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SOME RAMANUJAN HYPERGRAPHS ASSOCIATED TO $GL(n, \mathbb{F}_q)$

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ABSTRACT. We present examples of hypergraphs constructed from homogeneous spaces of finite general linear groups. These hypergraphs are constructed using an invariant analogue of a hypervolume and their spectra are analyzed to see if they are Ramanujan in the sense of W.-C. W. Li and P. Solé.

1. INTRODUCTION

In this paper we study some hypergraphs associated to homogeneous spaces of the general linear group $GL(n, \mathbb{F}_q)$ of all non-singular $n \times n$ matrices with entries in the finite field $\mathbb{F}_q$ with q elements. These hypergraphs are somewhat easier to study than the analogues of the finite upper half plane graphs that form the subject of [6]. Here we can completely analyze the spectrum of our hypergraphs. In fact, we will be able to tell which of them are Ramanujan hypergraphs in the sense of Winnie Li and Patrick Solé in [4]. Ultimately we find an infinite family of Ramanujan hypergraphs.

In [9] we found Ramanujan graphs by considering finite analogues of real symmetric spaces along with finite analogues of Euclidean and non-Euclidean distances. Here we will obtain hypergraphs using finite analogues of volume elements.

Before proceeding let us give some background on hypergraphs. Every hypergraph has an underlying ordinary graph and many of its properties depend on the properties of this graph. We refer the reader to Biggs [1] and Terras [8] for the graph theory background. For further hypergraph background, the reader is referred to Fan Chung [2] as well as Feng and Li [3].

Definition 1.1. A **hypergraph** X consists of a vertex set $V(X)$ and an edge set $E(X)$ such that each element of $E(X)$ is a non-empty subset of $V(X)$. Our hypergraphs will be *(d,r)*-**regular** and finite. This means that any given edge contains r elements of $V(X)$ and each vertex of X is contained in d edges. We will usually denote $V(X)$ and $E(X)$ just by V and E, respectively, when no ambiguity can arise.

Note that for ordinary (undirected) graphs $r = 2$.

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Definition 1.2. The **adjacency matrix**, $A = A(X)$, is a $|V| \times |V|$ matrix with diagonal entries $A_{x,x} = 0$, for $x \in V$, and off-diagonal entries

$$A_{x,y} = \#\{e \in E | \{x, y\} \subset e\}, \text{ for } x, y \in V, \ x \neq y. \tag{1.1}$$

Thus, when we speak of the spectrum of a hypergraph X we are referring to the spectrum of $A(X)$. Since A is symmetric, it may be viewed as the adjacency matrix of a multi-graph X' called the *associated graph* of X. This may not be a $0, 1$ matrix, however. Note that Chung [2] considers a different adjacency matrix, which seems more difficult to analyze but which may be more useful in the long run. However, we will not consider Chung's adjacency matrix in this paper.

The sum of the entries in each row and column of A is $k = d(r-1)$=degree of X'. Thus k is an eigenvalue of A. Moreover, the eigenvalues λ of the adjacency matrix of a (d, r)-regular hypergraph satisfy the inequality

$$|\lambda| \leq d(r-1).$$

We have the following definition of a Ramanujan hypergraph given by Winnie Li and P. Solé [4], which is in the same spirit as that given by A. Lubotzky, R. Phillips, and P. Sarnak [5] for an ordinary regular graph.

Definition 1.3 (W.-C. W. Li and P. Solé)**.** A finite connected (d, r)-regular hypergraph X is a **Ramanujan hypergraph** if every eigenvalue λ of $A(X)$, $|\lambda| \neq d(r-1)$, satisfies

$$|\lambda - (r-2)| \leq 2\sqrt{(d-1)(r-1)}. \tag{1.2}$$

Some motivation for this definition comes from the following facts about regular hypergraphs. The diameter of a hypergraph is just the maximum of the minimum of all lengths of paths between any two vertices. The proofs are in Feng and Li [3].

