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C^1 SMOOTHNESS OF LIOUVILLE ARCS IN ARNOL'D TONGUES

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ABSTRACT. For the generic two parameter family of C^r circle diffeomorphisms of a general form we prove that the bifurcation arcs which correspond to Liouville irrational rotation numbers are C^1 smooth. As a consequence, we give an explicit formula for the derivative of all non-resonance arcs. Results of Arnol'd, Herman, and others give greater smoothness for a more restricted class of rotation numbers using KAM techniques.

1. INTRODUCTION

Throughout this paper $f_{a,\alpha}$ is a two parameter family of degree one orientation preserving circle diffeomorphisms of the form

$$f_{a,\alpha} = \{x + a + \alpha\gamma(x)\} \pmod 1,$$

where $x \in [0,1), a \in [0,1], \alpha \in [0,1)$; and $\gamma \in C^r(S^1)$ for $1 < r \leq \infty$, with $|\gamma'(x)| \leq 1$, γ is periodic with period 1, γ has average 0, i.e., $\overline{\gamma} = \int_0^1 \gamma(x)dx = 0$; and $S^1 = \mathbf{R}/\mathbf{Z}$ is the circle with unit length. Families of this form have been extensively studied. See for example Arnold [1], Boyland [2], Hall [5], Herman [6].

Hall [5] proved that at $\alpha = 0$, all bifurcation arcs in the Arnol'd Tongue Picture (rotation number diagram) of $f_{a,\alpha}$ are C^1 and that non-resonance arcs that correspond to Liouville rotation numbers are not C^2, at $\alpha = 0$. In the same paper, Hall also conjectured:

Conjecture. *For the generic family, $f_{a,\alpha}$, all non-resonance arcs are C^1 smooth.*

In this paper we prove the conjecture of Hall. This is significant because previous results (cf. Arnol'd [1], Herman [6], Khanin and Sinai [8], Stark [11], Yoccoz [12]) imply greater smoothness but for a more restricted set of rotation numbers. All such results relating the smoothness of the non-resonance arcs to the smoothness of $f_{a,\alpha}$ involve some Diophantine conditions on the class of admissable rotation numbers, which exclude the Liouville ones. Our methods do not use such number theoretic restrictions, and thus include all irrational rotation numbers.

The main idea of the proof is to approximate irrational arcs with rational arcs in the C^1 topology. This is shown to be equivalent to the approximation of probability

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measures in the weak* topology. We use a theorem of Douady and Yoccoz (cf. De Melo and Pugh [3]), that any C^2 diffeomorphism of the circle with irrational rotation number determines a unique continuous measure with special properties. Because of the existence of isolated points where the rational arcs are not even C^1 smooth, we suspect that C^1 is the best we can do.

We give an explicit formula for the derivative of an irrational arc in terms of the unique measure of Douady and Yoccoz. For the standard family we conclude that Liouville arcs in particular are C^1-smooth.

The term *generic* here means that there exists a residual subset in the ambient space of γ, i.e., the Bananch space $C_0^r(S^1) = \{\gamma \in C^r(S^1) : \bar{\gamma} = 0, |\gamma'(x)| \leq 1\}$, whose elements give rise to families for which their corresponding rational arcs have finitely many isolated non-smooth points. (A point (a, α) on a rational arc is *non-smooth* if the corresponding diffeomorphism $f_{a,\alpha}$ has more than one degenerate periodic orbit.) Because of the particular form of $f_{a,\alpha}$ the C^r-smoothness of $f_{a,\alpha}$ depends only on γ.

The *rotation number* of $f_{a,\alpha}(.)$ is defined to be

$$\rho(f_{a,\alpha}) = \lim_{n\to\infty} \frac{f_{a,\alpha}^n(x) - x}{n}.$$

Poincaré showed that $\rho(f_{a,\alpha})$ exists, is independent of the point x, and is continuous in (a, α). Moveover, $\rho(f_{a,\alpha}) = \frac{p}{q} \in \mathbf{Q}$, if and only if $\exists x \in [0, 1)$ such that $f_{a,\alpha}^q(x) = x + p$, i.e., f has periodic points of period $\frac{p}{q}$. (Cf. Herman [6].)

