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PROOF OF WANG'S CONJECTURE ON SUBSPACES OF AN INNER PRODUCT SPACE

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ABSTRACT. B.Y. Wang conjectured that if R_t and S_t $(t = 1, \dots, k)$ are subspaces of an n-dimensional complex inner product space V, and their dimensions are i_t and $n - i_t + 1$, respectively, where $1 \le i_1 < i_2 < \cdots < i_k \le n$, then there exists a k-dimensional subspace W having two orthonormal bases $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ with $x_t \in R_t$ and $y_t \in S_t$ for all t.

We prove this conjecture and its real counterpart. The proof is in essence an application of a real version of the Bézout Theorem for the product of several projective spaces.

1. INTRODUCTION

Let V be an n-dimensional complex inner product space and $(i_1, i_2, \dots, i_k)$ a sequence of integers such that $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $R_1, R_2, \dots, R_k$ and $S_1, S_2, \dots, S_k$ be subspaces of V whose dimensions are

$$\dim R_t = i_t, \quad \dim S_t = n - i_t + 1, \quad t = 1, 2, \dots, k.$$

Unless stated otherwise, these hypotheses and notations will be used throughout the paper and will not be repeated (even in the statements of the theorems). Later on we shall allow V to be a real inner product space.

A.R. Amir-Moéz proved in 1956 (see [1]) the following result (which he attributed to A. Horn).

Theorem 1. *Assume that $R_1 \subset R_2 \subset \cdots \subset R_k$ and $S_1 \supset S_2 \supset \cdots \supset S_k$. Then there exists a k-dimensional subspace W of V having two bases $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ such that $x_t \in R_t$ and $y_t \in S_t$ for all t.*

Clearly the two bases mentioned in this theorem can be chosen to be orthonormal (just apply the Gram-Schmidt orthogonalization procedure).

In 1986 B.Y. Wang [3] proved the following stronger result. (He used it to obtain a new version of a theorem of Wielandt and Lidskiĭ (see [1]) giving max-min formulas for sums of subsequences of the sequence of all eigenvalues of a Hermitian matrix arranged in nonincreasing order.)

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Theorem 2. *Assume that* $R_1 \subset R_2 \subset \cdots \subset R_k$ *(or* $S_1 \supset S_2 \supset \cdots \supset S_k$*). Then there exists a* k*-dimensional subspace* W *of* V *having two orthonormal bases* $\{x_1, \ldots, x_k\}$ *and* $\{y_1, \ldots, y_k\}$ *such that* $x_t \in R_t$ *and* $y_t \in S_t$ *for all* t*.*

In 1990 he proposed a conjecture (see [4]) which is the main topic of our paper. Before stating the conjecture, we mention yet another result of Wang which he claims can be proved by a method similar to that used in [3].

Theorem 3. *There exists a* k*-dimensional subspace* W *of* V *having an orthonormal basis* $\{x_1, \ldots, x_k\}$ *and another basis* $\{y_1, \ldots, y_k\}$ *such that* $x_t \in R_t$ *and* $y_t \in S_t$ *for all* t*.*

We now state the above mentioned conjecture.

Wang's conjecture. *There exists a* k*-dimensional subspace* W *of* V *having two orthonormal bases* $\{x_1, \ldots, x_k\}$ *and* $\{y_1, \ldots, y_k\}$ *such that* $x_t \in R_t$ *and* $y_t \in S_t$ *for all* t*. (In other words, the inclusion conditions for the* R_i*'s in Theorem 2 can be dropped.)*

Remarks. 1) The case $k = 1$ of this conjecture is trivial.

2) The conjecture is true if $i_t = t$ for $t = 1, 2, \ldots, k$. Wang [5] states that the proof of this statement (as well as of another special case) was sent to him in 1994 by M.A.S. Shojaei and M. Shahryari (from Iran). This proof can be safely left to the reader.

3) The conjecture is also true if $i_t = n - k + t$ for $t = 1, 2, \ldots, k$. This follows from the previous remark by interchanging R_t and S_{k+1-t} for $t = 1, 2, \ldots, k$.