Theorem 1.1 (K. Feng and W.-C. W. Li)**.** *Let X be a (d, r)-regular hypergraph with diameter greater than or equal to $2l+2 \geq 4$. Set $q = (d-1)(r-1) = k-(r-1)$, where $k = d(r-1)$ is the degree of the underlying graph. Let $\lambda_2(X)$ denote the second largest eigenvalue of the adjacency matrix of X. Then,*

$$\lambda_2(X) > r - 2 + 2\sqrt{q} - \frac{2\sqrt{q}-1}{l}.$$

Corollary 1.2. *Let $\{X_m\}_{m=1}^{\infty}$ be a family of connected (d, r)-regular hypergraphs with $|V(X_m)| \to \infty$ as $m \to \infty$. Then*

$$\liminf_{m\to\infty} \lambda_2(X_m) \geq r - 2 + 2\sqrt{q}\,.$$

2. Some homogeneous spaces of the general linear group

We want to consider a quotient G/H, where G denotes a subgroup of $GL(n, \mathbb{F}_q)$ and H is a subgroup of G. Such quotients are often called homogenous spaces.

Definition 2.1. Let

$$A(1, n-1) = \left\{ \left(\begin{array}{cc} 1 & {}^t b \\ 0 & c \end{array} \right) \middle| \ c \in GL(n-1, \mathbb{F}_q), b \in \mathbb{F}_q^{n-1} \right\}, \tag{2.1}$$

which may be viewed as an analogue of the *affine group.* Here $b \in \mathbb{F}_q^{n-1}$, the space of $(n-1)$-dimensional column vectors with entries in the finite field, and ${}^t b$ denotes the transpose of b.

Definition 2.2. The $\mathcal{A}_q^n$ space is defined to be

$$\mathcal{A}_q^n = GL(n,\mathbb{F}_q)/A(1,n-1) \cong \mathbb{F}_q^n - \{0\}. \tag{2.2}$$

The mapping from $GL(n,\mathbb{F}_q)$ to $\mathbb{F}_q^n - \{0\}$, which induces the identification in (2.2), is that of sending $g \in GL(n,\mathbb{F}_q)$ to its first column.

Definition 2.3. The $\mathcal{M}_q^n$ **space** is defined for $n \geq 3$ as

$$\mathcal{M}_q^n = \left\{ \left(\begin{array}{c} 1 \\ x \end{array} \right) \middle| \, x \in \mathbb{F}_q^{n-1} \right\}. \tag{2.3}$$

An element $g \in A(1,n-1)$, defined in (2.1), acts on a column vector $v \in \mathcal{M}_q^n$ by $v \mapsto {}^t g v$, where ${}^t g$ denotes the transpose of g.

Next define a subgroup of $A(1,n-1)$,

$$A_M(n) = \{ g \in A(1,n-1) |\, det(g) = \pm 1 \}. \tag{2.4}$$

The *origin* of $\mathcal{M}_q^n$ is defined to be the vector $o_n = {}^t(1\ 0)$. The group $A_M(n)$ acts transitively on $\mathcal{M}_q^n$ since if ${}^t\,(1\ x) \in \mathcal{M}_q^n$ and

$$\eta_x = \left(\begin{array}{cc} 1 & {}^t x \\ 0 & I_{n-1} \end{array} \right) \in A_M(n), \tag{2.5}$$

then ${}^t\eta_x o_n = {}^t(1\ x)$. Here I_{n-1} is the $(n-1)\times(n-1)$ identity matrix.

The *stabilizer* of the origin o_n is easily seen to be

$$\mathcal{L}_n = \left\{ \left(\begin{array}{cc} 1 & 0 \\ 0 & c \end{array} \right) \in A_M(n) \right\}.$$

Thus $\mathcal{L}_n$ is isomorphic to the subgroup of $GL(n-1,\mathbb{F}_q)$ consisting of matrices of determinant ± 1. Further $\mathcal{M}_q^n$ can be identified with $\mathcal{L}_n \backslash A_M(n)$.

We now define *n-point invariant functions* on $\mathcal{M}_q^n$ and $\mathcal{A}_q^n$, which are invariant under the appropriate group actions. These functions will be used to create hypergraphs with the elements of $\mathcal{A}_q^n$ (or $\mathcal{M}_q^n$) as the vertices. See Definitions 3.1 and 3.2 below.

