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INVARIANT SUBSPACES OF ARBITRARY MULTIPLICITY FOR THE SHIFT ON ℓ^1

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ABSTRACT. It is shown that if n is a positive integer or $n = \infty$, then the unilateral shift on ℓ^1 has an invariant subspace such that its restriction to it has multiplicity n.

1

Let T be a bounded linear operator on a Banach space X. Recall that a subset C of X is called cyclic for T if the linear span of the set $\{T^n x : x \in C, n = 0, 1, \cdots\}$ is dense in X. The minimal cardinality of a cyclic set for T is called the multiplicity of T, and will be denoted by $m(T)$. It is readily verified that $m(T)$ is not less than the dimension of the quotient space $X/\overline{TX}$.

For $1 \leq p < \infty$, ℓ^p denotes the Banach space of all complex sequences a on the set $\mathbf{Z}_+$ (of nonnegative integers) such that the norm $\|a\|_p = (\sum_{n=0}^{\infty} |a(n)|^p)^{1/p}$ is finite. The unilateral (right) shift on this space, that is, the operator

$$a \to \big(0, a(0), a(1), \cdots\big), \quad a \in \ell^p \ ,$$

will be denoted by S_p.

Assume that V is a (closed) subspace of ℓ^p which is invariant under S_p. We shall also call the multiplicity of the operator $S_p|_V$ the multiplicity of V and denote it by $m(V)$. If $m(V) = 1$, we shall say that V is singly generated. The dimension of the quotient space V/S_pV is called the index of V, and will be denoted by $\mathrm{ind}V$. By the preceding observation, $m(V) \geq \mathrm{ind}V$.

By a well-known result of Beurling [5, Theorem IV], every invariant subspace of S_2 is singly generated. On the other hand, Abakumov and Borichev [1] recently proved that, if $2 < p < \infty$ and if n is a positive integer or $n = \infty$, then there exists an invariant subspace V of S_p such that $\mathrm{ind}V = n$. Thus, in particular, $m(V) \geq n$. They also proved a similar result for the shift on some other Banach spaces of sequences on $\mathbf{Z}_+$, in particular for the shift on the Banach space c_0 of all sequences of complex numbers on $\mathbf{Z}_+$ converging to zero, with the supremum norm. Other results of this type can be found in [6] and the references listed there.

For $1 < p < 2$, the problems of whether the operator S_p has an invariant subspace of index greater than one, and whether it has an invariant subspace which is not singly generated, are still open.

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Every invariant subspace of S_1 has index one. This follows from a general result of Richter [13, Corollary 3.8], and can also easily be seen directly. For the sake of completeness, we give a proof in section 3. The purpose of this note is to establish the following.

Theorem 1. *If n is a positive integer, or $n = \infty$, then the operator S_1 has an invariant subspace of multiplicity n.*

It seems that this result provides the first example of a shift invariant Banach space of sequences on $\mathbf{Z}_+$, on which the shift is bounded, has closed range, and each of its invariant subspaces has index one, but has invariant subspaces which are not singly generated.

2

To prove the theorem it is convenient to reformulate it first in an equivalent form. This requires some notation and definitions.

We shall denote by $\mathbf{T}$ the "circle" group $\mathbf{R}/2\pi\mathbf{Z}$ represented by the interval $[0, 2\pi]$ with addition modulo 2π. The Wiener algebra on $\mathbf{T}$ of absolutely convergent Fourier series will be denoted by $A(\mathbf{T})$. That is, $A(\mathbf{T})$ is the Banach algebra (with respect to pointwise multiplication) of all complex continuous functions f on $\mathbf{T}$ such that the norm

$$\|f\|_{A(\mathbf{T})} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|$$

is finite (as usual $\{\hat{f}(n)\}$ are the Fourier coefficients of f).

The closed subalgebra of all functions f in $A(\mathbf{T})$ such that $\hat{f}(n) = 0$ for $n < 0$ will be denoted by $A^+(\mathbf{T})$. It can be identified in the obvious way with the Banach algebra of holomorphic functions in the open unit disc whose Taylor series are absolutely convergent in its closure.

