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IMMERSED SURFACES OF PRESCRIBED GAUSS CURVATURE INTO MINKOWSKI SPACE

YUXIN GE

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ABSTRACT. Given a positive real valued function $k(x)$ on the disc, we will immerse the disc into three dimensional Minkowski space in such a way that Gauss curvature at the image point of x is $-k(x)$. Our approach lies on the construction of Gauss map of surfaces.

1. INTRODUCTION

The classical Minkowski problem is an embedding problem of differential geometry. This problem is the following: Given a positive function $K(u)$ defined on the unit sphere, does there exist a closed convex surface in $\mathbb{R}^3$ having $K(u)$ as its Gauss curvature at the point on the surface where the inner normal is u? In [11], Lewy has shown the existence of such a surface, under the condition that the function $K(u)$ is analytic. Later, by using a similar procedure, Nirenberg [13] published a paper in which he solved the Minkowski problem under the assumption that the function $K(u)$ possesses partial derivatives on the sphere up to second order. In [3], the author considers an analogous problem by using an approach, suggested in [8]: Given a positive real valued function $k(x)$ on the disc, we immerse the disc in $\mathbb{R}^3$ in such a way that Gauss curvature at the image point of x is $k(x)$. In this paper, we continue exploiting this method to immerse the disc into three dimensional Minkowski space. Namely we propose and use a method for constructing immersions of surfaces in the Minkowski space $\mathbb{R}^{2,1}$ by prescribing the Gauss curvature to be a negative function of the variable x in the surface. Notice that in [3] we had an analogous construction for surfaces in the Euclidean space but with a positive Gauss curvature.

Let $B = \{x \in \mathbb{R}^2,\ |\, x \,| < 1\}$ be a disc in $\mathbb{R}^2$. Let $\mathbb{R}^{2,1}$ be three dimensional Minkowski space with the standard metric $g = (dx_1)^2 + (dx_2)^2 - (dx_3)^2$. Let $\mathbb{H}^2 = \{x \in \mathbb{R}^3,\ g(x,x) = -1\}$ be the unit hyperboloid of two sheets and let $\mathbb{H}^2_+ = \{x \in \mathbb{H}^2,\ x_3 > 0\}$ be the upper sheet contained in the half-space $\{x_3 > 0\}$.

Let $l : \partial B \longrightarrow \mathbb{H}^2_+$ be a prescribed $C^{2,\gamma}$ mapping with $\gamma > 0$. We consider the space $H^1_l(B, \mathbb{H}^2_+)$ of functions u in $H^1(B, \mathbb{R}^3)$ satisfying that $u \in \mathbb{H}^2_+$ a.e. and $u = l$

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on ∂B. We define on $H^1_l(B,\mathbb{H}^2_+)$ the following energy functional E:

$$E(u)=\frac{1}{2}\int_B\sum_{i,j=1}^{2}a_{ij}(x)g\left(\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial x_j}\right)dx, \tag{1}$$

where $a_{ij}(x)$ satisfy the following conditions:

$$\exists\,\alpha>0,\ \text{such that}\ a_{ij}(x)\xi^i\xi^j\geq\alpha|\ \xi\ |^2,\qquad\forall\,x\in B,\forall\,\xi\in\mathbb{R}^2; \tag{2}$$

$$a_{ij}(x)\in C^{1,\gamma}(\bar{B},\mathbb{R}),\qquad\forall\,1\leq i,j\leq 2; \tag{3}$$

$$a_{ij}=a_{ji},\qquad\forall\,1\leq i,j\leq 2. \tag{4}$$

Here, it is easy to check that the critical points of E satisfy in the sense of distributions the following Euler equation:

$$\begin{cases}\displaystyle\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\lambda u=0, & \text{in } B,\\ u=l, & \text{on } \partial B,\end{cases} \tag{5}$$

where $\lambda=-\displaystyle\sum_{i,j=1}^{2}a_{ij}(x)g\left(\frac{\partial u}{\partial x_i}(x),\frac{\partial u}{\partial x_j}(x)\right)$.

