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VARIATIONAL REPRESENTATIONS OF VARADHAN FUNCTIONALS

HAROLD BELL AND WLODZIMIERZ BRYC

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Abstract. Motivated by the theory of large deviations, we introduce a class of non-negative non-linear functionals that have a variational "rate function" representation.

1. Introduction

Let $(\mathbf{X}, d)$ be a Polish space with metric $d()$ and let $\mathbf{C}_b(\mathbf{X})$ denote the space of all bounded continuous functions $F : \mathbf{X} \to \mathbb{R}$. In his work on large deviations of probability measures μ_n, Varadhan [12] introduced a class of non-linear functionals $\mathbb{L}$ defined by

$$\mathbb{L}(F) = \lim_{n\to\infty} \frac{1}{n} \log \int_{\mathbf{X}} \exp(nF(\mathbf{x}))d\mu_n \tag{1}$$

and used the large deviations principle of μ_n to prove the variational representation

$$\mathbb{L}(F) = L_0 + \sup_{\mathbf{x}\in\mathbf{X}} \{F(\mathbf{x}) - \mathbb{I}(\mathbf{x})\}, \tag{2}$$

where $\mathbb{I} : \mathbf{X} \to [0, \infty]$ is the *rate function* governing the large deviations, and $L_0 := \mathbb{L}(0) = 0$.

Several authors [1, 3, 4, 9, 10, 11] abstracted non-probabilistic components from the theory of large deviations. In particular, in [3] (see also [10, Theorem 3.1]) we give conditions which imply the rate function representation (2) when the limit (1) exists, and we show that the rate function is determined from the dual formula

$$\mathbb{I}(\mathbf{x}) = \mathbb{L}(0) + \sup_{F\in\mathbf{C}_b(\mathbf{X})} \{F(\mathbf{x}) - \mathbb{L}(F)\}. \tag{3}$$

In fact, one can reverse Varadhan's approach, and show that large deviations of probability measures μ_n follow from the variational representation (2) for (1) (see [8, Theorem 1.2.3]). In this context we have $\mu_n(\mathbf{X}) = 1$ which implies $\mathbb{L}(0) = 0$ in (3) and correspondingly $L_0 = 0$ in (2).

"Asymptotic values" in [3] are essentially what we call Varadhan Functionals here; the theorems in that paper are not entirely satisfying because the assumptions are in terms of the underlying probability measures. In this paper we present a more

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satisfying approach which relies on the theory of probability for motivation purposes only.

Definition 1.1. A function $\mathbb{L} : \mathbf{C}_b(\mathbf{X}) \to \mathbb{R}$ is a Varadhan Functional if the following conditions are satisfied:

$$\text{If } F \leq G, \text{ then } \mathbb{L}(F) \leq \mathbb{L}(G) \text{ for all } F, G \in \mathbf{C}_b(\mathbf{X}), \tag{4}$$

$$\mathbb{L}(F + \mathit{const}) = \mathbb{L}(F) + \mathit{const} \text{ for all } F \in \mathbf{C}_b(\mathbf{X}), \mathit{const} \in \mathbb{R}. \tag{5}$$

Expression (1) provides an example of Varadhan Functional, if the limit exists.

Another example is given by variational representation (2).

Condition (4) is equivalent to $\mathbb{L}(F \vee G) \geq \mathbb{L}(F) \vee \mathbb{L}(G)$, where $a \vee b$ denotes the maximum of two numbers. Varadhan Functionals like (1) satisfy a stronger condition.

Definition 1.2. A Varadhan Functional $\mathbb{L}$ is *maximal* if $\mathbb{L}(\cdot)$ is a lattice homomorphism

$$\mathbb{L}(F \vee G) = \mathbb{L}(F) \vee \mathbb{L}(G). \tag{6}$$

It is easy to see that each Varadhan Functional $\mathbb{L}(\cdot)$ satisfies the Lipschitz condition $|\mathbb{L}(F) - \mathbb{L}(G)| \leq \|F - G\|_\infty$; compare (9). Thus $\mathbb{L}$ is a continuous mapping from the Banach space $\mathbf{C}_b(\mathbf{X})$ of all bounded continuous functions into the real line. We will need the following stronger continuity assumption, motivated by the definition of the countable additivity of measures.

