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THE PQ-CONDITION FOR 3-MANIFOLD GROUPS

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Abstract. We give an elementary, topological proof of the fact that any subgroup of order pq of a finite 3-manifold group is cyclic if p and q are distinct odd primes. This condition, together with related results of Milnor and Reidemeister, implies that such a group acts orthogonally on some sphere.

Our aim here is to understand the restrictions on finite groups that are fundamental groups of 3-manifolds. Two necessary conditions are well known; namely that if G is a such a group, then its subgroups of order $2p$, for p a prime, are cyclic [2] (the $2p$-*condition*) and so are its subgroups of order p^2 [3], [4] (the p^2-*condition*). The latter is equivalent to the statement that all abelian subgroups of G are cyclic. The p^2-condition for 3-manifold groups is an immediate consequence of the fact that $(\mathbb{Z}/p\mathbb{Z})^2$ does not have a balanced presentation.

The above conditions together with the *pq-condition* characterise when finite groups act orthogonally on some sphere without fixed points [5]. The *pq-condition* is the condition that every subgroup of order pq of G, where p and q are distinct, odd primes, is abelian (hence cyclic). Our main result is that finite fundamental groups of 3-manifolds satisfy this condition. This was previously known, but proofs in the literature (for example in [2]) involve using the fact that G has cohomology with periodicity 4 together with some deep group theory to restrict the class of finite fundamental groups, and then observing that this condition is satisfied. It seems desirable to deduce this result using a direct topological argument, especially given the fundamental role of this condition in characterising orthogonal free actions. The proof given here is elementary and topological.

Theorem 0.1. *Suppose the finite group G is the fundamental group of a 3-manifold M, and p and q are distinct, odd primes. Then every subgroup of G of order pq is cyclic.*

Proof. First note that, by passing to a cover, we may assume that G itself has order pq. We need to show that G is abelian. Suppose it is not. Then by elementary group theory (interchanging p and q if necessary), we have a short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to G \to \mathbb{Z}/q\mathbb{Z} \to 0.$$

In particular, we have a normal subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Let M_p be the Galois cover of M corresponding to this subgroup. Then $\pi_1(M_p) = \mathbb{Z}/p\mathbb{Z}$ and M_p is a homology lens space.

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The group $\mathbb{Z}/q\mathbb{Z}$ is the group of deck transformations acting on M_p. The action of Z_q on $H_1(M_p)$ is induced by conjugation in G. Thus, to show that G is abelian, it suffices to show that this action is trivial. Let $\rho : M_p \to M_p$ denote a generator of the group of deck transformations. First, we prove the following more general lemma.

Lemma 0.2. *Let $f : M_p \to M_p$ be a fixed point free homeomorphism of M_p. Then the map $f_* : H_1(M_p) \to H_1(M_p)$ is either the identity map or the involution $\gamma \mapsto -\gamma$.*

Proof. As in the classification of lens spaces up to homotopy equivalence, we shall define a function $\zeta : H_1(M_p) \to \mathbb{Z}/p\mathbb{Z}$. This is defined as follows.

The short exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

induces the Bockstein cohomology long-exact sequence

$$\to H^1(M_p, \mathbb{Z}) \to H^1(M_p, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} H^2(M_p, \mathbb{Z}) \to H^2(M_p, \mathbb{Z}) \to H^2(M_p, \mathbb{Z}/p\mathbb{Z}) \to$$

from which we obtain the isomorphisms $\delta : H^1(M_p, \mathbb{Z}/p\mathbb{Z}) \to H^2(M_p, \mathbb{Z})$ and $\phi : H^2(M_p, \mathbb{Z}) \to H^2(M_p, \mathbb{Z}/p\mathbb{Z})$.

Fixing an orientation for M_p, we define the function ζ. Namely, given an element $\gamma \in H^2(M_p, \mathbb{Z}) \cong H_1(M_p, \mathbb{Z})$ we define $\zeta(\gamma) \in \mathbb{Z}/p\mathbb{Z}$ by

$$\zeta(\gamma) = (\delta^{-1}(\gamma) \cup \phi(\gamma))[H_3(M_p, \mathbb{Z})],$$

where $[H_3(M_p, \mathbb{Z})]$ is the image of the fundamental class of M_p in $H_3(M_p, \mathbb{Z}/p\mathbb{Z})$.

Observe that $\zeta(k\gamma) = k^2\zeta(\gamma)$, since both δ and ϕ are homomorphisms. Further, the fact that the cup product is a perfect pairing shows that $\zeta(\gamma)$ is non-zero for a generator γ of $H_1(M_p)$. We fix such a generator γ_0.

Note that all our maps are natural. Further the homeomorphism takes the fundamental class to itself since it must be orientation preserving by the Lefschetz fixed point theorem. Thus, we have the identity $\zeta(f_*(\gamma)) = \zeta(\gamma) \forall \gamma \in H_1(M_p)$. Now, as f_* is a homomorphism, it is of the form $f_* : \gamma \mapsto k\gamma$ for some $k \in \mathbb{Z}$. However, the above identities show that $\zeta(\gamma_0) = \zeta(f_*(\gamma_0)) = k^2\zeta(\gamma_0)$; hence $k^2 \cong 1 (\mathrm{mod}\ p)$ (as $\zeta(\gamma_0) \neq 0$). Since p is a prime, $k \cong \pm 1 (\mathrm{mod\ p})$, which proves the lemma. ∎

To prove the theorem, it suffices to show that $\rho_* : H_1(M) \to H_1(M)$ is the identity. By the lemma, we only have to rule out the action being an involution. However, this follows immediately as ρ has odd order. ∎

Remark 0.3. The first non-trivial application of the pq-condition is that the non-abelian group of order 21 is not a 3-manifold group. This has also been proved by Mennike [1] using different techniques. It is known that this group acts on higher dimensional spheres without fixed points.

References

1. E. G. Mennike *Finite fundamental groups of three-dimensional manifolds* Mat. Zametki **57** (1995), 105–117 MR **97e:**57001
2. J. Milnor *Groups which act on S^n without fixed points* Jour. Amer. Math. Soc. **79** (1957), 623–630. MR **19:**761d
3. K. Reidemeister *Kommutative Fundamentalgrüppen* Monatsch. Math. Phy. **43** (1935), 20–28.

4. P. Smith *Permutable periodic transformations* Proc. Nat. Acad. Sci. U.S.A. **30** (1944), 105–108. MR **5:**274d
5. H. Zassenhaus *Über endliche Fastkörper* Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **79** (1936), 187–220.

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