Notice that $\lambda<0$. We deduce from (5) the following equality:

$$\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(u\times\sum_{j=1}^{2}a_{ij}(x)\frac{\partial u}{\partial x_j}\right)=0, \tag{6}$$

where $\xi\times\eta=(\xi_2\eta_3-\xi_3\eta_2,\xi_3\eta_1-\xi_1\eta_3,\xi_2\eta_1-\xi_1\eta_2)$ for all ξ, $\eta\in\mathbb{R}^{2,1}$ is vectorial product in $\mathbb{R}^{2,1}$. Assume that u is an immersion. Thus, we obtain a new immersion G from B to $\mathbb{R}^{2,1}$ satisfying

$$\frac{\partial G}{\partial x_2}=\sum_{j=1}^{2}a_{1j}(x)u\times\frac{\partial u}{\partial x_j},\qquad\frac{\partial G}{\partial x_1}=-\sum_{j=1}^{2}a_{2j}(x)u\times\frac{\partial u}{\partial x_j}. \tag{7}$$

Our aim here is to prove that G has the prescribed Gauss curvature. More precisely, we will show the following theorem.

Theorem. *Under the above assumptions, the metric induced by g on $G(B)$ is Riemannian. Moreover, $G(B)$ has the Gauss curvature equal to $-det(a_{ij})^{-1}$ at each point $G(x)$.* □

This paper is organized as follows. We first prove that there exists a solution of (5) in the $C^{2,\gamma}$ norm. Then, we show that the solution u is unique. By the same strategy as in [9], we deduce that u is a diffeomorphism. Hence, using the above approach, we will establish our result.

2. Existence and regularity

Let us first give the existence and regularity results.

Proposition 1. *Under the above hypothesis, there exists a minimum $u\in H^1_l(B,H^2_+)$ of E which satisfies (5). Furthermore, one has the estimate:*

$$\|u\|_{C^{2,\gamma}}\leq C_1(\|u\|_{H^1}+\|l\|_{C^{2,\gamma}}), \tag{8}$$

where C_1 is a constant depending only on α, γ and $\|a_{ij}\|_{C^{1,\gamma}}$. □

Remark. Notice that the metric induced by g on $\mathbb{H}^2_+$ is Riemannian. So it is natural for us to look for the minimum of E.

Proof. We will make use of the stereographic projection:

$$\tag{9} \begin{aligned} P: \quad & \mathbb{H}^2_+ && \longrightarrow B, \\ & (x,y,z) && \longmapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right). \end{aligned}$$

With these stereographic coordinates, we can write the the functional E as follows:

$$\tag{10} E(v) = 2\int_B \sum_{i,j=1}^{2} \frac{a_{ij}(x)}{(1-\mid v\mid^2)^2}\left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j}\right\rangle dx,$$

where $v \in H^1_h(B,\mathbb{R}^2)$ with $h = P \circ l$ and $\langle , \rangle$ denotes the standard Euclidian inner product. Assume that $|h| \le r$ with some $r < 1$. Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a decreasing continuous map satisfying

$$\tag{11} f(z) = \begin{cases} \dfrac{1}{(1-z^2)^2}, & \text{if } 0 \le z \le r; \\ \dfrac{1}{(1-r^2)^2}, & \text{if } z \ge r. \end{cases}$$