Definition 1.3. A Varadhan Functional is *σ-continuous* if the following condition is satisfied:

$$\text{If } F_n \searrow 0, \text{ then } \mathbb{L}(F_n) \to \mathbb{L}(0). \tag{7}$$

Notice that if $\mathbf{X}$ is compact, then by Dini's theorem and the Lipschitz property, all Varadhan Functionals are σ-continuous.

Maximal Varadhan Functionals are convex; this follows from the proof of Theorem 2.1, which shows that formula (2) holds true for all Varadhan Functionals when the supremum is extended to all $\mathbf{x}$ in the Čech-Stone compactification of $\mathbf{X}$.

A simple example of convex and maximal but not σ-continuous Varadhan Functional is $\mathbb{L}(F) = \limsup_{x\to\infty} F(x)$, where $F \in \mathbf{C}_b(\mathbb{R})$. This Varadhan Functional cannot be represented by variational formula (2). Indeed, (2) implies that $\mathbb{I}(\mathbf{x}) \geq F(\mathbf{x}) - \mathbb{L}(F) = F(\mathbf{x})$ for all $F \in \mathbf{C}_b(\mathbb{R})$ that vanish at ∞; hence $\mathbb{I}(\mathbf{x}) = \infty$ for all $\mathbf{x} \in \mathbb{R}$ and (2) gives $\mathbb{L}(F) = -\infty$ for all $F \in \mathbf{C}_b(\mathbb{R})$, a contradiction.

An example of a convex and σ-continuous but not maximal Varadhan Functional is $\mathbb{L}(F) = \log \int_{\mathbf{X}} \exp F(\mathbf{x}) \nu(d\mathbf{x})$, where ν is a finite non-negative measure.

2. Variational representations

The main result of this paper is the following.

Theorem 2.1. *If a maximal Varadhan Functional $\mathbb{L} : \mathbf{C}_b(\mathbf{X}) \to \mathbb{R}$ is σ-continuous, then there is $L_0 \in \mathbb{R}$ such that variational representation (2) holds true and the rate function $\mathbb{I} : \mathbf{X} \to [0, \infty]$ is given by the dual formula (3). Furthermore, $\mathbb{I}(\cdot)$ is a tight rate function: sets $\mathbb{I}^{-1}([0, a]) \subset \mathbf{X}$ are compact for all $a > 0$.*

The next result is closely related to Bryc [3, Theorem T.1.1] and Deuschel & Stroock [6, Theorem 5.1.6]. Denote by $\mathcal{P}(\mathbf{X})$ the metric space (with Prokhorov metric) of all probability measures on a Polish space $\mathbf{X}$ with the Borel σ-field generated by all open sets.

Theorem 2.2. *If a convex Varadhan Functional $\mathbb{L} : \mathbf{C}_b(\mathbf{X}) \to \mathbb{R}$ is σ-continuous, then there is a lower semicontinuous function $\mathbb{J} : \mathcal{P}(\mathbf{X}) \to [0, \infty]$ and a constant L_0 such that*

$$\mathbb{L}(F) = L_0 + \sup_{\mu \in \mathcal{P}} \{ \int F d\mu - \mathbb{J}(\mu) \} \tag{8}$$

for all bounded continuous functions F.

A well-known example in large deviations is the convex σ-continuous functional $\mathbb{L}(F) := \log \int \exp F(\mathbf{x}) \nu(d\mathbf{x})$ with the rate function in (8) given by the relative entropy functional

$$\mathbb{J}(\mu) = \begin{cases} \int \log \frac{d\mu}{d\nu} d\mu & \text{if } \mu \ll \nu \text{ is absolutely continuous,} \\ \infty & \text{otherwise.} \end{cases}$$

Remark 2.1. Deuschel & Stroock [6, Section 5.1] consider convex functionals $\Phi : \mathbf{C}_b(\mathbf{X}) \to \mathbb{R}$ such that $\Phi(\mathit{const}) = \mathit{const}$. Such functionals satisfy condition (5). Indeed, write $F + \mathit{const}$ as a convex combination

$$F + \mathit{const} = (1 - \theta) F + \frac{\theta}{2} \left(\frac{2\mathit{const}}{\theta} \right) + \frac{\theta}{2} (2F),$$

where $0 < \theta < 1$. Using convexity and $\Phi(\mathit{const}) = \mathit{const}$ we get $\Phi(F + \mathit{const}) \leq \Phi(F) + \mathit{const} + \theta(\frac{\Phi(2F)}{2} - \Phi(F))$. Since $\theta > 0$ is arbitrary this proves that $\Phi(F + \mathit{const}) \leq \Phi(F) + \mathit{const}$. By routine symmetry considerations (replacing $F \mapsto F - \mathit{const}$, and then $\mathit{const} \mapsto -\mathit{const}$), (5) follows.

