IMMERSED SURFACES OF PRESCRIBED GAUSS CURVATURE INTO MINKOWSKI SPACE

YUXIN GE

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Abstract. Given a positive real valued function \( k(x) \) on the disc, we will immerse the disc into three dimensional Minkowski space in such a way that Gauss curvature at the image point of \( x \) is \(-k(x)\). Our approach lies on the construction of Gauss map of surfaces.

1. Introduction

The classical Minkowski problem is an embedding problem of differential geometry. This problem is the following: Given a positive function \( K(u) \) defined on the unit sphere, does there exist a closed convex surface in \( \mathbb{R}^3 \) having \( K(u) \) as its Gauss curvature at the point on the surface where the inner normal is \( u \)? In [11], Lewy has shown the existence of such a surface, under the condition that the function \( K(u) \) is analytic. Later, by using a similar procedure, Nirenberg [13] published a paper in which he solved the Minkowski problem under the assumption that the function \( K(u) \) possesses partial derivatives on the sphere up to second order. In [3], the author considers an analogous problem by using an approach, suggested in [8]: Given a positive real valued function \( k(x) \) on the disc, we immerse the disc in \( \mathbb{R}^3 \) in such a way that Gauss curvature at the image point of \( x \) is \( k(x) \). In this paper, we continue exploiting this method to immerse the disc into three dimensional Minkowski space. Namely we propose and use a method for constructing immersions of surfaces in the Minkowski space \( \mathbb{R}^{2,1} \) by prescribing the Gauss curvature to be a negative function of the variable \( x \) in the surface. Notice that in [3] we had an analogous construction for surfaces in the Euclidean space but with a positive Gauss curvature.

Let \( B = \{ x \in \mathbb{R}^2, \ | \ x \ | < 1 \} \) be a disc in \( \mathbb{R}^2 \). Let \( \mathbb{R}^{2,1} \) be three dimensional Minkowski space with the standard metric \( g = (dx_1)^2 + (dx_2)^2 - (dx_3)^2 \). Let \( \mathbb{H}_2 = \{ x \in \mathbb{R}^3, \ g(x,x) = -1 \} \) be the unit hyperboloid of two sheets and let \( \mathbb{H}_2^+ = \{ x \in \mathbb{H}_2, \ x_3 > 0 \} \) be the upper sheet contained in the half-space \( \{ x_3 > 0 \} \).

Let \( l: \partial B \rightarrow \mathbb{H}_2^+ \) be a prescribed \( C^{2,\gamma} \) mapping with \( \gamma > 0 \). We consider the space \( H^1(B, \mathbb{H}_2^+) \) of functions \( u \) in \( H^1(B, \mathbb{R}^3) \) satisfying that \( u \in \mathbb{H}_2^+ \) a.e. and \( u = l \).
on \( \partial B \). We define on \( H^1_1(B, \mathbb{H}^2_1) \) the following energy functional \( E \):

\[
E(u) = \frac{1}{2} \int_B \sum_{i,j=1}^{2} a_{ij}(x) \left( \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right) dx,
\]

where \( a_{ij}(x) \) satisfy the following conditions:

1. \( \exists \alpha > 0, \text{ such that } a_{ij}(x)\xi^i\xi^j \geq \alpha |\xi|^2, \quad \forall x \in B, \forall \xi \in \mathbb{R}^2; \)
2. \( a_{ij}(x) \in C^{1,\gamma}(B, \mathbb{R}), \quad \forall 1 \leq i, j \leq 2; \)
3. \( a_{ij} = a_{ji}, \quad \forall 1 \leq i, j \leq 2. \)

Here, it is easy to check that the critical points of \( E \) satisfy in the sense of distributions the following Euler equation:

\[
\begin{cases}
\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \lambda u = 0, & \text{in } B, \\
u = 1, & \text{on } \partial B,
\end{cases}
\]

where \( \lambda = -\sum_{i,j=1}^{2} a_{ij}(x)g \left( \frac{\partial u}{\partial x_i}(x), \frac{\partial u}{\partial x_j}(x) \right). \)