4) The conjecture is true if $k = n$. This follows immediately from the second remark.

Wang's conjecture is meaningful also for real inner product spaces. We shall refer to that variant as the *real Wang conjecture* although Wang himself did not formulate that version. The above remarks are also valid for the real Wang conjecture.

In Section 2 we prove both versions of the Wang conjecture when $k = 2$. In Section 3 we give the proof of the Wang conjectures in the general case. Although the case $k = 2$ is not used in the general proof, we have decided to include that proof because it is simpler and different from the proof in the general case. We point out that the main ideas involved in our proof are quite different from those in [3].

We thank Professor B.Y. Wang for the notes [5] on his conjecture which he kindly gave to the first author during a short visit to Waterloo in the Summer of 1998.

2. The case $k = 2$

We shall prove that real and complex Wang conjectures are true in the case $k = 2$. Our proof is based on the following (not-so-well-known) real variant of Bézout's theorem from algebraic geometry (see [2, Chapter IV, §2, Theorem 2]).

Theorem 4. *Consider the system of equations*

$$F_i(x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m) = 0, \quad i = 1, 2, \ldots, m+n,$$

where F_i *is a polynomial with real coefficients which is homogeneous of degree* k_i *in the variables* $x_0, x_1, \ldots, x_n$ *and homogeneous of degree* ℓ_i *in the variables*

$y_0, y_1, \dots, y_m$. If the sum

$$\sum k_{i_1} k_{i_2} \cdots k_{i_n} \ell_{j_1} \ell_{j_2} \cdots \ell_{j_m}, \tag{1}$$

taken over all permutations $(i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_m)$ of $(1, 2, \dots, n+m)$ such that $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_m$, is odd, then the above system has a nontrivial real solution, i.e., a solution such that all x_i's and y_j's are real numbers and at least one x_i and at least one y_j are not zero.

Shafarevich illustrated the usefulness of this theorem by giving a simple proof of the well-known fact that the dimension of a finite-dimensional real nonassociative division algebra (i.e., the one having no zero divisors) is a power of 2. (The harder fact that this dimension is 1, 2, 4, or 8 is not proved there.)

The sum (1) will be referred to herein as the *Bézout number* for a system of equations.

Theorem 5. *The real Wang conjecture is true if $k = 2$.*

Proof. We have to prove that there exists a 2-dimensional subspace W of V having two orthonormal bases $\{x_1, x_2\}$ and $\{y_1, y_2\}$ such that $x_1 \in R_1$, $x_2 \in R_2$, $y_1 \in S_1$, and $y_2 \in S_2$.

Assume that $R_1 \cap S_2 \neq 0$. We choose a unit vector $x_1 = y_2 \in R_1 \cap S_2$. As the sum of the dimensions of R_2 and S_1 is $n + 1 + i_2 - i_1 \geq n + 2$, we can choose a unit vector $x_2 = y_1 \in R_2 \cap S_1$ such that $x_1 \perp x_2$. By taking $W = \langle x_1, x_2 \rangle$, the subspace spanned by x_1 and x_2, we see that the theorem holds. From now on we assume that $R_1 \cap S_2 = 0$.

Fix a basis $\{a_1, a_2, \dots, a_{i_1}\}$ of R_1 and a basis $\{b_1, b_2, \dots, b_{n-i_2+1}\}$ of S_2. We write arbitrary vectors $x_1 \in R_1$ and $y_2 \in S_2$ as linear combinations:

$$x_1 = \xi_1 a_1 + \xi_2 a_2 + \cdots + \xi_{i_1} a_{i_1}, \quad y_2 = \eta_1 b_1 + \eta_2 b_2 + \cdots + \eta_{n-i_2+1} b_{n-i_2+1}.$$

As R_2 has codimension $n - i_2$, it is the zero set of $n - i_2$ linear functions, say $f_1, f_2, \dots, f_{n-i_2}$. Similarly, let $g_1, g_2, \dots, g_{i_1-1}$ be linear functions defining the subspace S_1.