For a closed subset E of $\mathbf{T}$, we shall denote by $A(E)$ and $A^+(E)$ the algebras of all functions on E which are restrictions to this set of functions in $A(\mathbf{T})$ and $A^+(\mathbf{T})$, respectively. These are Banach algebras with respect to the quotient norms

$$\|f\|_{A(E)} = \inf\left\{\|g\|_{A(\mathbf{T})} : g \in A(\mathbf{T}), g|_E = f\right\}$$

and

$$\|f\|_{A^+(E)} = \inf\left\{\|g\|_{A^+(\mathbf{T})} : g \in A^+(\mathbf{T}), g|_E = f\right\}.$$

If B is a commutative Banach algebra with unit, and $x_1, x_2, \ldots, x_n$ are elements in B, we shall denote by $[x_1, x_2, \ldots, x_n]$ the closure of the ideal generated by them algebraically. If I is a closed ideal in B which for some $n = 1, 2, \ldots,$ contains elements $x_1, x_2, \ldots, x_n$ such that $I = [x_1, x_2, \ldots, x_n]$, we shall say that I is finitely generated, and if there are no $n-1$ elements with that property, we shall say that I has exactly n generators.

The mapping $L : f \to \{\hat{f}(n)\}_{n\in\mathbf{Z}^+}$ is an isometric isomorphism of the Banach space $A^+(\mathbf{T})$ onto the Banach space ℓ^1, which carries the operator S of multiplication by the function $e^{i\theta}$ on $A^+(\mathbf{T})$ into the operator S_1. In formal terms, $S_1 = LSL^{-1}$; hence the operators S_1 and S are similar. Thus, observing that the invariant subspaces of the operator S are precisely the closed ideals in the Banach algebra $A^+(\mathbf{T})$, we see that Theorem 1 is equivalent to:

Theorem 2. *For every positive integer n, $A^+(\mathbf{T})$ has a closed ideal with exactly n generators, and also a closed ideal which is not finitely generated.*

Proof. We shall show that there exists a closed subset E of $\mathbf{T}$ such that the Banach algebra $A^+(E)$ has, for every positive integer n, a closed ideal with exactly n generators, and also has a closed ideal which is not finitely generated. This will imply the assertion of the theorem, since if $f_1, f_2, \dots, f_n$ are functions in $A^+(\mathbf{T})$ such that the closed ideal $[f_1|_E, f_2|_E, \dots, f_n|_E]$ in $A^+(E)$ has exactly n generators, then the same is true for the closed ideal $[f_1, f_2, \dots, f_n]$ in $A^+(\mathbf{T})$, and if J is a closed ideal in $A^+(E)$ which is not finitely generated, then $I = \{f \in A^+(\mathbf{T}) : f|_E \in J\}$ is a closed ideal in $A^+(\mathbf{T})$ with the same property.

We establish now the existence of such a set E. Let $\mathbb{D}$ be the cartesian product of countably many copies of the set $\{0,1\}$, that is, $\mathbb{D} = \{0,1\}^N$. This is a compact Hausdorff space with respect to the product topology (when $\{0,1\}$ is given the discrete topology). Let $C(\mathbb{D})$ denote the Banach algebra of complex continuous functions on $\mathbb{D}$, with the maximum norm $\| \|_\infty$, and consider the Varopoulos algebra $V(\mathbb{D})$, which consist of all complex continuous functions f on $\mathbb{D} \times \mathbb{D}$, which admit a representation of the form

$$(*) \qquad f(x,y) = \sum_{n=1}^{\infty} u_n(x) v_n(y), \quad (x,y) \in \mathbb{D} \times \mathbb{D},$$

where $u_n, v_n \in C(\mathbb{D})$, $n = 1, 2, \dots$, and

$$(**) \qquad \sum_{n=1}^{\infty} \|u_n\|_\infty \|v_n\|_\infty < \infty \ .$$

With the norm of f defined as the infimum over all the sums of the form $(**)$, for all possible representations of f in the form $(*)$, $V(\mathbb{D})$ is a Banach algebra with respect to pointwise addition and multiplication. (For further details on this algebra we refer to [7, Ch. 11], [10, Ch. 8], and [14].)