Notice that \( \lambda < 0 \). We deduce from (5) the following equality:

\[
\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( u \times \sum_{j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0,
\]

where \( \xi \times \eta = (\xi_2\eta_3 - \xi_3\eta_2, \xi_3\eta_1 - \xi_1\eta_3, \xi_1\eta_2 - \xi_2\eta_1) \) for all \( \xi, \eta \in \mathbb{R}^{2,1} \) is vectorial product in \( \mathbb{R}^{2,1} \). Assume that \( u \) is an immersion. Thus, we obtain a new immersion \( G \) from \( B \) to \( \mathbb{R}^{2,1} \) satisfying

\[
\frac{\partial G}{\partial x_2} = \sum_{j=1}^{2} a_{1j}(x)u \times \frac{\partial u}{\partial x_j}, \quad \frac{\partial G}{\partial x_1} = -\sum_{j=1}^{2} a_{2j}(x)u \times \frac{\partial u}{\partial x_j}.
\]

Our aim here is to prove that \( G \) has the prescribed Gauss curvature. More precisely, we will show the following theorem.

**Theorem.** Under the above assumptions, the metric induced by \( g \) on \( G(B) \) is Riemannian. Moreover, \( G(B) \) has the Gauss curvature equal to \(-\det(a_{ij})^{-1}\) at each point \( G(x) \).

This paper is organized as follows. We first prove that there exists a solution of (5) in the \( C^{2,\gamma} \) norm. Then, we show that the solution \( u \) is unique. By the same strategy as in \([9]\), we deduce that \( u \) is a diffeomorphism. Hence, using the above approach, we will establish our result.

### 2. Existence and regularity

Let us first give the existence and regularity results.

**Proposition 1.** Under the above hypothesis, there exists a minimum \( u \in H^1_1(B, H^2_1) \) of \( E \) which satisfies (5). Furthermore, one has the estimate:

\[
\|u\|_{C^{2,\gamma}} \leq C_1(\|u\|_{H^1} + \|l\|_{C^{2,\gamma}}),
\]

where \( C_1 \) is a constant depending only on \( \alpha, \gamma \) and \( \|a_{ij}\|_{C^{1,\gamma}} \).

\[\square\]
Remark. Notice that the metric induced by $g$ on $\mathbb{H}^2_+$ is Riemannian. So it is natural for us to look for the minimum of $E$.

Proof. We will make use of the stereographic projection:

$$P : \mathbb{H}^2_+ \longrightarrow B,$$

$$(x, y, z) \longrightarrow \left(\frac{x}{1 + z}, \frac{y}{1 + z}\right).$$

With these stereographic coordinates, we can write the functional $E$ as follows:

$$E(v) = 2 \int_B \sum_{i,j=1}^2 a_{ij}(x) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j}\right) dx,$$

where $v \in H^1_b(B, \mathbb{R}^2)$ with $h = P \circ l$ and $\langle , \rangle$ denotes the standard Euclidean inner product. Assume that $|h| \leq r$ with some $r < 1$. Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a decreasing continuous map satisfying

$$f(z) = \begin{cases} 
\frac{1}{(1 - z^2)^2}, & \text{if } 0 \leq z \leq r; \\
\frac{1}{(1 - r^2)^2}, & \text{if } z \geq r.
\end{cases}$$