Consider the second energy functional E_1

$$E_1(v) = 2\int_B \sum_{i,j=1}^{2} a_{ij}(x) f(|v|)\left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j}\right\rangle dx,$$

where $v \in H^1_h(B,\mathbb{R}^2)$. Obviously,

$$E_1(v) \le E(v).$$

By coerciveness and lower semi-continuity of E_1 (see [5] and [15]), it is clear that there exists $w \in H^1_h(B,\mathbb{R}^2)$ minimizing E_1. We define $\tilde{w}$ by

$$\tag{12} \tilde{w}_i(x) = \begin{cases} w_i(x), & \text{if } |w_i(x)| \le r; \\ r, & \text{if } w_i(x) \ge r; \\ -r, & \text{if } w_i(x) \le -r, \end{cases}$$

for $i = 1, 2$. Obviously, $\tilde{w} \in H^1_h(B,\mathbb{R}^2)$ and $E_1(\tilde{w}) \le E_1(w)$. Thus, $\mid w_i(x) \mid \le r$ a.e. for $i = 1, 2$. Replacing w by $(w_1 cos\theta - w_2 sin\theta, w_1 sin\theta + w_2 cos\theta)$ for any $\theta \in \mathbb{R}$, we deduce that $\mid w(x) \mid \le r$ a.e. So w is also a minimizer of E. Thanks to a result due to Jost and Meier [10], Lemma 1 (see also [3], Lemma 1), we conclude that there exists $q > 2$ such that

$$\tag{13} \|w\|_{W^{1,q}(B,\mathbb{R}^2)} \le C_4(\|w\|_{H^1(B,\mathbb{R}^2)} + \|h\|_{C^1}),$$

where the constants C_4 and q depend only on α and $\|a_{ij}\|_{C^1}$. Now we consider $u = P^{-1} \circ w$ and return to equation (5). From L^p-estimates and using Sobolev embedding theorem, we have

$$\|u\|_{W^{1,\frac{2q}{4-q}}} \le C\|u\|_{W^{2,\frac{q}{2}}} \le C(\|u\|_{H^1} + \|l\|_{C^2}), \qquad \text{if } q < 4.$$

Iterating the above procedure and using Schauder estimates, we complete the proof (cf. [5]). $\square$

3. Uniqueness

In this part, our main result is the following:

Proposition 2. *The solution for equation* (5) *in* $C^2(\bar{B}, \mathbb{H}^2_+)$ *is unique.* □

Remark. This result and the proof we propose generalize an analogous result for harmonic maps due independently to [6] and [1].

Denote ∇ the Levi-Civita connection on $\mathbb{H}^2_+$ for the metric g. Let $u_1 \in C^2(\bar{B}, \mathbb{H}^2_+)$ be a map with the same boundary condition as u. For any $x \in \bar{B}$, let $\gamma_x(s)$ denote the unique geodesic arc in $\mathbb{H}^2_+$ parametrized with constant speed (depending on x) for $s \in [0,1]$, and connecting $u(x)$ with $u_1(x)$. The uniqueness of $\gamma_x(s)$ follows from $\mathbb{H}^2_+$ having nonpositive curvature and simply connected. Define a C^2 map $F : \bar{B} \times [0,1] \longrightarrow \mathbb{H}^2_+$ by $F(x,s) = \gamma_x(s)$ and let $u_s \in C^2(\bar{B}, \mathbb{H}^2_+)$ be given by $u_s(x) = F(x,s)$. Then, F is a deformation of u. We will write the first and second variations of the energy E (see [2]).

Lemma 1. *Under the above hypothesis, we have the following formulas:*

$$\frac{dE(u_s)}{ds} = -\int_B g\left(\frac{\partial u_s}{\partial s}, \nabla_{\frac{\partial}{\partial x_i}}(\sum_{i,j=1}^2 a_{ij}\frac{\partial u_s}{\partial x_j})\right) \tag{14}$$

and

$$\begin{aligned}\frac{d^2E(u_s)}{ds^2} &= -\int_B \sum_{i,j=1}^2 a_{ij} R(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}, \frac{\partial F}{\partial s}) \\ &\quad + \int_B \sum_{i,j=1}^2 a_{ij} g\left(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial u_s}{\partial s}, \nabla_{\frac{\partial}{\partial x_j}}\frac{\partial u_s}{\partial s}\right),\end{aligned} \tag{15}$$

where R is the curvature of $\mathbb{H}^2_+$. □

Proof. First, we suppose that F is C^∞. By definition,

$$E(u_s) = \frac{1}{2}\int_B \sum_{i,j=1}^2 a_{ij}(x) g(\frac{\partial u_s}{\partial x_i}, \frac{\partial u_s}{\partial x_j})dx.$$