3. Proofs

Let $L_0 := \mathbb{L}(0)$. Passing to $\mathbb{L}'(F) := \mathbb{L}(F) - L_0$ if necessary, without losing generality, we assume $\mathbb{L}(0) = 0$.

Lemma 3.1. *Let $\hat{\mathbf{X}}$ be a compact Hausdorff space. Suppose $\mathbf{X} \subset \hat{\mathbf{X}}$ is a separable metric space in the relative topology. If $\mathbf{x}_0 \in \hat{\mathbf{X}} \setminus \mathbf{X}$, then there are bounded continuous functions $F_n : \hat{\mathbf{X}} \to \mathbb{R}$ such that*

(i) $F_n(\mathbf{x}) \searrow 0$ *for all* $\mathbf{x} \in \mathbf{X}$,
(ii) $F_n(\mathbf{x}_0) = 1$ *for all* $n \in \mathbb{N}$.

Proof. Since $\hat{\mathbf{X}}$ is Hausdorff, for every $\mathbf{x} \in \mathbf{X}$ there is an open set $U_\mathbf{x} \ni \mathbf{x}$ such that its closure $\bar{U}_\mathbf{x}$ does not contain $\mathbf{x}_0$.

By Lindelöf property for separable metric space $\mathbf{X}$, there is a countable subcover $\{U_n\}$ of $\{U_\mathbf{x}\}$.

A compact Hausdorff space $\hat{\mathbf{X}}$ is normal. So there are continuous functions $\phi_n : \hat{\mathbf{X}} \to \mathbb{R}$ such that $\phi_n \big|_{\bar{U}_n} = 0$ and $\phi_n(\mathbf{x}_0) = 1$.

To end the proof take $F_n(\mathbf{x}) = \min_{1 \leq k \leq n} \phi_k(\mathbf{x})$. ∎

The following lemma is contained implicitly in [3, Theorem T.1.2].

Lemma 3.2. *Theorem 2.1 holds true for compact $\mathbf{X}$.*

Proof. Let $\mathbb{I}(\cdot)$ be defined by (3). Thus $\mathbb{I}(\mathbf{x}) \geq F(\mathbf{x}) - \mathbb{L}(F)$ which implies $\mathbb{L}(F) \geq \sup_{\mathbf{x}\in\mathbf{X}}\{F(\mathbf{x}) - \mathbb{I}(\mathbf{x})\}$. To end the proof we need therefore to establish the converse inequality. Fix a bounded continuous function $F \in \mathbf{C}_b(\mathbf{X})$ and $\epsilon > 0$. Let $s = \sup_{\mathbf{x}\in\mathbf{X}}\{F(\mathbf{x}) - \mathbb{I}(\mathbf{x})\}$. Clearly $F(\mathbf{x}) - \mathbb{I}(\mathbf{x}) \leq s \leq \mathbb{L}(F)$. By (3) again, for every $\mathbf{x} \in \mathbf{X}$, there is $F_{\mathbf{x}} \in \mathbf{C}_b(\mathbf{X})$ such that $\mathbb{I}(\mathbf{x}) < F_{\mathbf{x}}(\mathbf{x}) - \mathbb{L}(F_{\mathbf{x}}) + \epsilon$. Therefore

$$F(\mathbf{x}) \leq s + \mathbb{I}(\mathbf{x}) < s + \epsilon + F_{\mathbf{x}}(\mathbf{x}) - \mathbb{L}(F_{\mathbf{x}}).$$

This means that the sets $U_{\mathbf{x}} = \{\mathbf{y} \in \mathbf{X} : F(\mathbf{y}) - F_{\mathbf{x}}(\mathbf{y}) < s + \epsilon - \mathbb{L}(F_{\mathbf{x}})\}$ form an open covering of $\mathbf{X}$. Using compactness of $\mathbf{X}$, we choose a finite covering $U_{\mathbf{x}(1)}, \dots, U_{\mathbf{x}(k)}$. Then, writing $F_i = F_{\mathbf{x}(i)}$ we have

$$F(\mathbf{x}) < \max_{1\leq i\leq k}\{F_i(\mathbf{x}) - \mathbb{L}(F_i)\} + s + \epsilon$$

for all $\mathbf{x} \in \mathbf{X}$.