Consider the second energy functional $E_1$

$$E_1(v) = 2 \int_B \sum_{i,j=1}^2 a_{ij}(x) f(|v|) \left(\frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j}\right) dx,$$

where $v \in H^1_b(B, \mathbb{R}^2)$. Obviously,

$$E_1(v) \leq E(v).$$

By coerciveness and lower semi-continuity of $E_1$ (see [5] and [15]), it is clear that there exists $w \in H^1_b(B, \mathbb{R}^2)$ minimizing $E_1$. We define $\tilde{w}$ by

$$\tilde{w}_i(x) = \begin{cases} 
w_i(x), & \text{if } |w_i(x)| \leq r; \\
r, & \text{if } w_i(x) \geq r; \\
-r, & \text{if } w_i(x) \leq -r,
\end{cases}$$

for $i = 1, 2$. Obviously, $\tilde{w} \in H^1_b(B, \mathbb{R}^2)$ and $E_1(\tilde{w}) \leq E_1(w)$. Thus, $|w_i(x)| \leq r$ a.e. for $i = 1, 2$. Replacing $w$ by $(w_1 \cos \theta - w_2 \sin \theta, w_1 \sin \theta + w_2 \cos \theta)$ for any $\theta \in \mathbb{R}$, we deduce that $|w(x)| \leq r$ a.e. So $w$ is also a minimizer of $E$. Thanks to a result due to Jost and Meier [10], Lemma 1 (see also [3], Lemma 1), we conclude that there exists $q > 2$ such that

$$\|w\|_{W^{1,\alpha}(B, \mathbb{R}^2)} \leq C_4(\|w\|_{H^1(B, \mathbb{R}^2)} + \|h\|_{C^1}),$$

where the constants $C_4$ and $q$ depend only on $\alpha$ and $\|a_{ij}\|_{C^1}$. Now we consider $u = P^{-1} \circ w$ and return to equation (5). From $L^p$-estimates and using Sobolev embedding theorem, we have

$$\|u\|_{W^{1,2+\frac{2r}{q-2}}} \leq C\|u\|_{W^{2,\frac{q}{2}}} \leq C(\|u\|_{H^1} + \|l\|_{C^2}),$$

if $q < 4$.

Iterating the above procedure and using Schauder estimates, we complete the proof (cf. [5]).
3. Uniqueness

In this part, our main result is the following:

**Proposition 2.** The solution for equation (5) in $C^2(\tilde{B}, \mathbb{H}^2_+)$ is unique.

Remark. This result and the proof we propose generalize an analogous result for harmonic maps due independently to [6] and [1].

Denote $\nabla$ the Levi-Civita connection on $\mathbb{H}^2_+$ for the metric $g$. Let $u_1 \in C^2(\tilde{B}, \mathbb{H}^2_+)$ be a map with the same boundary condition as $u$. For any $x \in \tilde{B}$, let $\gamma_x(s)$ denote the unique geodesic arc in $\mathbb{H}^2_+$ parametrized with constant speed (depending on $x$) for $s \in [0,1]$, and connecting $u(x)$ with $u_1(x)$. The uniqueness of $\gamma_x(s)$ follows from $\mathbb{H}^2_+$ having nonpositive curvature and simply connected. Define a $C^2$ map $F: \tilde{B} \times [0,1] \rightarrow \mathbb{H}^2_+$ by $F(x,s) = \gamma_x(s)$ and let $u_s \in C^2(\tilde{B}, \mathbb{H}^2_+)$ be given by $u_s(x) = F(x,s)$. Then, $F$ is a deformation of $u$. We will write the first and second variations of the energy $E$ (see [2]).

**Lemma 1.** Under the above hypothesis, we have the following formulas:

\begin{equation}
\frac{dE(u_s)}{ds} = -\int_{\tilde{B}} \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} \left( \frac{\partial u_s}{\partial x_i} \cdot \frac{\partial u_s}{\partial x_j} \right) \nabla_{\frac{\partial}{\partial s}} (2 \sum_{i,j=1}^{2} a_{ij} \frac{\partial u_s}{\partial x_j})
\end{equation}

\begin{equation}
\frac{d^2E(u_s)}{ds^2} = -\int_{\tilde{B}} \sum_{i,j=1}^{2} a_{ij} R \left( \frac{\partial F}{\partial s} \cdot \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_j} \cdot \frac{\partial F}{\partial s} \right) + \int_{\tilde{B}} \sum_{i,j=1}^{2} a_{ij} g \left( \nabla_{\frac{\partial}{\partial s}} \frac{\partial u_s}{\partial x_i}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial u_s}{\partial x_j} \right),
\end{equation}

where $R$ is the curvature of $\mathbb{H}^2_+$.