Differentiating under the integral sign and using the symmetry of the Riemannian connection, we obtain

$$\begin{aligned}\frac{dE(u_s)}{ds} &= \int_B \sum_{i,j=1}^2 a_{ij}(x) g\left(\nabla_{\frac{\partial}{\partial s}}\frac{\partial u_s}{\partial x_i}, \frac{\partial u_s}{\partial x_j}\right)dx \\ &= \int_B \sum_{i,j=1}^2 a_{ij}(x) g\left(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial u_s}{\partial s}, \frac{\partial u_s}{\partial x_j}\right)dx \\ &= -\int_B g\left(\frac{\partial u_s}{\partial s}, \nabla_{\frac{\partial}{\partial x_i}}(\sum_{i,j=1}^2 a_{ij}\frac{\partial u_s}{\partial x_j})\right)dx,\end{aligned}$$

since $\frac{\partial F}{\partial s}=0$ on ∂B. Therefore, we obtain (14). Taking the derivative of (14), we have

$$\begin{aligned}
\frac{d^2E(u_s)}{ds^2} &= -\int_B g\left(\frac{\partial u_s}{\partial s}, \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial x_i}}(\sum_{i,j=1}^{2} a_{ij}\frac{\partial u_s}{\partial x_j})\right)dx \\
&= -\int_B \sum_{i,j=1}^{2} a_{ij}R(\frac{\partial F}{\partial s},\frac{\partial F}{\partial x_i},\frac{\partial F}{\partial x_j},\frac{\partial F}{\partial s}) \\
&\quad -\int_B g\left(\frac{\partial u_s}{\partial s}, \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial s}}(\sum_{i,j=1}^{2} a_{ij}\frac{\partial u_s}{\partial x_j})\right)dx \\
&= -\int_B \sum_{i,j=1}^{2} a_{ij}R(\frac{\partial F}{\partial s},\frac{\partial F}{\partial x_i},\frac{\partial F}{\partial x_j},\frac{\partial F}{\partial s}) \\
&\quad +\int_B \sum_{i,j=1}^{2} a_{ij}g\left(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial u_s}{\partial s}, \nabla_{\frac{\partial}{\partial x_j}}\frac{\partial u_s}{\partial s}\right)dx.
\end{aligned}$$

Thus, we establish (15). By density, we finish the proof. □

Proof of Proposition 2. Suppose that $u_1 \in C_l^2(\bar{B},\mathbb{H}_+^2)$ is another solution for equation (5). Putting $s=0$ and $s=1$ in (14), we obtain

$$\frac{dE(u_s)}{ds}\,|_{s=0,1} = 0.$$

On the other hand, we have

$$\frac{d^2E(u_s)}{ds^2} \geq 0$$

since $-R(\dfrac{\partial F}{\partial s},\cdot,\cdot,\dfrac{\partial F}{\partial s})$ is a positive quadratic form, that is, $E(u_s)$ is convex. Thus, $E(u_s)\equiv E(u_0)$. Thanks to formula (15), we infer that $\frac{\partial F}{\partial s}\equiv 0$. This contradiction completes the proof. □

4. The diffeomorphism property

Let $l:\partial B \longrightarrow \mathbb{H}^2\cap\{x_3=\alpha_1,\ \alpha_1>1\}$ be a C^2 diffeomorphism with $\deg(l,\partial B) = 1$. We will prove the following result.

