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LOCAL DERIVATIONS OF REFLEXIVE ALGEBRAS II

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ABSTRACT. Let $\mathcal{A}$ be a reflexive algebra in Banach space X such that both $0_+ \neq 0$ and $X_- \neq X$ in Lat$\mathcal{A}$. Then every local derivation of $\mathcal{A}$ into itself is a derivation.

1. Introduction

When attempting to find sufficient conditions for a linear mapping to be a derivation, an obvious candidate is the concept of a local derivation. Let $\mathcal{B}$ be a Banach algebra. We say that a linear transformation $\phi : \mathcal{B} \to \mathcal{B}$ is a local derivation if for every $B \in \mathcal{B}$ there is a derivation $\delta_B : \mathcal{B} \to \mathcal{B}$ depending on B, such that $\phi(B) = \delta_B(B)$. The theory of local derivation originates from Kadison's theorem, which says that each norm-continuous local derivation of a von Neumann algebra $\mathcal{B}$ on Hilbert space into a dual $\mathcal{B}$-bimodule $\mathcal{M}$ is a derivation. In the last few years interest in local derivations of operator algebras in Banach space has also been growing.

In [6] Larson and Sourour proved that every local derivation of $B(X)$ into itself is a derivation, where $B(X)$ denotes the algebra of all bounded linear operators on a complex Banach space X.

Han Deguang and Wei Shuyun [2] showed that every norm-continuous local derivation of some nest algebras is a derivation.

In our previous publication [3] we proved that if $\mathcal{A}$ is a reflexive algebra in reflexive Banach space X such that both $0_+ \neq 0$ and $X_- \neq X$ in Lat$\mathcal{A}$, then every norm-continuous local inner derivation of $\mathcal{A}$ into itself is a derivation.

We ought perhaps to mention that the conditions in [3] such as the reflexivity of X and the assumptions that δ is continuous and inner are very strong. It is the purpose of this paper to drop the assumptions on X and δ in the result of [3], thus extending the result to a more general setting. More precisely, we shall see that every local derivation of $\mathcal{A}$ into itself is a derivation. Note that our approach is quite different from that of [3] but is simple.

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2. Preliminaries and notations

In what follows we denote by X a fixed complex Banach space. The usual notation Lat$\mathcal{B}$ will denote the lattice of invariant subspaces for a subset $\mathcal{B} \subseteq B(X)$, and Alg$\mathcal{L}$ will denote the algebra of bounded linear operators leaving invariant every member of a family $\mathcal{L}$ of subspaces. $\mathcal{B}$ is reflexive if $\mathcal{B}$ =ref $\mathcal{B}$, where ref $\mathcal{B} = \{T \in B(X) : Tx \in [\mathcal{B}x], x \in X\}$ and $[\cdot]$ denotes the norm closure.

For a lattice $\mathcal{L}$ of subspaces of X, if $N \in \mathcal{L}$, we denote $\bigvee\{M \in \mathcal{L} : N \nsubseteq M\}$ by N_- and $\bigwedge\{M \in \mathcal{L} : M \nsubseteq N\}$ by N_+.

For a subset $\mathcal{S} \subseteq X$, $\mathcal{S}^\perp = \{f \in X^* : f(\mathcal{S}) = \{0\}\}$, where X^* is the dual space of X. If $x \in X$ and $f \in X^*$, the rank one operator $u \mapsto f(u)x$ is denoted by $x \otimes f$.

Let $L(X)$ denote the algebra of all linear transformations from X into itself. If $\mathcal{I}$ is a subset of $B(X)$, we write $\mathrm{ref}_a(\mathcal{I}) = \{T \in L(X) : Tx \in \mathcal{I}x, x \in X\}$, where $\mathcal{I}x = \{Sx : S \in \mathcal{I}\}$. The set $\mathcal{I}$ is said to be algebraically reflexive if $\mathcal{I} = \mathrm{ref}_a(\mathcal{I})$.

The following lemma is taken from [7]; it will get repeated use.

Lemma 2.1. *If $\mathcal{L}$ is a subspace lattice, then $x \otimes f \in$Alg$\mathcal{L}$ if and only if there exists an element $L \in \mathcal{L}$ such that $x \in L$ and $f \in (L_-)^\perp$.*

3. Local derivations

Throughout this section, $\mathcal{A}$ will be a reflexive algebra in Banach space X such that both $0_+ \neq 0$ and $X_- \neq X$ in Lat$\mathcal{A}$.