Definition 2.4. For $x_1,...,x_n \in \mathbb{F}_q^n$, define the n**-point invariant function**

$$D(x_1,...,x_n) = det([x_1,...,x_n])^2, \tag{2.6}$$

where $[x_1,...,x_n]$ is the matrix whose ith column is x_i, for $i = 1,...,n$.

Notice that (at least when q is odd) we must square the determinant so that D will be invariant under permutation of entries. D is invariant under the action by elements g in the subgroup of $GL(n,\mathbb{F}_q)$ consisting of matrices of determinant ± 1 on vectors $x_j \in \mathbb{F}_q^n$, $j = 1,...,n$, since

$$\begin{aligned} D(gx_1,...,gx_n) &= det([gx_1,...,gx_n])^2 = det(g)^2 det([x_1,...,x_n])^2 \\ &= D(x_1,...,x_n), \text{ if } det(g) = \pm 1. \end{aligned} \tag{2.7}$$

3. Hypergraphs

In this section we consider two types of hypergraphs. The first (in Definition 3.1) is associated with $\mathcal{A}_q^n$ space. The second (in Definition 3.2) is associated with $\mathcal{M}_q^n$ space.

Definition 3.1. For $a \in \mathbb{F}_q$, **the alpha hypergraph** $\alpha_q^n(a)$ has as its vertex set the elements of $\mathcal{A}_q^n$, and an edge between $x_1, \dots, x_n \in \mathcal{A}_q^n$, pairwise distinct, if the n-point invariant $D(x_1, \dots, x_n) = a^2$. The definitions of $\mathcal{A}_q^n$ and D are found in equations (2.2) and (2.6), respectively.

Theorem 3.1. *For $n \in \mathbb{Z}$, $n \geq 3$, and $a \in \mathbb{F}_q^*$, $\alpha_q^n(a)$ is a (d_n, n)-regular hypergraph, where*

$$d_n = \frac{2q^{n-1}(q^n - q^1)(q^n - q^2)\dots(q^n - q^{n-2})}{\delta(n-1)!}$$

and $\delta = 2$ if q is even, $\delta = 1$ if q is odd. Further for $n \geq 3$, $a \neq 0$, $m = \frac{q^n-1}{q-1}$, the adjacency matrix of the alpha hypergraph $\alpha_q^n(a)$ is $A = \gamma_n C \otimes D$, where $C = J_m - I_m$, J_m is an $m \times m$ matrix of 1s, I_m is the $m \times m$ identity matrix, and $D = J_{q-1}$. Here $\gamma_n = \frac{d_n(n-1)}{q^n - q}$.

Proof. Given a vertex x_1 of $\alpha_q^n(a)$, i.e., a non-zero vector in $\mathbb{F}_q^n$, we seek x_2 linearly independent of x_1 over $\mathbb{F}_q$, x_3 linearly independent of x_1 and x_2, and continuing in this manner to choose the $x_j, \dots$, finally choose x_n linearly independent of $x_1, x_2, ..., x_{n-1}$ over $\mathbb{F}_q$. The number of such ordered $(n-1)$-tuples $x_2, ..., x_n$ is

$$\prod_{j=1}^{n-1}(q^n - q^j) = q^{n-1}(q-1)\prod_{j=1}^{n-2}(q^n - q^j).$$

To get the correct determinant, we must rescale the last column which introduces a factor of $\frac{2}{\delta(q-1)}$. We have now counted each edge $(n-1)!$ times and this gives the formula for d_n.