Consider the system of equations

$$\begin{aligned} &(f_i(y_2)x_1 - f_i(x_1)y_2 | x_1) = 0, && i = 1, 2, \dots, n - i_2, \\ &(g_i(y_2)x_1 - g_i(x_1)y_2 | y_2) = 0, && i = 1, 2, \dots, i_1 - 1, \end{aligned}$$

in the real variables $\xi_1, \dots, \xi_{i_1}$ and $\eta_1, \dots, \eta_{n-i_2+1}$, where $(x|y)$ denotes the inner product of x and y.

The first set of equations (the set with f_i's) is homogeneous of degree two in the ξ's and homogeneous of degree one in the η's. The second set of equations is homogeneous of degree one in the ξ's and homogeneous of degree two in the η's. Thus, the sum defining the Bézout number for this system has a summand equal to 1 which corresponds to the case when all the degrees $d(i, \sigma(j))$ are one. All other terms in the sum are even. Therefore, the Bézout number is odd, so by Theorem 4 we can choose nonzero x_1 and y_2 which solve the equations above.

As $R_1 \cap S_2 = 0$, x_1 and y_2 are linearly independent and so if we set $W = \langle x_1, y_2 \rangle$, then $\dim W = 2$. Define $x_2, y_1 \in W$ by

$$x_2 = \|x_1\|^2 y_2 - (y_2|x_1)x_1, \quad y_1 = \|y_2\|^2 x_1 - (x_1|y_2)y_2.$$

Then $x_1 \perp x_2$, $y_1 \perp y_2$, and x_2 and y_1 are nonzero since $x_1, y_2 \neq 0$. Moreover, $x_2 \in R_2$ and $y_1 \in S_1$, since $f_i(x_2) = 0$ for $i = 1, 2, \dots, n - i_2$ and $g_i(y_1) = 0$ for $i = 1, 2, \dots, i_1 - 1$ by our equations above.

By normalizing x_1, x_2, y_1, and y_2 we have our orthonormal bases. Hence, the real Wang conjecture holds for $k = 2$. □

Theorem 6. *The complex Wang conjecture is true if* $k = 2$.

Proof. The proof for the complex case is essentially the same as in the real case. However, when applying Theorem 4 and calculating the Bézout number we must separate our variables and equations into real and imaginary parts. We set the imaginary parts of ξ_1 and η_1 to zero. Then the number of ξ variables is $2i_1 - 1$, and the number of η variables is $2(n - i_2) + 1$. The first set of equations is of size $2(n - i_2)$, and the second set of equations is of size $2(i_1 - 1)$. Again, there is only one odd number in the sum defining the Bézout number, so the Bézout number is odd. □

3. The general case

The version of the Bézout theorem given in the previous section is not general enough for our use. We shall state a more general version which can be proved by the same method as the one used in [2].

More elaborate notation is necessary. Let

$$x_i = (\xi_0^{(i)}, \xi_1^{(i)}, \dots, \xi_{n_i}^{(i)}), \quad 1 \leq i \leq m,$$

be m sets of variables, and let $n = n_1 + n_2 + \cdots + n_m$. We consider the system of n equations

$$F_j(x_1, x_2, \dots, x_m) = 0, \quad 1 \leq j \leq n,$$

where F_j is a polynomial with real coefficients which is homogeneous in each of the m sets of variables separately and has degree $d(i, j)$ in the variables x_i.