Lemma 3.1. *For $A \in \mathcal{A}$,*

(1) *if $RA = 0$ for every rank one operator $R \in \mathcal{A}$ of the form $x \otimes f$ with $x \in 0_+$ and $f \in X^*$, then $A = 0$;*

(2) *if $AR = 0$ for every rank one operator $R \in \mathcal{A}$ of the form $x \otimes f$ with $x \in X$ and $f \in (X_-)^\perp$, then $A = 0$.*

The proof of this lemma is straightforward so we omit it.

The following lemma can be found in [8] but we present the proof for the sake of completeness.

Lemma 3.2. *Let δ be a local derivation from a Banach algebra $\mathcal{B}$ into a $\mathcal{B}$-module $\mathcal{M}$. Then $\delta(PAQ) = \delta(PA)Q + P\delta(AQ) - P\delta(A)Q$ holds for each $A \in \mathcal{B}$ and any idempotents P and Q in $\mathcal{B}$.*

Proof. Let $A \in \mathcal{B}$. As δ is a local derivation, it is easy to see that

$$P^\perp \delta(PAQ)Q^\perp = 0$$

for any idempotents P, Q in $\mathcal{B}$, where $P^\perp = I - P$ and $Q^\perp = I - Q$ are also idempotents. Since

$$\begin{aligned}
&\delta(PAQ)Q^\perp - P\delta(AQ)Q^\perp \\
&\quad = [\delta(PAQ) - P\delta(AQ)]Q^\perp \\
&\quad = [(P^\perp\delta(PAQ) + P\delta(PAQ)) - (P\delta(P^\perp AQ) + P\delta(PAQ))]Q^\perp \\
&\quad = P^\perp\delta(PAQ)Q^\perp - P\delta(P^\perp AQ)Q^\perp \\
&\quad = 0,
\end{aligned}$$

hence

$$\delta(PAQ)Q^\perp = P\delta(AQ)Q^\perp.$$

Since $Q^{\perp} = I - Q$ is also an idempotent, similarly we have

$$\delta(PAQ^{\perp})Q = P\delta(AQ^{\perp})Q.$$

Then we obtain that

$$\begin{aligned} \delta(PAQ) - P\delta(AQ) &= (\delta(PAQ) - P\delta(AQ))Q^{\perp} + (\delta(PAQ) - P\delta(AQ))Q \\ &= (\delta(PAQ) - P\delta(AQ))Q \\ &= [(\delta(PA) - \delta(PAQ^{\perp})) - (P\delta(A) - P\delta(AQ^{\perp}))]Q \\ &= \delta(PA)Q - P\delta(A)Q + (P\delta(AQ^{\perp})Q - \delta(PAQ^{\perp})Q) \\ &= \delta(PA)Q - P\delta(A)Q, \end{aligned}$$

and so $\delta(PAQ) = \delta(PA)Q + P\delta(AQ) - P\delta(A)Q$. □

Theorem 3.3. *Let $\mathcal{A}$ be a reflexive algebra in Banach space X such that both $0_+ \neq 0$ and $X_- \neq X$ in Lat$\mathcal{A}$. Then every local derivation of $\mathcal{A}$ into itself is a derivation.*

Proof. Let $\delta : \mathcal{A} \to \mathcal{A}$ be a local derivation. Then $\delta(I) = 0$. By Lemma 3.2 we have that

$$\delta(PQ) = \delta(P)Q + P\delta(Q)$$

holds for any rank one projections P and Q in $\mathcal{A}$.

To prove that $\delta(AB) = \delta(A)B + A\delta(B)$ for every $A, B \in \mathcal{A}$, we divide the proof into several steps.

Step 1. For any rank one operator $P, Q \in \mathcal{A}$, where $P = x \otimes f$ with $x \in 0_+$ and $f \in X^*$, $Q = y \otimes g$ with $y \in X$ and $g \in (X_-)^{\perp}$, we have $\delta(PQ) = \delta(P)Q + P\delta(Q)$.

Case 1. If $f(x) \neq 0$ and $g(y) \neq 0$. Let $P' = P/f(x)$ and $Q' = Q/g(y)$. Then both P' and Q' are rank one projections, so we have

$$\delta(P'Q') = \delta(P')Q' + P'\delta(Q').$$

By the linearity of δ, we obtain that $\delta(PQ) = \delta(P)Q + P\delta(Q)$.