The Arnol'd Tongue Picture for $f_{a,\alpha}$ is best described by:

1.1 Theorem (cf. Boyland [2])**.** (1) *For each rational $\frac{p}{q}$ there exist Lipschitz functions $a^+, a^- : [0, 1) \to \mathbf{R}$ such that*:

(a) $\forall \alpha \in [0, 1)$ $a^-(\alpha) \leq a^+(\alpha)$,
(b) $a^-(0) = \frac{p}{q} = a^+(0)$,
(c) $(a, \alpha) \in H_{\frac{p}{q}} = \{(a, \alpha) : \rho(f_{a,\alpha}) = \frac{p}{q}\}$ *if and only if* $a^-(\alpha) \leq a^+(\alpha)$.

Moverover the Lipschitz constant of a^+, a^- is independent of $\frac{p}{q}$ and depends only on γ.

(2) *For any given irrational η there exists a Lipschitz function $a_\eta(\alpha) : [0, 1) \to \mathbf{R}$ such that $\rho(f_{a,\alpha}) = \eta$ if and only of $a = a_\eta(\alpha)$. Again the Lipschitz constant is independent of η.*

The Arnol'd Tongue Picture exhibits three distinct bifurcation phenomena: Resonance horns (Arnol'd Tongues), Diophantine and Liouville arcs. Resonance horns correspond to diffeomorphisms with rational rotation numbers and periodic behavior. Such diffeomorphisms are typically Morse-Smale (structurally stable) and so determine regions in the parameter plane with non-empty interior. The boundary of a resonance horn consists of a pair of arcs, referred to as *rational arcs*. Irrational arcs, i.e., Diophantine and Liouville arcs correspond to Diophantine and Liouville irrational rotation numbers respectively. They exist inbetween the resonance regions and correspond to non-resonance. (For definitions of these number theoretic properties see for example Herman [6], or Khinchin [9].)

2.1 s-invariant measures. The rational arcs are given by the Implicit Function Theorem, so they are piecewise as smooth as γ. Following [10], at a smooth point

(a, α) on a $\frac{p}{q}$-rational arc $a(\alpha)$, the derivative is given by

$$a'(\alpha) = -\frac{\sum_{i=0}^{q-1} \frac{1}{Df^i_{a,\alpha}(x)} \gamma(f^{i-1}_{a,\alpha}(x))}{\sum_{i=0}^{q-1} \frac{1}{Df^i_{a,\alpha}(x)}}$$

where $f_{a,\alpha}$ is the diffeomorphism corresponding to (a, α), and $x \in Per(f_{a,\alpha})$. This formula can be written as a discrete integral

$$a'(\alpha) = -\int \gamma(f^{-1}_{a,\alpha}(x) d\mu_\alpha(x)$$

where μ_α is an *rdf-measure*, i.e., a discrete Borel probability measure which is supported on the degenerate orbit of $f_{a,\alpha}$, and defined by

$$\mu_\alpha(A) = k \sum_{i=0}^{q-1} \frac{1}{Df^i_{a,\alpha}(x)} \delta_{f^i_{a,\alpha}(x)}(A).$$

Here, A is a Borel set, $\delta_{f^i_{a,\alpha}(x)}$ is the δ-measure on $f^i_{a,\alpha}(x)$, and k is a normalizing constant, i.e.,

$$\frac{1}{k} = \sum_{i=0}^{q-1} \frac{1}{Df^i_{a,\alpha}(x)}.$$

The formula for $a'(\alpha)$ is a weighted average, where the weights are given by the reciprocal of the derivatives of the iterates along the orbit. Hence the acronym, *rdf-measure*. It is easy to see that the assignment of weights is independent of the initial choice of a point x in the orbit, and so a degenerate periodic orbit supports a unique *rdf*-measure.