We define the *Bézout number* of this system to be the nonnegative integer

$$N = \sum \prod_{1 \leq j \leq n_1} d(1, \sigma(j)) \cdot \prod_{n_1 < j \leq n_1 + n_2} d(2, \sigma(j)) \cdots \prod_{n_1 + \cdots + n_{m-1} < j \leq n} d(m, \sigma(j)),$$

where the sum is taken over all permutations σ of the set $\{1, 2, \dots, n\}$ such that

$$\begin{gathered} \sigma(1) < \sigma(2) < \cdots < \sigma(n_1), \\ \sigma(n_1 + 1) < \sigma(n_1 + 2) < \cdots < \sigma(n_1 + n_2), \\ \vdots \\ \sigma(n_1 + \cdots + n_{m-1} + 1) < \sigma(n_1 + \cdots + n_{m-1} + 2) < \cdots < \sigma(n). \end{gathered}$$

There is an alternative way of writing the above formula. Let F denote the set of all functions

$$f : \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$$

such that $|f^{-1}(i)| = n_i$ for each i. Then

$$N = \sum_{f \in F} \prod_{1 \leq j \leq n} d(f(j), j). \tag{2}$$

We can now state the generalization.

Theorem 7. *If the Bézout number N is odd, then the above system of polynomial equations has at least one nontrivial real solution. ("Nontrivial" means here that in each set of variables x_i at least one of the variables is assigned a nonzero value.)*

We are going to show that the Wang conjectures can be derived from this theorem by constructing a suitable system of polynomial equations.

We consider first the real Wang conjecture. We write a variable vector $x_t \in R_t$ as a linear combination of some basis vectors of that subspace, for $t = 1, \dots, k$. This introduces a set of i_t real variables, the coordinates of x_t. Similarly, each variable vector $y_t \in S_t$ introduces a set of $n - i_t + 1$ real variables. In addition to these vectors we introduce $n - k$ variable vectors $z_1, z_2, \dots, z_{n-k} \in V$. More precisely, we fix a basis of V, say $v_1, v_2, \dots, v_n$, and choose $z_i \in \langle v_1, v_2, \dots, v_{n-i+1}\rangle$. Thus z_i introduces a set of $n - i + 1$ real variables. Altogether we have introduced $n + k$ sets of variables. The total number of variables is

$$\mu = \frac{1}{2}(n^2 + 2nk - k^2 + n + k).$$

Next we set up our system (S) of bilinear equations:

$$\begin{aligned}
&(x_i|x_j) = (y_i|y_j) = 0, && 1 \le i < j \le k;\\
&(z_i|z_j) = 0, && 1 \le i < j \le n-k;\\
&(x_i|z_j) = (y_i|z_j) = 0, && 1 \le i \le k,\ 1 \le j \le n-k.
\end{aligned}$$

The number of these equations is

$$\nu = \frac{1}{2}(n^2 + 2nk - k^2 - n - k).$$

Note that $\mu - \nu = n + k$ is the number of sets of variables.

We are going to prove that the Bézout number N for this system of equations is odd. For that purpose we introduce a graph Γ. The set of nodes, say $\mathcal{N}$, of Γ is the set of indeterminates

$$X_1, X_2, \dots, X_k, Y_1, Y_2, \dots, Y_k, Z_1, Z_2, \dots, Z_{n-k} \tag{3}$$

which are in one-to-one correspondence with the variable vectors x_t, y_t, and z_s. The edges of this graph will be the equations of the system (S). Each equation, when considered as an edge of Γ, connects the nodes of the variable vectors involved with that equation. For instance the equation $(x_i|x_j) = 0$ with $i < j$ connects the nodes X_i and X_j. We shall denote the set of edges of Γ by $\mathcal{E}$.

To each node of our graph we associate a weight: $w(X_t) = i_t - 1$, $w(Y_t) = n - i_t$ for $1 \le t \le k$, and $w(Z_s) = n - s$ for $1 \le s \le n - k$.

The formula (2), in the present case, takes the form

$$N = \sum_{f \in F} \prod_{e \in \mathcal{E}} d(f(e), e),$$

where

$$F = \{f : \mathcal{E} \to \mathcal{N} : |f^{-1}(v)| = w(v),\ \forall v \in \mathcal{N}\}.$$

Each summand in this formula is 0 or 1 because all the degrees $d(f(e), e)$ are 0 or 1. By dropping the zero terms, we obtain that

$$N = |F \cap \mathcal{O}|, \tag{4}$$

where $\mathcal{O}$ is the set of all functions $f : \mathcal{E} \to \mathcal{N}$ such that for each edge e the node $f(e)$ is incident with e. We can identify $\mathcal{O}$ with the set of all orientations of Γ: an $f \in \mathcal{O}$ determines the orientation by directing each edge e to point towards the node $f(e)$.