When $x = \lambda y$, for some $\lambda \in \mathbb{F}_q^*$, we have $A_{x,y} = 0$. Otherwise $A_{x,y}$ is the number of $\{z_3, ..., z_n\}$ such that $det(x, y, z_3, ..., z_n) = \pm a$. As above, this number is

$$\gamma_n = \frac{2}{\delta(q-1)(n-2)!}\prod_{j=2}^{n-1}(q^n - q^j).$$

It follows that $A_{x,y} = \gamma_n$, when $x \neq \lambda y$. Partition the elements of $\mathbb{F}_q^n - \{0\}$ into one-dimensional subspaces. Then C is the $m \times m$ matrix indexed by the one-dimensional subspaces, and D is the matrix indexed by $\mathbb{F}_q^*$. ∎

Corollary 3.2. *For $n \geq 3$, and $a \neq 0$, with γ_n and m as in the preceding theorem, the spectrum of the adjacency matrix of $\alpha_q^n(a)$ is given by the table below.*

eigenvalue	*multiplicity*
$d_n(n-1)$	1
0	$m(q-2)$
$-\gamma_n(q-1)$	$m-1$

Proof. The eigenvalues of D are $q-1$, with multiplicity 1, and 0, with multiplicity $q-2$. The eigenvalues of C are $m-1$, with multiplicity 1, and -1, with multiplicity $m-1$. The spectrum of $C \otimes D$ is the set of $\lambda\mu$, where $\lambda \in spec(C)$ and $\mu \in spec(D)$. ∎

Note that the spectrum is independent of $a \in \mathbb{F}_q^*$. In fact, the hypergraphs $\alpha_q^n(a)$ and $\alpha_q^n(1)$ are isomorphic for all non-zero a. Take an $n \times n$ matrix g with determinant a. Map a vector x in $\alpha_q^n(1)$ to gx in $\alpha_q^n(a)$. Use (2.7) to see that an edge $\{x_1, ..., x_n\}$ in $\alpha_q^n(1)$ is mapped to an edge $\{gx_1, ..., gx_n\}$ in $\alpha_q^n(a)$.

Theorem 3.3. *For $n \geq 3$, and $a \neq 0$, the hypergraph $\alpha_q^n(a)$ is Ramanujan only when $n = 3$ and $q = 2, 3, 4$ or when $n = 4$ and $q = 2$.*

Proof. The alpha hypergraph is Ramanujan if and only if we have

$$\gamma_n(q-1) + n - 2 \leq 2\sqrt{\gamma_n(q^n - q) - n + 1}. \tag{3.1}$$

When $q \geq 3$ and $n \geq 4$ inequality (3.1) fails since it is easy to see that $\gamma_n \geq q^n$. Here use the definition of γ_n in Theorem 3.1 and note that

$$\gamma_n = \frac{2}{\delta} q^{\frac{(n-2)(n+1)}{2}} \frac{q^{n-2}-1}{n-2} \frac{q^{n-3}-1}{n-3} \cdots \frac{q^2-1}{2}.$$

It follows that $\gamma_n \geq q^{\frac{(n-2)(n+1)}{2}}$. If $q = 2$ and $n \geq 5$, the inequality (3.1) also fails as one can see that $\gamma_n \geq 4q^n$.

For $q \geq 5$ and $n = 3$, the inequality (3.1) fails since $\gamma_3 = \frac{2q^2}{\delta}$ and $\gamma_3(q-1) > 4\frac{q^3-1}{q-1}$. It is easy to check that the inequality (3.1) holds in the remaining 4 cases. □

Now we turn to a study of the second type of hypergraph.

Definition 3.2. For $a \in \mathbb{F}_q$, the **mu hypergraph** $\mu_q^n(a)$ has as its vertex set $\mathcal{M}_q^n$, and an edge between the n distinct elements $x_1, \ldots, x_n \in \mathcal{M}_q^n$ if the n-point invariant $D(x_1, \ldots, x_n) = a^2$. The definitions of $\mathcal{M}_q^n$ and D are found in equations (2.3) and (2.6), respectively.

Theorem 3.4. *For $n \in \mathbb{Z}$, $n \geq 3$, and $a \in \mathbb{F}_q^*$, $\mu_q^n(a)$ is a (d_n, n)-regular hypergraph where*

$$d_n = \frac{2(q^{n-1}-1)(q^{n-1}-q)\ldots(q^{n-1}-q^{n-2})}{\delta(q-1)(n-1)!},$$

and $\delta = 2$ if q is even, $\delta = 1$ if q is odd. Furthermore, the adjacency matrix of the mu hypergraph $\mu_q^n(a)$ is $\kappa_n(J_m - I_m)$, where $\kappa_n = \frac{d_n(n-1)}{(q^{n-1}-1)}$, $m = q^{n-1}$, J_m is the matrix of ones, and I_m is the $m \times m$ identity matrix.