Using (4), (5), and (6) we have

$$\begin{aligned} \mathbb{L}(F) &\leq \mathbb{L}\left(\max_{1\leq i\leq k}\{F_i - \mathbb{L}(F_i)\} + s + \epsilon\right) \\ &= \mathbb{L}\left(\max_i \{F_i - \mathbb{L}(F_i)\}\right) + s + \epsilon \\ &= \max_i \{\mathbb{L}\left(F_i - \mathbb{L}(F_i)\right)\} + s + \epsilon. \end{aligned}$$

Since (5) implies $\mathbb{L}\left(F_i - \mathbb{L}(F_i)\right) = \mathbb{L}\left(F_i\right) - \mathbb{L}(F_i) = 0$ this shows that $s \leq \mathbb{L}(F) < s + \epsilon$. Therefore $\mathbb{L}(F) = s$, proving (2). □

Proof of Theorem 2.1. Let $\hat{\mathbf{X}}$ be the Čech-Stone compactification of $\mathbf{X}$. Since the inclusion $\mathbf{X} \subset \hat{\mathbf{X}}$ is continuous, we define $\hat{\mathbb{L}} : \mathbf{C}_b(\hat{\mathbf{X}}) \to \mathbb{R}$ by $\hat{\mathbb{L}}(\hat{F}) := \mathbb{L}(\hat{F}|_{\mathbf{X}})$. It is clear that $\hat{\mathbb{L}}$ is a maximal Varadhan Functional, so by Lemma 3.2 there is $\mathbb{I} : \hat{\mathbf{X}} \to [0, \infty]$ such that $\hat{\mathbb{L}}(\hat{F}) = \sup\{\hat{F}(\mathbf{x}) - \mathbb{I}(\mathbf{x}) : \mathbf{x} \in \hat{\mathbf{X}}\}$.

Using σ-continuity(7) it is easy to check that $\mathbb{I}(\mathbf{x}) = \infty$ for all $\mathbf{x} \in \hat{\mathbf{X}} \setminus \mathbf{X}$. Indeed, given $\mathbf{x}_0 \in \hat{\mathbf{X}} \setminus \mathbf{X}$ by Lemma 3.1 there are $F_n \in \mathbf{C}_b(\hat{\mathbf{X}})$ such that $F_n \searrow 0$ on $\mathbf{X}$, but $F_n(\mathbf{x}_0) = C > 0$. Then from (3) we get $\mathbb{I}(\mathbf{x}_0) \geq \hat{\mathbb{L}}(0) + F_n(\mathbf{x}_0) - \hat{\mathbb{L}}(F_n) \to \hat{\mathbb{L}}(0) + C$. Since $C > 0$ is arbitrary, $\mathbb{I}(\mathbf{x}_0) = \infty$.

This shows that $\hat{\mathbb{L}}(\hat{F}) = \sup\{\hat{F}(\mathbf{x}) - \mathbb{I}(\mathbf{x}) : \mathbf{x} \in \mathbf{X}\}$ for all $\hat{F} \in \mathbf{C}_b(\hat{\mathbf{X}})$. It remains to observe that since $\hat{\mathbf{X}}$ is a Čech-Stone compactification, every function $F \in \mathbf{C}_b(\mathbf{X})$ is a restriction to $\mathbf{X}$ of some $\hat{F} \in \mathbf{C}_b(\hat{\mathbf{X}})$ (see [7, IV.6.22]). Therefore (2) holds true for all $F \in \mathbf{C}_b(\mathbf{X})$.