Case 2. If one of $f(x)$ and $g(y)$ is 0. Without loss of generality, suppose that $f(x) = 0$ and $g(y) \neq 0$. Choose $f' \in X^*$ such that $(f + f')(x) = f'(x) \neq 0$. Then by Case 1,

$$\begin{aligned} \delta(PQ) &= \delta(x \otimes (f + f') \cdot y \otimes g) - \delta(x \otimes f' \cdot y \otimes g) \\ &= \delta(x\otimes(f + f'))y\otimes g + x\otimes(f + f')\delta(y\otimes g) - \delta(x\otimes f')y\otimes g - x\otimes f'\delta(y\otimes g) \\ &= \delta(x \otimes f)y \otimes g + x \otimes f\delta(y \otimes g) \\ &= \delta(P)Q + P\delta(Q). \end{aligned}$$

Case 3. If both $f(x)$ and $g(y)$ are 0. Choose $f' \in X^*$ and $y' \in X$ such that $(f + f')(x) = f'(x) \neq 0$ and $g(y + y') = g(y') \neq 0$. With the same argument as in Case 2, we also have that $\delta(PQ) = \delta(P)Q + P\delta(Q)$.

Step 2. For each $A \in \mathcal{A}$ and any rank one operator $Q \in \mathcal{A}$, we have

$$\delta(AQ) = \delta(A)Q + A\delta(Q)$$

where $Q = y \otimes g$ with $y \in X$ and $g \in (X_-)^{\perp}$.

For any rank one operator $P \in \mathcal{A}$ of the form $x \otimes f$ with $x \in 0_+$ and $f \in X^*$, by using an argument similar to that used in Step 1, we know that Lemma 3.2 holds for each $A \in \mathcal{A}$ and any rank one operators $P, Q \in \mathcal{A}$, where $P = x \otimes f$ with $x \in 0_+$ and $f \in X^*$, $Q = y \otimes g$ with $y \in X$ and $g \in (X_-)^{\perp}$.

On the other hand, by Step 1,

$$\delta(PAQ) = \delta(PA \cdot Q) = \delta(PA)Q + PA\delta(Q).$$

Then we get that

$$P\delta(AQ) = P\delta(A)Q + PA\delta(Q);$$

by Lemma 3.1, this yields that

$$\delta(AQ) = \delta(A)Q + A\delta(Q).$$

Step 3. For any $A, B \in \mathcal{A}$, we have that $\delta(AB) = \delta(A)B + A\delta(B)$, i.e. δ is a derivation.

For each rank one operator $Q = y \otimes g \in \mathcal{A}$ with $y \in X$ and $g \in (X_-)^\perp$, by Step 2,

$$\delta(ABQ) = \delta(AB)Q + AB\delta(Q).$$

On the other hand, also by Step 2,

$$\delta(ABQ) = \delta(A \cdot BQ) = \delta(A)BQ + A\delta(BQ) = \delta(A)BQ + A\delta(B)Q + AB\delta(Q).$$

Thus we have that $\delta(AB)Q = \delta(A)BQ + A\delta(B)Q$, i.e. $(\delta(AB) - \delta(A)B - A\delta(B))Q = 0$.

By Lemma 3.1, we obtain that $\delta(AB) = \delta(A)B + A\delta(B)$, i.e. δ is a derivation. This completes the proof. □

Corollary 3.4. *The set of all derivations of $\mathcal{A}$ into itself is algebraically reflexive.*

Corollary 3.5. *If $\mathcal{N}$ is a nest on X such that both $0_+ \neq 0$ and $X_- \neq X$ in $\mathcal{N}$, then the set of all derivations of nest algebra Alg$\mathcal{N}$ into itself is algebraically reflexive.*

Corollary 3.6. *If $\mathcal{L}$ is a commutative subspace lattice on X such that both $0_+ \neq 0$ and $X_- \neq X$ in $\mathcal{L}$, then the set of all derivations of the CSL algebra Alg$\mathcal{L}$ into itself is algebraically reflexive.*

For the local derivations of standard algebras we have

Corollary 3.7. *If $\mathcal{B}$ is a standard operator algebra in X, then every local derivation of $\mathcal{B}$ into itself is a derivation.*

The following corollary is well known.

Corollary 3.8. *Every local derivation of $B(X)$ is a derivation.*

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