Proof. First, we suppose that $F$ is $C^\infty$. By definition,

$$E(u_s) = \frac{1}{2} \int_{\tilde{B}} \sum_{i,j=1}^{2} a_{ij}(x) g \left( \frac{\partial u_s}{\partial x_i}, \frac{\partial u_s}{\partial x_j} \right) dx.$$ 

Differentiating under the integral sign and using the symmetry of the Riemannian connection, we obtain

\begin{align*}
\frac{dE(u_s)}{ds} &= \int_{\tilde{B}} \sum_{i,j=1}^{2} a_{ij}(x) g \left( \nabla_{\frac{\partial}{\partial s}} \frac{\partial u_s}{\partial x_i}, \frac{\partial u_s}{\partial x_j} \right) dx \\
&= \int_{\tilde{B}} \sum_{i,j=1}^{2} a_{ij}(x) g \left( \frac{\partial u_s}{\partial s}, \frac{\partial u_s}{\partial x_j} \right) dx \\
&= -\int_{\tilde{B}} g \left( \frac{\partial u_s}{\partial s}, \nabla_{\frac{\partial}{\partial s}} (2 \sum_{i,j=1}^{2} a_{ij} \frac{\partial u_s}{\partial x_j}) \right) dx,
\end{align*}
since \( \frac{\partial F}{\partial s} = 0 \) on \( \partial B \). Therefore, we obtain (14). Taking the derivative of (14), we have

\[
\frac{d^2 E(u_s)}{ds^2} = -\int_B g \left( \frac{\partial u_s}{\partial s}, \nabla \frac{\partial u_s}{\partial s}, \sum_{i,j=1}^2 a_{ij} \frac{\partial u_s}{\partial x_j} \right) dx
\]

Thus, we establish (15). By density, we finish the proof.

**Proof of Proposition 2.** Suppose that \( u_1 \in C^2(\overline{B}, \mathbb{H}^2) \) is another solution for equation (5). Putting \( s = 0 \) and \( s = 1 \) in (14), we obtain

\[
\frac{dE(u_s)}{ds} \bigg|_{s=0,1} = 0.
\]

On the other hand, we have

\[
\frac{d^2 E(u_s)}{ds^2} \geq 0
\]

since \(-R(\frac{\partial F}{\partial s}, \ldots, \frac{\partial F}{\partial s})\) is a positive quadratic form, that is, \( E(u_s) \) is convex. Thus, \( E(u_s) \equiv E(u_0) \). Thanks to formula (15), we infer that \( \frac{\partial F}{\partial s} \equiv 0 \). This contradiction completes the proof.

**4. The Diffeomorphism Property**

Let \( l : \partial B \rightarrow \mathbb{H}^2 \cap \{ x_3 = \alpha_1, \alpha_1 > 1 \} \) be a \( C^2 \) diffeomorphism with \( \text{deg}(l, \partial B) = 1 \). We will prove the following result.

**Proposition 3.** Under the above assumptions, the unique minimizer \( u \) of \( E \) is a diffeomorphism and \( \text{rank}(\nabla u(x)) = 2 \) for all \( x \in \overline{B} \).

