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COMPOSITION OPERATORS WITH CLOSED RANGE ON THE BLOCH SPACE

PRATIBHA GHATAGE, JUN YAN, AND DECHAO ZHENG

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Abstract. In this note we investigate conditions under which a holomorphic self-map of the unit disk induces a composition operator with closed range on the Bloch space.

1. Introduction

For each $z \in D$, let φ_z denote the Möbius transformation of D

$$\varphi_z(w) = \frac{z-w}{1-\overline{z}w},$$

for $w \in D$. The pseudo-hyperbolic distance on D is defined by

$$\rho(z,w) = |\varphi_z(w)|, \quad z,w \in D.$$

The pseudohyperbolic distance is Möbius invariant, that is,

$$\rho(gz,gw) = \rho(z,w),$$

for all $g \in Aut(D)$, the Möbius group of D, and all $z,\ w \in D$. It has the following useful property:

$$1-\rho(z,w)^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}. \tag{1.1}$$

A function f is called a Bloch function if it is analytic in D and if

$$\|f\|_B = \sup_{z\in D}(1-|z|^2)|f'(z)| < \infty.$$

This defines a seminorm, and the Bloch functions form a complex Banach space B with the norm

$$\|f\|_\beta = |f(0)| + \|f\|_B.$$

We will show that $(1-|z|^2)|f'(z)|$ is Lipschitz with respect to pseudo-hyperbolic metric and use this result to study composition operators on the Bloch space in Section 3.

The little Bloch space of D, denoted B_0, is the closed subspace of B consisting of functions f with $(1-|z|^2)|f'(z)| \to 0$ as $|z| \to 1^-$.

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The composition operator $C_\varphi : H(D) \to H(D)$ is defined by $C_\varphi f = f \circ \varphi$ on the space of analytic funtions on D. It is well known exactly when C_φ is compact on B or B_0 [2]. To our knowledge no such study has been done to determine when C_φ has closed range. In this note we give a set of necessary conditions and a partial converse. For a comprehensive treatment of composition operators we refer the reader to [3].

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2. Necessary conditions

We assume that φ is a holomorphic self-map of the unit disk D and C_φ is the composition operator on the Bloch space B. We write $G = \varphi(D)$, and $\tau_\varphi(z) = (1-|z|^2)\varphi'(z)/(1-|\varphi(z)|^2)$. The following are two simple consequences of the Schwarz-Pick Lemma [1, p. 2].

(1) C_φ maps B into B.
(2) $0 \le |\tau_\varphi(z)| \le 1$.

Lemma 1. *If C_φ is bounded below on B, then, for all $f \in B$,*

$$\|f\|_B \le k\{sup(1-|w|^2)|f'(w)|, w \in G\}$$

for some k.

The proof is a trivial consequence of (2) above.

As Smith has shown in [4] there exists a univalent function $\varphi : D \to D$ with $\varphi(D)$ dense in D for which $\tau_\varphi(z) \to 0$ as $|z| \to 1$ [4, 6.5]. This is known to be equivalent to the fact that C_φ is compact on B_0 [2].

Proposition 1. *If C_φ is bounded below on B, then there exist positive constants ϵ, r with $r < 1$ such that, for all $z \in D$, $\rho(\varphi(\Omega_\epsilon), z) \le r$ where $\Omega_\epsilon = \{z \in D, |\tau_g(z)| > \epsilon\}$.*

Proof. Since $C_\varphi : B \to B$ is bounded below, there is a constant $0 < k \le 1$ such that

$$\|C_\varphi f\|_\beta \ge k\|f\|_\beta,$$

for $f \in B_0$. For each $w \in D$, let $f_w(z) = \frac{w-z}{1-\overline{w}z} - \frac{w-\varphi(0)}{1-\overline{w}\varphi(0)}$. Clearly, f_w is a bounded and continuous analytic function on the closed unit disk and so is in B_0. Moreover, an easy computation gives $\|f_w\|_\beta \ge 1$. Thus

$$\|C_\varphi f_w\|_\beta \ge k\|f_w\|_\beta = k.$$

On the other hand, we also have

$$(1-|z|^2)|(C_\varphi f_w)'(z)| = (1-|\varphi_w(\varphi(z))|^2)|\tau_\varphi(z)|,$$

