treat the cases $\dim(A)$ equal to zero, one and two, thereby answering in the negative a question raised in Lyubeznik's paper. In fact, we will show that $\lambda_{p,i}(A)$ has in the two-dimensional case a topological interpretation.

1. Introduction

Throughout let k be a field and let A be a local k-algebra. It is shown in [4] that if A is the quotient of a regular local ring $(R, \mathfrak{m}, k)$ of dimension n containing k, $\phi : R \twoheadrightarrow A$, $\ker \phi = I$, then the Bass number $\lambda_{p,i}(A) = \mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \dim_k \operatorname{Ext}_R^p(k, H_I^{n-i}(R))$ is finite and a function of A, i, p alone but not of R or ϕ.

Only little is known about the $\lambda_{p,i}$ so far, but they carry interesting information. For example, if $R = \mathbb{C}[x_0, \dots, x_n]$, $\hat{R} = \mathbb{C}[[x_0, \dots, x_n]]$ and $I \subseteq R$ is the defining ideal of a smooth variety $V \subseteq \mathbb{P}^n_{\mathbb{C}}$, then, for $i < n - \operatorname{codim}(V)$, $\lambda_{0,i}(\hat{R}/I \cdot \hat{R}) = \dim_{\mathbb{C}} \left(H_x^i(\tilde{V}, \mathbb{C}) \right)$ where $H_x^i(\tilde{V}, \mathbb{C})$ stands for the i-th singular cohomology group of the affine cone $\tilde{V}$ over V with support in the vertex x of $\tilde{V}$ and with coefficients in $\mathbb{C}$.

Since completion does not change $\lambda_{p,i}(A)$ ([4], Lemma 4.2) one may assume that $R = k[[x_1, \dots, x_n]]$. As $H_I^0(-) = H^0_{\sqrt{I}}(-)$, $\lambda_{p,i}(A) = \lambda_{p,i}(A_{red})$. Hence we assume that I is radical. One has $H_I^{n-i}(R) = 0$ for $i > \dim(A)$ and $\lambda_{p,i}(A) = 0$ for $p > i$ by [4], (4.4i) and (4.4ii).

We define the type of the ring $A = R/I$ to be the matrix $\Lambda(A)$ where $\Lambda(A)_{i,j} = \lambda_{i,j}(A)$ for $0 \leq i, j \leq n$.

Recall the Hartshorne-Lichtenbaum vanishing theorem ([2], Theorem 3.1) which we denote by HLVT and in essence states that $H_I^n(R) = 0$ if and only if I is not $\mathfrak{m}$-primary. As is well known, $H^n_{\mathfrak{m}}(R) = E_R(k)$, the R-injective hull of k.

Note that by virtue of the spectral sequence

$$(1.1) \qquad E_2^{pq} = H^p_{\mathfrak{m}}(H^q_I(R)) \Rightarrow E_\infty^{pq} = H^{p+q}_{\mathfrak{m}}(R)$$

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and HLVT we have $\Lambda(A) = (1)$ if A is Artinian, and $\Lambda(A) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ if $\dim(A) = 1$.

G. Lyubeznik asked in [4] whether $\lambda_{d,d}(A) = 1$ for any A and proved it to be true for A normal. We shall show that this is not the case in general.

2. Pure dimension two

We shall assume that k is separably closed. This means that in R we can use the second vanishing theorem, due to Ogus, Hartshorne-Speiser and Huneke-Lyubeznik (see [3], Theorem 1.1.): for $\sqrt{I} \subsetneq \mathfrak{m}$, we have $H^{n-1}_I(R) = 0$ if and only if the punctured spectrum of R/I is connected.

Lemma 2.1. *Let $I = \bigcap_1^s P_i$ such that $V(I) \setminus \{\mathfrak{m}\}$ is connected and all P_i are prime ideals of dimension 2. Then $H^{n-1}_I(R) = 0$ and I is of type* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proof. The second vanishing theorem shows that $H^{n-1}_I(R) = 0$. The lemma follows from the spectral sequence (1.1). □

Proposition 2.2. *Let I be radical of pure dimension 2. Let a be the number of connected components of the punctured spectrum of R/I. Then I is of type* $\begin{pmatrix} 0 & a-1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$ *and $H^{n-1}_I(R) = E_R(k)^{a-1}$.*

Proof. If $a = 1$, the claim follows from the previous lemma. If $a > 1$, write $I = \bigcap_1^a J_k$ where each J_k is radical and defines a connected component of $\operatorname{Spec}(R/I) \setminus \{\mathfrak{m}\}$. Set $J = \bigcap_1^{a-1} J_k$. By induction, $\lambda_{2,2}(R/J) = a-1$ and $\lambda_{2,2}(R/J_a) = 1$. Since $\mathfrak{m}$ is minimal to $J + J_a$, $H^{n-1}_{J+J_a}(R) = H^{n-2}_{J+J_a}(R) = 0$. Hence by the Mayer-Vietoris sequence to J and J_a, $H^{n-2}_{JJ_a}(R) = H^{n-2}_J(R) \oplus H^{n-2}_{J_a}(R)$ so that $\lambda_{2,2}(R/I) = a-1+1$. Moreover, the Mayer-Vietoris sequence to J and J_a also contains a piece

$$0 \to H^{n-1}_J(R) \oplus H^{n-1}_{J_a}(R) \to H^{n-1}_{JJ_a}(R) \to H^n_{J+J_a}(R) \to 0$$

where the last zero comes from HLVT. By induction, the term on the left is isomorphic to $E_R(k)^{a-1}$ and in particular injective. The sequence splits and the proposition follows. □