Proposition 3. *Under the above assumptions, the unique minimizer u of E is a diffeomorphism and rank$(\nabla u(x))=2$ for all $x\in\bar{B}$.* □

The proof here is the same as in [3]. To prove this fact, we will consider the following energy functional:

$$E_t(u) = \frac{1}{2}\int_B \sum_{i,j=1}^{2}\Big[(1-t)\delta_{ij}+ta_{ij}(x)\Big]g\left(\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial x_j}\right)dx. \tag{16}$$

Let $I_t=\inf_{v\in H_l^1(B,\mathbb{H}_+^2)}E_t(v)$. Denote $u^t\in H_l^1(B,\mathbb{H}_+^2)$ the unique minimum of E_t in $H_l^1(B,\mathbb{H}_+^2)$ given by Propositions 1 and 2, then u^t satisfies:

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(\Big[(1-t)\delta_{ij}+ta_{ij}(x)\Big]\frac{\partial u^t}{\partial x_j}\right)+\lambda_t u^t = 0, & \text{in } B,\\ u^t = l, & \text{on } \partial B,\end{cases} \tag{17}$$

where $\lambda_t = -\sum_{i,j=1}^{2}\Big[(1-t)\delta_{ij}+ta_{ij}(x)\Big]g\left(\frac{\partial u^t}{\partial x_i}(x),\frac{\partial u^t}{\partial x_j}(x)\right)$. u^t is in $C^2(\bar{B},\mathbb{H}^2_+)$ by Proposition 1. Define a mapping F_*:

$$F_*:\quad \begin{array}{ll}[0,1] & \longrightarrow C^2(\bar{B},\mathbb{H}^2_+),\\ t & \longmapsto u^t.\end{array}$$

We need also several technical lemmas.

Lemma 2. *With the above notations, we have rank$(\nabla u^t(x))=2$, for any $t\in[0,1]$ and $x\in\partial B$.* □

The proof is the same as that of Lemma 5 in [3].

Lemma 3. *F_* is continuous.* □

Proof. First we notice that I_t is continuous. Indeed, for some fixed $v\in H^1_l(B,\mathbb{H}^2_+)$

$$0\le I_t\le\frac{1}{2}\left(4+\sum_{i,j=1}^{2}\|a_{ij}\|_{C^0}\right)\|\nabla v\|^2_{L^2}\le C,\qquad \forall 0\le t\le 1.$$

On the other hand, we have that for any $0\le t,\,t'\le 1$

$$|I_t-I_{t'}|\quad\le\frac{1}{\min(1,\alpha)}|t-t'|\left(4+\sum_{i,j=1}^{2}\|a_{ij}\|_{C^0}\right)\max\{I_t,I_{t'}\}.$$

Then the claim yields. Now let t be fixed. Assume that $\{t_n\}_{n\in\mathbb{N}}$ is a sequence converging to t. It follows from Proposition 1 that $\{u^{t_n}\}_{n\in\mathbb{N}}$ is compact in $C^2(\bar{B},\mathbb{H}^2_+)$. Modulo a subsequence, we can assume that $u^{t_n}\longrightarrow u$ in $C^2(\bar{B},\mathbb{H}^2_+)$ for $u\in C^2(\bar{B},\mathbb{H}^2_+)\cap H^1_l(B,\mathbb{H}^2_+)$. Clearly,

$$E_t(u)=I_t.$$

Now by Proposition 2, we terminate the proof. □

Proof of Proposition 3. We define a set

$$T_1=\{t\in[0,1],\ u^t\text{ is a diffeomorphism}\}.$$

***Step* 0 : T_1 is not empty.**
In view of Theorem 5.1.1 in [9] (see also [3], Lemma 7), we have $0\in T_1$.
***Step* 1 : T_1 is open.**
Let $t_1\in T_1$. Applying Lemmas 2 and 3, we get

$$\exists\,\tau_1>0,\text{ s.t. }\forall\,t\in\,]t_1-\tau_1,t_1+\tau_1[\,\cap[0,1]\Longrightarrow\text{rank}(\nabla u^t(x))=2,\qquad\forall x\in\bar{B}.$$