The proof here is the same as in [3]. To prove this fact, we will consider the following energy functional:

\[
E_t(u) = \frac{1}{2} \int_B \sum_{i,j=1}^2 [(1-t)\delta_{ij} + ta_{ij}(x)] g \left( \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right) dx.
\]

Let \( I_t = \inf_{v \in H^1_t(B, \mathbb{H}^2_+)} E_t(v) \). Denote \( u^t \in H^1_t(B, \mathbb{H}^2_+) \) the unique minimum of \( E_t \) in \( H^1_t(B, \mathbb{H}^2_+) \) given by Propositions 1 and 2, then \( u^t \) satisfies:

\[
\left\{ \begin{array}{l}
\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( [(1-t)\delta_{ij} + ta_{ij}(x)] \frac{\partial u^t}{\partial x_j} \right) + \lambda_t u^t = 0, \quad \text{in } B, \\
u^t = l, \quad \text{on } \partial B,
\end{array} \right.
\]
where \( \lambda_t = - \sum_{i,j=1}^2 \left[ (1-t) \delta_{ij} + ta_{ij}(x) \right] g \left( \frac{\partial u^t}{\partial x_i}(x), \frac{\partial u^t}{\partial x_j}(x) \right) \). \( u^t \) is in \( C^2(\overline{B}, \mathbb{R}^2_+) \) by Proposition 1. Define a mapping \( F_* : [0,1] \to C^2(\overline{B}, \mathbb{R}^2_+), t \mapsto u^t \).

We need also several technical lemmas.

**Lemma 2.** With the above notations, we have \( \text{rank}(\nabla u^t(x)) = 2 \), for any \( t \in [0,1] \) and \( x \in \partial B \).

The proof is the same as that of Lemma 5 in [3].

**Lemma 3.** \( F_* \) is continuous.

**Proof.** First we notice that \( I_t \) is continuous. Indeed, for some fixed \( v \in H^1_0(B, \mathbb{R}^2_+) \)

\[
0 \leq I_t \leq \frac{1}{2} \left( 4 + \sum_{i,j=1}^2 \| a_{ij} \|_{C^0} \right) \| \nabla v \|_{L^2}^2 \leq C, \quad \forall 0 \leq t \leq 1.
\]

On the other hand, we have that for any \( 0 \leq t, t' \leq 1 \)

\[
|I_t - I_{t'}| \leq \frac{1}{\min(1, \alpha)} |t - t'| \left( 4 + \sum_{i,j=1}^2 \| a_{ij} \|_{C^0} \right) \max\{I_t, I_{t'}\}.
\]

Then the claim yields. Now let \( t \) be fixed. Assume that \( \{ t_n \}_{n \in \mathbb{N}} \) is a sequence converging to \( t \). It follows from Proposition 1 that \( \{ u^{t_n} \}_{n \in \mathbb{N}} \) is compact in \( C^2(\overline{B}, \mathbb{R}^2_+) \). Modulo a subsequence, we can assume that \( u^{t_n} \to u \) in \( C^2(\overline{B}, \mathbb{R}^2_+) \cap H^1(B, \mathbb{R}^2_+) \). Clearly,

\[
E_t(u) = I_t.
\]

Now by Proposition 2, we terminate the proof.

**Proof of Proposition 3.** We define a set

\[
T_1 = \{ t \in [0,1], \; u^t \text{ is a diffeomorphism} \}.
\]

**Step 0 :** \( T_1 \) is not empty.

In view of Theorem 5.1.1 in [9] (see also [3], Lemma 7), we have \( 0 \in T_1 \).

**Step 1 :** \( T_1 \) is open.

Let \( t_1 \in T_1 \). Applying Lemmas 2 and 3, we get

\[
\exists \tau_1 > 0, \text{ s.t. } \forall t \in [t_1 - \tau_1, t_1 + \tau_1] \cap [0,1] \Rightarrow \text{rank}(\nabla u^t(x)) = 2, \quad \forall x \in \overline{B}.
\]

Now the claim follows from a result in [14] (see also [3], Lemma 6).