In general, if f is a circle diffeormorphism with a degenerate orbit, which supports an *rdf*-measure, μ, then μ transforms under f as follows:

$$\begin{aligned}\int \phi(f(x)) \frac{1}{Df(x)} d\mu(x) &= \frac{\sum_{i=0}^{q-1} \frac{1}{Df^i(y)} \phi(f(f^i(y))) \frac{1}{Df(f^i(y))}}{\sum_{i=0}^{q-1} \frac{1}{Df^i(y)}} \\ &= \frac{\sum_{i=0}^{q-1} \frac{1}{Df^i(y)} \phi(f^i(y))}{\sum_{i=0}^{q-1} \frac{1}{Df^i(y)}} \\ &= \int \phi(x) d\mu(x),\end{aligned}$$

for $y \in Per(f)$, and ϕ any test function in $C(S^1)$.

This kind of transformation rule has a more general significance.

2.2 Definition (De Melo and Pugh [3])**.** Let f be a circle diffeomorphism, and μ a Borel probability measure on S^1. Then μ is *s-invariant under* f if

$$\int_{S^1} \phi(x) d\mu(x) = \int_{S^1} \phi(f(x)) (Df(x))^s d\mu(x)$$

for all test functions $\phi \in C(S^1)$, for any $s \in \mathbf{R}$.

Thus *rdf*-measures are s-invariant when $s = -1$. In their study of measures that exhibit this kind of transformation rule, Douady and Yoccoz in unpublished work (cf. De Melo and Pugh [3]) proved the following result which we use:

2.3 Theorem (Douady-Yoccoz)**.** *If f is a $C^r (r \geq 2)$ circle diffeomorphism with irrational rotation number, then for each $s \in \mathbf{R}$ there exists a unique Borel probability measure μ_s such that*:

(a) *μ_s is s-invariant under f,*

(b) *μ_s is atomless,*

(c) *μ_s has support S^1, and*

(d) *f is μ_s-ergodic.*

Moreover, the uniqueness of μ_s is among all signed Borel probability measures on S^1 satisfying the above properties.

2.4 *Remarks.* (1) In the light of subsection 2.1 and Theorem 2.3, the case $s = -1$ is of special significance for circle diffeomorphisms. Other special cases are discussed in De Melo and Pugh [3].

(2) The property of s-invariance is natural under iteration. For any $k \in \mathbf{Z}$,

$$\int \phi(x) d\mu(x) = \int \phi(f^k(x))(Df^k(x))^s d\mu(x).$$

(3) For purposes of generalisation, a *discrete s-measure* is defined as in the case of an *rdf*-measure but with an adapted weighting system, i.e., replace $\frac{1}{Df^i(x)}$ with $[Df^i(x)]^s$ (for $s \in \mathbf{R}$) in the definition of an *rdf*-measure.

In the context of the two parameter families $f_{a,\alpha}$ we show that continuous s-measures can be approximated with discrete s-measures.

2.5 Theorem. *Let $f_{a,\alpha}$ be a generic two parameter family and $s \in \mathbf{R}$. Let $a_{r_n}(\alpha)$ be a sequence of rational arcs converging to an irrational arc $a_\eta(\alpha)$, and let $\mu_{\eta,\alpha}$ be the unique s-measure corresponding to (a_η, α) and $\mu_{n,\alpha}$ the s-measures corresponding to (a_{r_n}, α) for each n. Then*:

(a) *$\mu_{n,\alpha}$ converges in the weak*-topology to $\mu_{\eta,\alpha}$,*

(b) *for any test function ϕ,*

$$\int \phi(x) d\mu_{n,\alpha} \to \int \phi(x) d\mu_{\eta,\alpha},$$

(c) *for $s = -1, \phi(x) = \gamma(f_{a,\alpha}^{-1}(x)$,*

$$a_n'(\alpha) \to \int \phi(x) d\mu_{\eta,\alpha}(x)$$

as $n \to \infty$, for almost every α.