Now the claim follows from a result in [14] (see also [3], Lemma 6).
***Step* 2 : T_1 is also closed.**
Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence converging to t. Assume that u^{t_n} are diffeomorphisms, $\forall\,n\in\mathbb{N}$. We suppose that

$$\exists\,x_0\in\bar{B},\qquad\text{s.t.}\qquad\det(\nabla(P\circ u^t)(x_0))=0.$$

Denote $v = P \circ u^t$ and choose $\theta \in \mathbb{R}$ such that

$$((\nabla v_1)\cos\theta + (\nabla v_2)\sin\theta)(x_0) = 0.$$

Define $\omega_* = \omega_1 \cos\theta + \omega_2 \sin\theta$ for all continous functions $\omega : B \longrightarrow \mathbb{R}^2$. Thanks to the results of Hartman and Wintner [7], Theorems 1 and 2 (see also [3], Lemma 9), there exists some $n_1 \geq 1$ and $a \in \mathbb{C}^*$ such that

$$\partial_z v_*(z) = a(z - z_0)^{n_1} + o(|\ z - z_0\ |^{n_1})$$

where $z_0 = (x_0)_1 + i(x_0)_2$. Therefore, there exists $r_0 > 0$ and some n sufficiently large such that

$$\deg(\partial_z(P \circ u^{t_n})_*|_{\partial B(z_0,r_0)}, 0) = \deg(\partial_z v_*|_{\partial B(z_0,r_0)}, 0) = n_1 \geq 1.$$

However, by property of degree, this contradicts that u^{t_n} is a diffeomorphism. Hence, Proposition 3 is proved. □

5. Proof of the Theorem

Now, with the above results and preceding method, we can prove our main result. Note first that $g(\frac{\partial G}{\partial x_i}, u(x)) = 0$ for $i = 1, 2$. That is, $u(x)$ is the normal vector on $G(B)$ at point $G(x)$ for all $x \in \bar{B}$. On the other hand, we have

$$\begin{aligned} g(u \times u_{x_1}, u \times u_{x_1}) &= (u^3)^2(u^2_{x_1})^2 + (u^2)^2(u^3_{x_1})^2 - 2u^2u^2_{x_1}u^3u^3_{x_1} + (u^3)^2(u^1_{x_1})^2 \\ &\quad +(u^1)^2(u^3_{x_1})^2 - 2u^1u^1_{x_1}u^3u^3_{x_1} - (u^1)^2(u^2_{x_1})^2 - (u^2)^2(u^1_{x_1})^2 \\ &\quad +2u^2u^2_{x_1}u^1u^1_{x_1} \\ &= (u^3)^2((u^2_{x_1})^2 + (u^1_{x_1})^2) + (u^2)^2((u^3_{x_1})^2 - (u^1_{x_1})^2) \\ &\quad +(u^1)^2((u^3_{x_1})^2 - (u^2_{x_1})^2) - 2(u^3)^2(u^3_{x_1})^2 + 2u^2u^2_{x_1}u^1u^1_{x_1}, \end{aligned}$$

since $g(u, u_{x_1}) = 0$ (here subscripts denote partial differentiation with respect to coordinates). With help of the equalities $u^3 = \sqrt{1 + (u^1)^2 + (u^2)^2}$ and $(u^3_{x_1})^2 = \dfrac{(u^1u^1_{x_1} + u^2u^2_{x_1})^2}{1 + (u^1)^2 + (u^2)^2}$, we deduce

$$g(u \times u_{x_1}, u \times u_{x_1}) = g(u_{x_1}, u_{x_1}).$$

Replacing x_1 by x_2 and $x_1 + x_2$, implies

$$g(u \times u_{x_2}, u \times u_{x_2}) = g(u_{x_2}, u_{x_2}) \qquad \text{and} \qquad g(u \times u_{x_1}, u \times u_{x_2}) = g(u_{x_1}, u_{x_2}).$$

Therefore, we conclude that G is an immersion and that the metric induced on $G(B)$ is Riemannian.