It is proved in [2, Theorem 2.2] that $V(\mathbb{D})$ has a closed ideal which is not finitely generated, and the proof there also shows that, for every positive integer n, it also has a closed ideal with exactly n generators. (An explicit proof of this fact can be found in [7, Lemma 11.2.11].) Hence the theorem will be proved if we show that there exists a closed subset E of $\mathbf{T}$ such that the Banach algebras $A^+(E)$ and $V(\mathbb{D})$ are isomorphic (algebraically and topologically). To see this, consider the dilated Cantor set on $\mathbf{T}$,

$$C = \left\{ 2\pi \sum_{n=1}^{\infty} \varepsilon_n 3^{-n}, \ \varepsilon_n = 0, 2 \right\} .$$

This is a closed subset of $\mathbf{T}$ and is the algebraic sum of the perfect sets

$$C_1 = \left\{ 2\pi \sum_{n=1}^{\infty} \varepsilon_n 9^{-n}, \ \varepsilon_n = 0, 2 \right\}$$

and $C_2 = 3C_1$. Thus by a theorem of Varopoulos [14, Theorem 4.3.3] (see also [10], p. 110), there exists a closed subset E of C such that the Banach algebras $A(E)$ and $V(\mathbb{D})$ are isometrically isomorphic. On the other hand, by a result of Kahane and Katznelson [12, Theorem 3] and the example in [11, p. 58], $A^+(C) = A(C)$, with equivalent norms, and therefore also $A^+(E) = A(E)$, again with equivalent norms. This shows that the Banach algebras $A^+(E)$ and $V(\mathbb{D})$ are isomorphic, and the theorem is proved. $\square$

3

We now show that every invariant subspace of S_1 has index one. This is equivalent to the assertion that if I is a closed nonzero ideal in $A^+(\mathbf{T})$, then the quotient space I/SI has dimension one. Assume first that I contains a function u such that $\hat{u}(0) = 1$. We claim that $I = \operatorname{span}(SI, u)$, which clearly proves the assertion for this case. To show this, consider a function f in I. Since $\hat{u}(0) = 1$, there exist functions g and v in $A^+(\mathbf{T})$ such that $f - \hat{f}(0)u = Sg$ and $1 - u = Sv$. Since $g = ug + vSg$ and I is an ideal which contains the functions u and Sg, it follows that $g \in I$, and therefore $f \in \operatorname{span}(SI, u)$. In the general case, if I is a nonzero closed ideal in $A^+(\mathbf{T})$, there exists a nonnegative integer n such that $I = S^n J$, where J is a closed ideal in $A^+(\mathbf{T})$ that contains a function u such that $\hat{u}(0) = 1$, and therefore by the previous part J/SJ has dimension one. Since S is an isometry, the spaces I/SI and J/SJ have the same dimension, and the assertion is proved.

4

We conclude with some comments on bilateral shifts. For $1 \leq p < \infty$, let U_p denote the bilateral shift on $\ell^p(\mathbf{Z})$, that is, the operator defined by

$$U_p a = \left\{a(n-1)\right\}_{n \in \mathbf{Z}}, \quad a \in \ell^p(\mathbf{Z}) .$$

As before, if M is an invariant subspace of U_p, we shall call the multiplicity of $U_p|_M$ also the multiplicity of M, and if M has multiplicity one, we shall say that it is singly generated.

Combining a result of Wiener [8, Theorem 2] with a result of Helson and Lowdenslager [9] (or [8, Theorem 3]), it follows that every invariant subspace of U_2 is singly generated.

On the other hand since $U_p|_{\ell^p(\mathbf{Z}_+)}$ can be identified with S_p, it follows from the results of Abakumov and Borichev [1] mentioned before, that if $2 < p < \infty$, then U_p has invariant subspaces of arbitrarily large multiplicity. By Theorem 1, or directly by [2, Theorem 2.1], the same is true for S_1. (See also [7, Theorem 11.2.3].)

If $1 < p < 2$, it is shown in [3] and [4] (in a more general setting) that there exists a subspace of $\ell^p(\mathbf{Z})$ which is invariant under U_p and U_p^{-1} that contains no element x such that the linear span of the set $\{U_p^n x, n \in \mathbf{Z}\}$ is dense in it. Thus, in particular, this subspace is not singly generated. We do not know whether, for these values of p, the operators U_p also have invariant subspaces of infinite multiplicity, or of arbitrarily large finite multiplicity.

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