3. The mixed case

Let $I = J_1 \cap J_2$ where each J_i is radical and of pure dimension i, and let a be the number of connected components of $\operatorname{Spec}(R/J_2) \setminus \{\mathfrak{m}\}$. Let $x \in J_2 \setminus \bigcup\{P | P \in \operatorname{ass}(I), \dim(P) = 1\}$. Then $\operatorname{rad}(I + R \cdot x) = J_2$. Consider the long exact sequence of Proposition 8.1.2 in [1]:

$$\begin{aligned} 0 \quad \to \quad & H^{n-2}_{J_2}(R) \to H^{n-2}_I(R) \to \left(H^{n-2}_I(R)\right)_x \to \\ & H^{n-1}_{J_2}(R) \to H^{n-1}_I(R) \to \left(H^{n-1}_I(R)\right)_x \to H^n_{J_2}(R) = 0 \end{aligned}$$

where the zero on the right comes from HLVT. By [4] (4.4iii), the inclusion $H^{n-2}_{J_2}(R) \to H^{n-2}_I(R)$ is an isomorphism. Hence $\lambda_{2,2}(R/I) = \lambda_{2,2}(J_2)$ and we have a four piece exact sequence

$$0 \to \left(H^{n-2}_I(R)\right)_x \to H^{n-1}_{J_2}(R) \to H^{n-1}_I(R) \to \left(H^{n-1}_I(R)\right)_x \to 0.$$

Note that if M is an R-module and $x \in \mathfrak{m}$, then $\operatorname{Ext}^i_R(k, M_x) = 0$ for all i. Let F be the kernel of the map $H^{n-1}_I(R) \to (H^{n-1}_I(R))_x$ and split the sequence into two short exact sequences. Since a is the number of connected components of the punctured spectrum of R/J_2, application of $\operatorname{Ext}^\bullet_R(k, -)$ to the first sequence yields

$$\begin{aligned} 0 \quad \to \quad & 0 \to k^{a-1} \to \operatorname{Ext}^0_R(k, F) \to \\ & 0 \to 0 \to \operatorname{Ext}^1_R(k, F) \to \cdots \end{aligned}$$

according to Proposition 2.2. Hence $\operatorname{Ext}^0_R(k, F) = k^{a-1}$ and $\operatorname{Ext}^i_R(k, F) = 0$ for $i > 0$. Application of $\operatorname{Ext}^\bullet_R(k, -)$ to the second sequence then yields

$$\begin{aligned} 0 \quad \to \quad & k^{a-1} \to k^{\lambda_{0,1}(R/I)} \to 0 \to \\ & 0 \to k^{\lambda_{1,1}(R/I)} \to 0 \to \cdots . \end{aligned}$$

This proves that $\lambda_{1,1}(R/I) = 0$, $\lambda_{0,1}(R/I) = a - 1$ and the type of I equals the type of J_2.

We present our conclusions in form of the following

Proposition 3.1. *Let I be a radical two-dimensional mixed ideal of the complete regular ring $(R, \mathfrak{m}, k)$ where k is separably closed. Write $I = J_1 \cap J_2$ where each J_i is radical of pure dimension i. Let a be the number of connected components of* $\operatorname{Spec}(R/J_2) \setminus \{\mathfrak{m}\}$. *Then I is of type* $\begin{pmatrix} 0 & a-1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$. *In particular, the type is independent of the one-dimensional components of I.* □

Remark 3.2. We are not aware of results computing the type of A for general I if $\dim(A) > 2$. However, there are some results that relate to the invariants $\lambda_{p,i}(A)$. Known to us are the following:

In [5], the author gives a combinatorial algorithm to calculate the $\lambda_{p,i}(A)$ from a primary decomposition of I assuming that I is a monomial ideal.

In [8] the $\lambda_{p,i}(A)$ for monomial I are investigated in relation to certain Ext-modules. Related results have been obtained in [6], where certain combinatorial properties of $H^i_I(R)$ are studied in the monomial case.

In [7] an algorithm is explained that computes the local cohomology modules $H^i_J(S)$ if S is a ring of polynomials over a field of characteristic zero, and an algorithm to compute their Bass numbers with respect to a maximal ideal. In particular, the $\lambda_{p,i}(A)$ are computable if A is a quotient of S. However, these algorithms do not shed light on structural information about local cohomology in general.

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