Now we will calculate the curvature of $G(B)$. Denote D (resp. ∇) the Levi-Civita connection on $\mathbb{R}^{2,1}$ (resp. $G(B)$) and R the curvature. Obviously, we have

$$\nabla_X Y = D_X Y + g(D_X Y, u)u,$$

where X and Y are vector fields on $G(B)$. So, this implies

$$\begin{aligned}
& R\left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_1}\right) \\
= \ & g\left(\nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial G}{\partial x_2} - \nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_1}\right) \\
= \ & g\left(D_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial G}{\partial x_2} - D_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_1}\right) \\
= \ & g\left(D_{\frac{\partial}{\partial x_2}} \frac{\partial G}{\partial x_2}, u\right) g\left(D_{\frac{\partial}{\partial x_1}} u, \frac{\partial G}{\partial x_1}\right) - g\left(D_{\frac{\partial}{\partial x_1}} \frac{\partial G}{\partial x_2}, u\right) g\left(D_{\frac{\partial}{\partial x_2}} u, \frac{\partial G}{\partial x_1}\right) \\
= \ & -g\left(D_{\frac{\partial}{\partial x_2}} u, \frac{\partial G}{\partial x_2}\right) g\left(D_{\frac{\partial}{\partial x_1}} u, \frac{\partial G}{\partial x_1}\right) + g\left(D_{\frac{\partial}{\partial x_1}} u, \frac{\partial G}{\partial x_2}\right) g\left(D_{\frac{\partial}{\partial x_2}} u, \frac{\partial G}{\partial x_1}\right) \\
= \ & (-a_{11}a_{22} + a_{12}^2) g(u \times u_{x_1}, u_{x_2})^2,
\end{aligned}$$

since $g(u \times u_{x_i}, u_{x_i}) = 0$ for $i = 1, 2$. On the other hand,

$$\begin{aligned}
& g(G_{x_1}, G_{x_1}) g(G_{x_2}, G_{x_2}) - g(G_{x_1}, G_{x_2})^2 \\
= \ & g\Big(\sum_{j=1}^{2} a_{2j} u \times u_{x_j}, \sum_{k=1}^{2} a_{2k} u \times u_{x_k}\Big) g\Big(\sum_{j=1}^{2} a_{1j} u \times u_{x_j}, \sum_{k=1}^{2} a_{1k} u \times u_{x_k}\Big) \\
& - g\Big(\sum_{j=1}^{2} a_{1j} u \times u_{x_j}, \sum_{k=1}^{2} a_{2k} u \times u_{x_k}\Big)^2 \\
= \ & g\Big(\sum_{j=1}^{2} a_{2j} u_{x_j}, \sum_{k=1}^{2} a_{2k} u_{x_k}\Big) g\Big(\sum_{j=1}^{2} a_{1j} u_{x_j}, \sum_{k=1}^{2} a_{1k} u_{x_k}\Big) - g\Big(\sum_{j=1}^{2} a_{1j} u_{x_j}, \sum_{k=1}^{2} a_{2k} u_{x_k}\Big)^2 \\
= \ & \det(a_{ij})^2 \big(g(u_{x_1}, u_{x_1}) g(u_{x_2}, u_{x_2}) - g(u_{x_1}, u_{x_2})^2\big) \\
= \ & -\det(a_{ij})^2 g(u_{x_1} \times u_{x_2}, u_{x_1} \times u_{x_2}) \\
= \ & \det(a_{ij})^2 g(u \times u_{x_1}, u_{x_2})^2.
\end{aligned}$$

Hence, $K(G(x)) = -\det(a_{ij}(x))^{-1}$. □

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Département de Mathématiques, Faculté de Sciences et Technologie, Université Paris XII-Val de Marne, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France

C.M.L.A., E.N.S de Cachan, 61, avenue du Président Wilson, 94235 Cachan Cedex, France

E-mail address: `ge@cmla.ens-cachan.fr`