We need to consider several subgraphs of Γ. Let Γ_{XZ} denote the complete bipartite graph on the X and Z-nodes, and the edges connecting X's and Z's. Let Δ_X denote the complete graph on the X-nodes, and Δ_{YZ} the complete graph on the union of the sets of Y and Z-nodes.

Each orientation f of Γ induces an orientation of its subgraph Γ_{XZ} which we denote by $\rho(f)$. For each orientation g of Γ_{XZ} we set

$$\mathcal{O}_g = \{f \in \mathcal{O} : \rho(f) = g\},$$

i.e., $\mathcal{O}_g$ is the set of all extensions of g to orientations of the whole graph Γ. In view of (4), it is clear that

$$N = \sum_g |F \cap \mathcal{O}_g|, \tag{5}$$

where the sum is over all orientations g of Γ_{XZ}.

Let g denote an orientation of Γ_{XZ} such that $|F\cap\mathcal{O}_g|$ is odd. In particular, $F\cap\mathcal{O}_g$ is nonempty. By choosing an $f \in F \cap \mathcal{O}_g$, we see that $|g^{-1}(X_t)| \leq |f^{-1}(X_t)| = w(X_t)$ for each t. Hence the new weights $w_g(X_t) = w(X_t) - |g^{-1}(X_t)|$ and $w_g(Z_s) = w(Z_s) - |g^{-1}(Z_s)|$ are nonnegative.

Assume that $w_g(X_r) = w_g(X_s)$ for some $r \neq s$. Let $f \in F \cap \mathcal{O}_g$ be arbitrary. We examine each node X_t with $t \neq r, s$ and the two edges that connect X_t to X_r and X_s. If exactly one of these two edges is directed towards X_t, then we reverse the orientation of each of them. We also reverse the orientation of the edge joining X_r and X_s. We denote this new orientation of Γ by f'. The assumption $w_g(X_r) = w_g(X_s)$ ensures that $f' \in F$. Clearly the map $f \to f'$ is a fixed-point-free involution on the set $F \cap \mathcal{O}_g$. Since $|F \cap \mathcal{O}_g|$ is odd, this is a contradiction.

We conclude that the weights $w_g(X_1), \ldots, w_g(X_k)$ are distinct. Since Δ_X has k nodes and $F \cap \mathcal{O}_g \neq \emptyset$, $w_g(X_t) \leq k-1$ for all t. It follows that

$$\{w_g(X_1), \ldots, w_g(X_k)\} = \{0, 1, \ldots, k-1\}. \tag{6}$$

The graph Δ_{YZ} is also complete, so a similar argument implies that $w_g(Z_r) \neq w_g(Z_s)$ for $r \neq s$ and $w_g(Z_r) \neq w(Y_t)$ for all r and t.

Thus the weights $w(Y_1), \ldots, w(Y_k), w_g(Z_1), \ldots, w_g(Z_{n-k})$ are distinct. Since all the weights $w(Y_t) = n - i_t$ and $w_g(Z_s)$ are nonnegative and $< n$, we must have

$$\{w_g(Z_1), \ldots, w_g(Z_{n-k})\} = \{0, 1, \ldots, n-1\} \setminus \{n - i_1, \ldots, n - i_k\}. \tag{7}$$

Let us say that an orientation f of Γ is *exceptional* if $f \in F$ and its restriction $g = \rho(f)$ satisfies (6) and (7).

We claim now that there exists a unique exceptional orientation f of Γ. The proof is by induction on n. The base case, $n = 1$, is trivial. Assume now that $n > 1$ and that the claim is true for values less than n. We set $g = \rho(f)$.