To prove that the rate function is tight, suppose that there is $a > 0$ such that $\mathbb{I}^{-1}[0, a]$ is not compact. Then there is $\delta > 0$ and a sequence $\mathbf{x}_n \in \mathbf{X}$ such that $\mathbb{I}(\mathbf{x}_n) \leq a$, and $d(\mathbf{x}_m, \mathbf{x}_n) > \delta$ for all $m \neq n$. Since Polish spaces have Lindelöf property, there is a countable number of open balls of radius $\delta/2$ which cover $\mathbf{X}$. For $k = 1, 2, \dots,$ denote by $B_k \ni \mathbf{x}_k$ one of the balls that contain $\mathbf{x}_k$, and let ϕ_k be a bounded continuous function such that $\phi_k(\mathbf{x}_k) = 2a$ and $\phi_k = 0$ on the complement of B_k. Then $F_n = \max_{k\geq n}\phi_k \searrow 0$ pointwise. On the other hand (2) implies $\mathbb{L}(F_n) \geq L_0 + F_n(\mathbf{x}_n) - \mathbb{I}(\mathbf{x}_n) \geq L_0 + a$, contradicting (7). □

Lemma 3.3. *If* $\mathbb{L}(\cdot)$ *is a Varadhan Functional, then*

$$\inf_{\mathbf{x}\in\mathbf{X}}\{F(\mathbf{x}) - G(\mathbf{x})\} \leq \mathbb{L}(F) - \mathbb{L}(G). \tag{9}$$

Proof. Let $const = \inf_{\mathbf{x}}\{F(\mathbf{x}) - G(\mathbf{x})\}$. Clearly, $F \geq G + const$. By positivity condition (4) this implies $\mathbb{L}(F) \geq \mathbb{L}(G + const) = \mathbb{L}(G) + const$. □

The next lemma is implicitly contained in the proof of [3, Theorem T.1.1]. Let $\mathcal{P}_a(\mathbf{X})$ denote all regular finitely-additive probability measures on $\mathbf{X}$ with the Borel field.

Lemma 3.4. *If $\mathbb{L}(\cdot)$ is a convex Varadhan Functional on $\mathbf{C}_b(\mathbf{X})$, then there exist a lower semicontinuous function $\mathbb{J} : \mathcal{P}_a(\mathbf{X}) \to [0,\infty]$ such that*

$$\mathbb{L}(F) = \mathbb{L}(0) + \sup\{\mu(F) - \mathbb{J}(\mu) : \mu \in \mathcal{P}_a(\mathbf{X})\}, \tag{10}$$

and the supremum is attained.

Proof. Let $\mathbb{J}(\cdot)$ be defined by

$$\mathbb{J}(\mu) = \mathbb{L}(0) + \sup\{\mu(F) - \mathbb{L}(F) : F \in \mathbf{C}_b(\mathbf{X})\} \tag{11}$$

and fix $F_0 \in \mathbf{C}_b(\mathbf{X})$. Recall that throughout this proof we assume $\mathbb{L}(0) = 0$.

By the definition of $\mathbb{J}(\cdot)$, we need to show that

$$\mathbb{L}(F_o) = \sup_{\mu} \inf_{F}\{\mu(F_0) - \mu(F) + \mathbb{L}(F)\}, \tag{12}$$

where the supremum is taken over all $\mu \in \mathcal{P}_a(\mathbf{X})$ and the infimum is taken over all $F \in \mathbf{C}_b(\mathbf{X})$. Moreover, since (11) implies that $\mathbb{J}(\mu) \geq \mu(F_0) - \mathbb{L}(F_0)$ for all $\mu \in \mathcal{P}_a(\mathbf{X})$, then $\mathbb{L}(F_0) \geq \sup_\mu \inf_F\{\mu(F_0) - \mu(F) + \mathbb{L}(F)\}$. Hence to prove (12), it remains to show that there is $\nu \in \mathcal{P}_a(\mathbf{X})$ such that

$$\mathbb{L}(F_0) \leq \nu(F_0) - \nu(F) + \mathbb{L}(F) \text{ for all } F \in \mathbf{C}_b(\mathbf{X}) \tag{13}$$

(also, for this ν, the supremum in (10) will be attained). To find ν, consider the following sets. Let

$$\mathcal{M} = \{F \in \mathbf{C}_b(\mathbf{X}) : \inf_{\mathbf{x}}[F(\mathbf{x}) - F_0(\mathbf{x})] > 0\}$$

and let $\mathcal{N}$ be a set of all finite convex combinations of functions $g(\mathbf{x})$ of the form $g(\mathbf{x}) = F(\mathbf{x}) + \mathbb{L}(F_0) - \mathbb{L}(F)$, where $F \in \mathbf{C}_b(\mathbf{X})$.