Proof. Since (b), (c) follow immediately from (a), we prove (a): Let $f_{n,\alpha}, f_{\eta,\alpha}$ be the corresponding diffeomorphisms of $(a_{r_n}, \alpha), (a_\eta, \alpha)$ respectively. As $a_n(\alpha) \to a_\eta(\alpha)$, $f_{n,\alpha} \to f_{\eta,\alpha}$ in the C^r-topology. The sequence of discrete probability s-measures $\mu_{n,\alpha}$ is included in the unit ball of the dual space of $C(S^1)$ (a space of measures, cf. Diestel [4]). By the Banach-Alaoglu Theorem, it has weak*-limit points. Let μ be such a limit point. Then there exists a subsequence $\mu_{n_k,\alpha}$ which converges to μ as $k \to \infty$. Therefore, for all test functions ϕ, as $k \to \infty$ we have

$$\begin{aligned} \mu_{n_k,\alpha} \to \mu &\iff \int \phi d\mu_{n_k,\alpha} \to \int \phi d\mu \\ &\iff \int \phi \circ f_{n_k,\alpha}(x)(Df_{n_k,\alpha}(x))^s d\mu_{n_k,\alpha}(x) \to \int \phi(x) d\mu(x). \end{aligned}$$

The last equivalence follows because each $\mu_{n_k,\alpha}$ is s-invariant. By continuity,

$$\int \phi \circ f_{n_k,\alpha}(x)(Df_{n_k,\alpha})_x^s d\mu_{n_k,\alpha}(x) \to \int \phi \circ f_{\eta,\alpha}(x)(Df_{\eta,\alpha}(x))^s d\mu(x).$$

So by the uniqueness of limits it follows that

$$\int \phi \circ f_{\eta,\alpha}(x)(Df_{\eta,\alpha}(x))^s d\mu(x) = \int \phi(x) d\mu(x).$$

Thus μ is s-invariant under f, and by the Douady-Yoccoz theorem, such a measure is unique. So $\mu = \mu_{\eta,\alpha}$. Since this is true for all subsequences, we must have that the sequence $\mu_{n,\alpha}$ itself converges to the s-measure $\mu_{\eta,\alpha}$.

For the generic family, there are only countably many non-smooth points on rational arcs which do not have unique s-measures (cf. [10]). Therefore, the convergence is pointwise almost everywhere. □

2.6 *Remark.* (1) Let I be any compact subset of $[0,1)$. Define a function $g : I \to \mathbf{R}$ by

$$g : \alpha \to -\int \gamma(f_{a,\alpha}^{-1}(x) d\mu_{\eta,\alpha}(x).$$

Then g is continuous everywhere. Indeed, if α_n is a sequence in I converging to α, then by the same argument used in Theorem 2.5, μ_{η,α_n} converges to $\mu_{\eta,\alpha}$ in the weak* topology. Therefore, $g(\alpha_n)$ converges to $g(\alpha)$ as $n \to \infty$

(2) It is possible that (a_η, α) is an accumulation point of non-smooth points (a_n, α) for fixed α. Generically, there are at most countably many accumulation points (Theorem 3.1). So even though the approximation scheme in Theorem 2.5 does not apply at such accumulation points, the continuity of g makes it possible to extend differentiability to such points (Theorem 2.8).

(3) Diophantine arcs have been shown to be smooth (cf. Herman [6], Yoccoz [12]). We give their derivatives in Theorem 2.8.

For the standard family, there are no non-smooth points as the following argument shows. Thus in particular the Liouville arcs are C^1 everywhere.