**Step 2 :** \( T_1 \) is also closed.

Let \( \{ t_n \}_{n \in \mathbb{N}} \) be a sequence converging to \( t \). Assume that \( u^{t_n} \) are diffeomorphisms, \( \forall n \in \mathbb{N} \). We suppose that

\[
\exists x_0 \in \overline{B}, \text{ s.t. } \det(\nabla(P \circ u^t)(x_0)) = 0.
\]
Denote \( v = P \circ u^t \) and choose \( \theta \in \mathbb{R} \) such that
\[
((\nabla v_1) \cos \theta + (\nabla v_2) \sin \theta)(x_0) = 0.
\]
Define \( \omega_* = \omega_1 \cos \theta + \omega_2 \sin \theta \) for all continuous functions \( \omega : B \to \mathbb{R}^2 \). Thanks to the results of Hartman and Wintner \([7]\), Theorems 1 and 2 (see also \([3]\), Lemma 9), there exists some \( n_1 \geq 1 \) and \( a \in \mathbb{C}^* \) such that
\[
\partial_z \omega_*(z) = a(z - z_0)^{n_1} + o(|z - z_0|^{n_1})
\]
where \( z_0 = (x_0)_1 + i(x_0)_2 \). Therefore, there exists \( r_0 > 0 \) and some \( n \) sufficiently large such that
\[
\deg(\partial_z (P \circ u^{tn})|_{\partial B(z_0, r_0)}, 0) = \deg(\partial_z \omega_*(B(z_0, r_0)), 0) = n_1 \geq 1.
\]
However, by property of degree, this contradicts that \( u^{tn} \) is a diffeomorphism. Hence, Proposition 3 is proved. 

5. Proof of the Theorem

Now, with the above results and preceding method, we can prove our main result. Note first that \( g(\frac{\partial G}{\partial u^i}, u(x)) = 0 \) for \( i = 1, 2 \). That is, \( u(x) \) is the normal vector on \( G(B) \) at point \( G(x) \) for all \( x \in B \). On the other hand, we have
\[
g(u \times u_{x_1}, u \times u_{x_1}) = (u^3)^2(u_{x_1}^2)^2 + (u^2)^2(u_{x_2}^3)^2 - 2u^2u_{x_1}^2u_{x_2}^2u_{x_3}^3 + (u^3)^2(u_{x_1}^2)^2
\]
\[
+ (u^1)^2(u_{x_1}^3)^2 - 2u^1u_{x_1}^1u_{x_2}^3u_{x_3}^3 - (u^1)^2(u_{x_2}^2)^2 - (u^2)^2(u_{x_2}^1)^2
\]
\[
+ 2u^1u_{x_1}^2u_{x_2}^2u_{x_3}^1u_{x_1}^1,
\]
since \( g(u, u_{x_1}) = 0 \) (here subscripts denote partial differentiation with respect to coordinates). With help of the equalities \( u^3 = \sqrt{1 + (u^1)^2 + (u^2)^2} \) and \( (u_{x_1}^3)^2 = \frac{(u_{x_1}^1)^2 + (u_{x_2}^2)^2}{1 + (u^1)^2 + (u^2)^2} \), we deduce
\[
g(u \times u_{x_1}, u \times u_{x_1}) = g(u_{x_1}, u_{x_1}).
\]
Replacing \( x_1 \) by \( x_2 \) and \( x_1 + x_2 \), implies
\[
g(u \times u_{x_2}, u \times u_{x_2}) = g(u_{x_2}, u_{x_2}) \quad \text{and} \quad g(u \times u_{x_1}, u \times u_{x_2}) = g(u_{x_1}, u_{x_2}).
\]
Therefore, we conclude that \( G \) is an immersion and that the metric induced on \( G(B) \) is Riemannian.