Suppose $i_1 > 1$, i.e., $w(X_1) > 0$. Then $n - i_t < n-1$ for $t = 1, \ldots, k$, so to satisfy (7) we must have $w_g(Z_s) = n-1$ for some s. But $w_g(Z_s) \leq w(Z_s) = n - s$, and so $s = 1$. This means that $|g^{-1}(Z_1)| = 0$, i.e., that all edges of Γ_{XZ} incident with Z_1 must point away from Z_1, towards the X_t nodes. It follows that $|f^{-1}(Z_1)| = n-1$, and so all edges of Δ_{YZ} incident with Z_1 must point towards Z_1. We now delete the node Z_1 and the edges of Γ incident with it. We also decrement n and each $w(X_t)$

by 1. By the induction hypothesis, this subgraph of Γ has a unique exceptional orientation, so we are done.

Otherwise, $i_1 = 1$, i.e., $w(X_1) = 0$. This forces all edges of Γ incident with X_1 to point away from X_1. As $w(Y_1) = n-1$, all edges incident with Y_1 must point towards Y_1. We now delete the nodes X_1 and Y_1 and all the edges of Γ incident with either of these nodes. We also decrement n, k, and each of the weights $w(X_t)$, $t > 1$, and $w(Z_s)$ by 1. By the induction hypothesis, this subgraph of Γ has a unique exceptional orientation. Thus our claim is proved.

The formula (5) now implies that N is odd.

Theorem 8. *The real Wang conjecture is true.*

Proof. Since the Bézout number N for the system (S) is odd, by Theorem 7, there exists a solution to our system with all vectors $x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_{n-k}$ nonzero.

Define $W = \langle z_1, \dots, z_{n-k}\rangle^\perp$. Since $z_s \perp z_t$ for $s \neq t$, W has dimension k. Our system of equations implies also that $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ are two orthogonal bases of W. Normalizing these bases gives us an answer. □

Now let us consider the complex Wang conjecture. We need to give yet another interpretation for the Bézout number N of the system (S). Let P be the polynomial with integer coefficients in $n+k$ indeterminates (3) defined by the formula

$$P = \prod_{1\le i<j\le k} (X_i + X_j)(Y_i + Y_j) \cdot \prod_{1\le i<j\le n-k} (Z_i + Z_j) \cdot \prod_{\substack{1\le i\le k \\ 1\le j\le n-k}} (X_i + Z_j)(Y_i + Z_j).$$

We observe that the coefficient of the monomial

$$M = X_1^{i_1-1} X_2^{i_2-1} \cdots X_k^{i_k-1} Y_1^{n-i_1} Y_2^{n-i_2} \cdots Y_k^{n-i_k} Z_1^{n-1} Z_2^{n-2} \cdots Z_{n-k}^{k}$$

in the expansion of the polynomial P is also equal to N. Indeed each orientation f of Γ corresponds to exactly one monomial, say M_f, in the uncollected expansion of the product defining P. If, say, the edge joining X_r and Z_s is directed towards Z_s, then from the factor $X_r + Z_s$ of P we choose the variable Z_s to be one of the factors in M_f. We have $M_f = M$ if and only if $f \in F \cap \mathcal{O}$. Therefore, $|F \cap \mathcal{O}|$ is equal to the coefficient of the monomial M.

Theorem 9. *The complex Wang conjecture is true.*

Proof. The complex case is similar to the real case except that we must separate the variables and equations into real and imaginary parts. We set the imaginary part of the first coordinate of each vector to zero. Then we have $n^2 + 2nk - k^2$ real variables, and $n^2 + 2nk - k^2 - n - k$ equations. The difference is again $n+k$, the number of sets of variables.

Just as in the real case, we can show that the Bézout number N' for this new system is the coefficient of the monomial M^2 in the expansion of the polynomial P^2. Since $(a+b)^2 \equiv a^2+b^2 \pmod 2$ for any polynomials a and b with integer coefficients, we have $N \equiv N' \pmod 2$, i.e., N' is odd. So by Theorem 7 our new system has a nontrivial solution. Hence the system (S) also has a nontrivial solution in the complex case as well. The remainder of the proof is identical to the real case. □

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