2.7 The standard family. Let $f_{a,\alpha}$ be the standard family

$$f_{a,\alpha}(x) = x + a + \frac{\alpha}{2\pi}\sin(2\pi x)$$

where $x \in [0,1), a \in [0,1], \alpha \in [0,1)$. Consider the complexification of this family as follows:

$$f_{a,\alpha}(z) = z + a + \frac{\alpha}{2\pi}\sin(2\pi z)$$

where $z \in \mathbf{C}$, and a and α are real parameters. Then by complex conjugation, $f(\overline{z}) = \overline{f(z)}$. Since $\alpha < 1$, the critical points of $f_{a,\alpha}(z)$ occur in conjugate pairs. Any neutral periodic point with derivative 1 must have at least one of these critical points in its basin of attraction. However, given the symmetry with respect to complex conjugation, all the critical points must lie in its basin of attraction. If we now project onto the circle by passing through the quotient $\mathbf{C}/\mathbf{Z}, t \to e^{2\pi i t}$ it follows that each $f_{a,\alpha}$ on a rational arc can have at most one degenerate periodic orbit (cf. Herman [7]).

This implies by Remark 2.6(1) that the map

$$g : \alpha \to \int \sin(f_{a,\alpha}^{-1}(2\pi x))d\mu_{a,\alpha}(x)$$

is uniformly continuous on any compact subset of $[0,1)$. So if a sequence of arcs $a_n(\alpha) \to a(\alpha)$ as $n \to \infty$, then their corresponding sequence of derivatives exist, are uniformly bounded, and equicontinuous in α. Hence by the Arzela-Ascoli theorem the convergence of the corresponding sequence of derivatives is uniform. Hence all the arcs are C^1 everywhere. □

2.8 Theorem. *Let $f_{a,\alpha}$ be a generic family. Let a_η be an irrational arc defined on any compact subset I of $[0,1)$. Let $\mu_{\eta,\alpha}$ be the unique measure corresponding to (a_η, α) for each α, given by Theorem 2.3. Then a_η is C^1-smooth everywhere, and*

$$a'_\eta(\alpha) = -\int \gamma(x)\frac{1}{Df_{a,\alpha}(x)}d\mu_{\eta,\alpha}(x).$$

In particular, the Liouville arcs are C^1 smooth everywhere.

Proof. Let $a_{r_n}(\alpha)$ be a sequence of rational arcs defined on I with corresponding *rdf*-measures $\mu_{n,\alpha}$, converging to $a_\eta(\alpha)$, and let g be the map as defined in Remark 2.6(1). Then by subsection 2.1 and Theorem 2.5 (for $s = -1$),

$$a'_{r_n}(\alpha) = -\int_0^\alpha \gamma \circ f_{a,\alpha}^{-1} d\mu_{n,\alpha} \to g(\alpha)$$

for almost every α, as $n \to \infty$. Since the arcs $a_{r_n}(\alpha)$ are uniformly Lipschitz, hence absolutely continuous, the sequence $\{a'_{r_n}(\alpha)\}$ is uniformly bounded by $\sup\gamma$,

$$a_{r_n}(\alpha) = \int_0^\alpha a'_{r_n}(t)dt \to \int_0^\alpha g(t)dt$$

for all α in I, as $n \to \infty$. However, $a_{r_n} \to a_\eta$ as $n \to \infty$, so that

$$a_\eta(\alpha) = \int_0^\alpha g(t)dt$$

for every α. Hence

$$a'_\eta(\alpha) = g(\alpha) = -\int \gamma(f_{a,\alpha}^{-1}(x))d\mu_{\eta,\alpha}(x) = -\int \gamma(x)\frac{1}{Df_{a,\alpha}(x)}d\mu_{\eta,\alpha}(x).$$

The last equality follows by $(s = -1)$-invariance of $\mu_{\eta,\alpha}$. □

The generic context of the above results is as follows.

3.1 Theorem ([10])**.** *Given the Banach space*

$$C_0^r(S^1) = \{\gamma \in C^r(S^1) : \overline{\gamma} = 0, |\gamma'(x)| \leq 1\},$$

for $1 < r \leq \infty$. Then there exists a residual class $\mathbf{T}$ in $C_0^r(S^1)$ such that for each $\gamma \in \mathbf{T}$, the family $f_{a,\alpha}(x) = x + a + \alpha\gamma(x)$ has the property that its corresponding rational arcs in the (a,α)-parameter plane have at most finitely many non-smooth points.

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