Now we will calculate the curvature of \( G(B) \). Denote \( D \) (resp. \( \nabla \)) the Levi-Civita connection on \( \mathbb{R}^{2,1} \) (resp. \( G(B) \)) and \( R \) the curvature. Obviously, we have
\[
\nabla_X Y = D_X Y + g(D_X u, u)u,
\]
where $X$ and $Y$ are vector fields on $G(B)$. So, this implies

$$R \left( \frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_1} \right)$$

$$= g \left( \nabla_{\frac{\partial G}{\partial x_1}} \nabla_{\frac{\partial G}{\partial x_2}} - \nabla_{\frac{\partial G}{\partial x_2}} \nabla_{\frac{\partial G}{\partial x_1}} \right)$$

$$= g \left( D_{\frac{\partial G}{\partial x_1}} \nabla_{\frac{\partial G}{\partial x_2}} \nabla_{\frac{\partial G}{\partial x_2}} - D_{\frac{\partial G}{\partial x_2}} \nabla_{\frac{\partial G}{\partial x_1}} \nabla_{\frac{\partial G}{\partial x_1}} \right)$$

$$= g \left( D_{\frac{\partial G}{\partial x_1}} \frac{\partial G}{\partial x_2}, u \right) g \left( D_{\frac{\partial G}{\partial x_2}} \frac{\partial G}{\partial x_2}, u \right) - g \left( D_{\frac{\partial G}{\partial x_2}} \frac{\partial G}{\partial x_1}, u \right) g \left( D_{\frac{\partial G}{\partial x_1}} \frac{\partial G}{\partial x_1}, u \right)$$

$$= -g \left( D_{\frac{\partial G}{\partial x_1}} u, \frac{\partial G}{\partial x_2} \right) g \left( D_{\frac{\partial G}{\partial x_2}} u, \frac{\partial G}{\partial x_2} \right) + g \left( D_{\frac{\partial G}{\partial x_2}} u, \frac{\partial G}{\partial x_1} \right) g \left( D_{\frac{\partial G}{\partial x_1}} u, \frac{\partial G}{\partial x_1} \right)$$

$$= (-a_{11}a_{22} + a_{12}^2)g(u \times u_{x_1}, u_{x_2})^2,$$

since $g(u \times u_{x_i}, u_{x_i}) = 0$ for $i = 1, 2$. On the other hand,

$$g(G_{x_1}, G_{x_1})g(G_{x_2}, G_{x_2}) - g(G_{x_1}, G_{x_2})^2$$

$$= g \left( \sum_{j=1}^{2} a_{2j} u \times u_{x_j}, \sum_{k=1}^{2} a_{2k} u \times u_{x_k} \right) g \left( \sum_{j=1}^{2} a_{1j} u \times u_{x_j}, \sum_{k=1}^{2} a_{1k} u \times u_{x_k} \right)$$

$$- g \left( \sum_{j=1}^{2} a_{1j} u \times u_{x_j}, \sum_{k=1}^{2} a_{2k} u \times u_{x_k} \right)^2$$

$$= g \left( \sum_{j=1}^{2} a_{2j} u_{x_j}, \sum_{k=1}^{2} a_{2k} u_{x_k} \right) g \left( \sum_{j=1}^{2} a_{1j} u_{x_j}, \sum_{k=1}^{2} a_{1k} u_{x_k} \right) - g \left( \sum_{j=1}^{2} a_{1j} u_{x_j}, \sum_{k=1}^{2} a_{2k} u_{x_k} \right)^2$$

$$= \det(a_{ij})^2(g(u_{x_1}, u_{x_1})g(u_{x_2}, u_{x_2}) - g(u_{x_1}, u_{x_2})^2)$$

$$= -\det(a_{ij})^2 g(u_{x_1} \times u_{x_2}, u_{x_1} \times u_{x_2})$$

$$= \det(a_{ij})^2 g(u \times u_{x_1}, u_{x_2})^2.$$

Hence, $K(G(x)) = -\det(a_{ij}(x))^{-1}$. \hfill \square

References


DÉPARTEMENT DE MATHEMATIQUES, FACULTÉ DE SCIENCES ET TECHNOLOGIE, UNIVERSITÉ PARIS XII-VAL DE MARNE, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 Créteil Cedex, FRANCE

C.M.L.A., E.N.S DE CACHAN, 61, AVENUE DU PRÉSIDENT WILSON, 94235 CACHAN CEDEX, FRANCE

E-mail address: ge@cmla.ens-cachan.fr