Proof. For $x_1, ..., x_n \in \mathbb{F}_q^n$ and $y_j = x_j - x_1$, for $2 \leq j \leq n$, we have

$$det(x_1, x_2..., x_n) = det(x_1, y_2, ..., y_n).$$

If x_1 is in $\mathcal{M}_q^n$, then each y_j has first coordinate 0 if and only if the corresponding $x_j \in \mathcal{M}_q^n$. Thus to count the number of edges in $\mu_q^n(a)$ containing a given x_1, we want to count the number of matrices in $GL(n-1, \mathbb{F}_q)$ of determinant $\pm a$ up to permutation of the columns. This leads to the given value of d_n as in the proof of Theorem 3.1. Likewise, to calculate an entry of the adjacency matrix A, the diagonal entries are 0, and the off-diagonal entries count the number of $GL(n-1, \mathbb{F}_q)$ matrices of determinant $\pm a$ with a given non-zero first column up to permutation of the last $n-2$ columns. This gives us κ_n. □

Corollary 3.5. *If $\mu_q^n(a)$ is as in Definition 3.2 and d_n and κ_n are as in Theorem 3.4, the spectrum of $\mu_q^n(a)$ is given by the following table.*

eigenvalue	*multiplicity*
$d_n(n-1)$	1
$-\kappa_n$	$q^{n-1}-1$

Proof. This goes just as in Corollary 3.2. □

The eigenvalues of $\mu_q^n(a)$ do not depend on $a \in \mathbb{F}_q^*$. Therefore, for fixed n and q, the $\mu_q^n(a)$ are isospectral for all $a \in \mathbb{F}_q^*$. Once more this happens because the graphs are isomorphic.

Our next question is: Is $\mu_q^n(a)$ Ramanujan? The following theorem classifies the hypergraphs $\mu_q^n(a)$, for non-zero a, with respect to the Ramanujan property. In this case we find an infinite number of Ramanujan hypergraphs.

Theorem 3.6. *For $n \geq 3$ and $a \in \mathbb{F}_q^*$, the hypergraph $\mu_q^n(a)$ is Ramanujan if and only if it is on the following list:*

$$\begin{array}{ll} (i) & n = 3\ , \quad or \\ (ii) & n = 4,\ q = 2. \end{array}$$

Proof. By Corollary 3.5, the only non-trivial eigenvalue of $\mu_q^n(a)$ is $-\kappa_n$. To check for the Ramanujan property we want to know if

$$\kappa_n + n - 2 \leq 2\sqrt{\kappa_n(q^{n-1} - 1) - n + 1}. \tag{3.2}$$

We consider first the case $n = 3$. In this case $\kappa_n = 2q/\delta$. So (3.2) becomes

$$\frac{2q}{\delta} + 1 \leq 2\sqrt{\frac{2q}{\delta}(q^2 - 1) - 2}.$$

It is easily seen that the Ramanujan bound holds.

When $n \geq 4$, we have

$$\kappa_n = 2\frac{q^{\frac{(n-2)(n-1)}{2}}(q^{n-2} - 1)(q^{n-3} - 1)\dots(q^2 - 1)}{\delta(n-2)!}\ ,$$

using Theorem 3.4. Thus if $n \geq 4$ and $q \geq 3$, or if $n \geq 5$ and $q = 2$, $\kappa_n \geq 7q^{n-1} > 4(q^{n-1} - 1)$, which implies that inequality (3.2) fails.

Now it only remains to check that (3.2) holds when $q = 2,\ n = 4$. □

Remark. The hypergraphs $\mu_q^n(0)$ are considered by Martínez in [7]. They give additional Ramanujan hypergraphs.

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