It is easily seen from the definitions that $\mathcal{M}$ and $\mathcal{N}$ are convex; also $\mathcal{M} \subset \mathbf{C}_b(\mathbf{X})$ is non-empty since $1 + F_0 \in \mathcal{M}$, and open since $\{F : \inf_{\mathbf{x}}[F(\mathbf{x}) - F_0(\mathbf{x})] \leq 0\} \subset \mathbf{C}_b(\mathbf{X})$ is closed. Furthermore, $\mathcal{M}$ and $\mathcal{N}$ are disjoint. Indeed, take arbitrary

$$\mathcal{N} \ni g = \sum \alpha_k F_k + \mathbb{L}(F_0) - \sum \alpha_k \mathbb{L}(F_k).$$

Then

$$\inf_x\{g(\mathbf{x}) - F_0(\mathbf{x})\}$$

$$= \inf_x\{\sum \alpha_k F_k(\mathbf{x}) - F_0(\mathbf{x})\} - \sum \alpha_k \mathbb{L}(F_k) + \mathbb{L}(F_0)$$

$$\leq \inf_x\{\sum \alpha_k F_k(\mathbf{x}) - F_0(\mathbf{x})\} - \mathbb{L}(\sum \alpha_k F_k) + \mathbb{L}(F_0) \leq 0,$$

where the first inequality follows from the convexity of $\mathbb{L}(\cdot)$ and the second one follows from (9) applied to $F = \sum \alpha_k F_k(\mathbf{x})$ and $G = F_0$.

Therefore $\mathcal{M}$ and $\mathcal{N}$ can be separated, i.e. there is a non-zero linear functional $f^* \in \mathbf{C}_b^*(\mathbf{X})$ such that for some $\alpha \in \mathbb{R}$

$$f^*(\mathcal{N}) \leq \alpha < f^*(\mathcal{M}) \tag{14}$$

(see e.g. [7, V. 2. 8]).

Claim: f^* is non-negative.

Indeed, it is easily seen that $F_0(\cdot)$ belongs to $\mathcal{N}$, and, as a limit of $\epsilon + F_0(\mathbf{x})$ as $\epsilon \to 0$, F_0 is also in the closure of $\mathcal{M}$. Therefore by (14) we have $\alpha = f^*(F_0)$. To end the proof take arbitrary F with $\inf_{\mathbf{x}} F(\mathbf{x}) > 0$. Then $F + F_0 \in \mathcal{M}$ and by (14)

$$f^*(F) = f^*(F + F_0) - f^*(F_0) > \alpha - f^*(F_0) = 0.$$

This ends the proof of the claim.

Without losing generality, we may assume $f^*(1) = 1$; then it is well known (see e.g. [2, Ch. 2 Section 4 Theorem 1]) that $f^*(F) = \nu(F)$ for some $\nu \in \mathcal{P}_a(\mathbf{X})$; for regularity of ν consult [7, IV.6.2]. It remains to check that ν satisfies (13). To this end observe that since $F+\mathbb{L}(F_0)-\mathbb{L}(F) \in \mathcal{N}$, by (14) we have $\nu(F)+\mathbb{L}(F_0)-\mathbb{L}(F) \leq \alpha = \nu(F_0)$ for all $F \in \mathbf{C}_b(\mathbf{X})$. This ends the proof of (10). □

Proof of Theorem 2.2. Lemma 3.4 gives the variational representation (10) with the supremum taken over a too large set. To end the proof we will show that $\mathbb{J}(\mu) = \infty$ on measures μ that fail to be countably-additive.

Suppose that μ is additive but not countably additive. Then Daniell-Stone theorem implies that there is $\delta > 0$ and a sequence $F_n \searrow 0$ of bounded continuous functions such that $\int F_n d\mu > \delta > 0$ for all n. By (11) and σ-continuity, $\mathbb{J}(\mu) \geq \mathbb{L}(0) + C \int F_n d\mu - \mathbb{L}(CF_n) \geq \mathbb{L}(0) + C\delta - \mathbb{L}(CF_n) \to \mathbb{L}(0) + C\delta$. Since $C > 0$ is arbitrary, then $\mathbb{J}(\mu) = \infty$ for all μ that are additive but not countably-additive. Thus (10) implies (8). □

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Department of Mathematics, University of Cincinnati, P.O. Box 210025, Cincinnati, Ohio 45221–0025

E-mail address: `bellh@math.uc.edu`

Department of Mathematics, University of Cincinnati, P.O. Box 210025, Cincinnati, Ohio 45221–0025

E-mail address: `brycwz@email.uc.edu`