and $C_\varphi f_w(0) = 0$. Then there is a point $z_w \in D$ such that

$$(1-|z_w|^2)(C_\varphi f_w)'(z_w)| \ge \|C_\varphi f_w\|_\beta/2 \ge k/2.$$

So we obtain that

$$(1-|\varphi_w(\varphi(z_w))|^2)|\tau_\varphi(z_w)| \ge k/2.$$

Thus

$$(1-|\varphi_w(\varphi(z_w))|^2) \geq k/2,$$

and

$$|\tau_\varphi(z_w)| \geq k/2,$$

so

$$|\varphi_w(\varphi(z_w))|^2 \leq 1-k/2.$$

Let $r=\sqrt{1-k/2}<1$, and $\epsilon = k/2$. Noting $\rho(\varphi(z_w),w)=|\varphi_w(\varphi(z_w))|$, we conclude that

$$\rho(\varphi(z_w),w)<r, \text{ and}$$

$$|\tau_\varphi(z_w)| \geq \epsilon.$$

This completes the proof. □

3. Sufficient conditions

In this section we will obtain a sufficient condition for C_φ to be bounded below on the Bloch space. First we show that for a Bloch function f, $(1-|z|^2)|f'(z)|$ is Lipschitz.

Theorem 1. *Let f be in the Bloch space. Then*

$$||(1-|z|^2)|f'(z)|-(1-|w|^2)|f'(w)|| \leq 3.31\rho(z,w)\|f\|_B,$$

for $z,\ w \in D$.

Proof. For $z,\ w$ in D, let $\lambda=\varphi_w(z)$. Then

$$(1-|w|^2)f'(w)=(f\circ\varphi_w)'(0),$$

and

$$(1-|z|^2)|(f\circ\varphi_w)'(z)|=(1-|\lambda|^2)|f'(\lambda)|.$$

Letting $g=f\circ\varphi_w$, we can rewrite the above equations as

$$g'(0)=(1-|w|^2)f'(w),$$

and

$$(1-|\lambda|^2)|g'(\lambda)|=(1-|z|^2)|(f\circ\varphi_w)'(z)|.$$

Thus

$$||(1-|z|^2)|f'(z)|-(1-|w|^2)|f'(w)||$$

$$=|(1-|\lambda|^2)|g'(\lambda)|-|g'(0)||$$

$$=|\lambda|^2|g'(0)|+(1-|\lambda|^2)|g'(\lambda)-g'(0)|.$$

Clearly by the definition of $\|g\|_B$ we have

$$|g'(0)| \leq \|g\|_B=\|f\circ\varphi_w\|_B=\|f\|_B,$$

where the last equation comes from the fact that the Bloch norm is the Möbius invariant. Now we turn to estimate $|g'(\lambda)-g'(0)|$. To do this, let $\lambda \in D$.

Note, for each $z \in D$,

$$(1-|z|^2)g''(z) = (g' \circ \varphi_z)'(0)$$

$$= \frac{1}{2\pi r}\int_0^{2\pi} g' \circ \varphi_z(re^{i\theta})e^{-i\theta}d\theta,$$

for all $0 < r < 1$. Then for each fixed $0 < r < 1$, we have

$$|\int_0^{2\pi} g' \circ \varphi_z(re^{i\theta})e^{-i\theta}d\theta|$$

$$\leq \int_0^{2\pi} \frac{1-|\varphi_z(re^{i\theta})|^2}{1-|\varphi_z(re^{i\theta})|^2}|g' \circ \varphi_z(re^{i\theta})|d\theta$$

$$\leq \|g\|_B \int_0^{2\pi} \frac{1}{1-|\varphi_z(re^{i\theta})|^2}d\theta.$$

On the other hand, we also have

$$\frac{1}{1-|\varphi_z(re^{i\theta})|^2} = \frac{|1-re^{i\theta}\overline{z}|^2}{(1-|z|^2)(1-r^2)}.$$

Thus we obtain

$$(1-|z|^2)^2|g''(z)| \leq \|g\|_B \frac{1+r^2|z|^2}{r(1-r^2)},$$

for $0 < r < 1$.

Now we consider two cases.

Case 1. If $|\lambda| \leq 1/2$, letting $h(r) = \frac{1+r^2/4}{r(1-r^2)}$, then the minimal value of $h(r)$ for $0 < r < 1$ approximately equals 2.81. Hence for $|z| \leq 1/2$, we have

$$(1-|z|^2)^2|g''(z)| \leq 2.81\|g\|_B.$$

Also, $|g'(\lambda) - g'(0)| \leq \int_0^1 |g''(t\lambda)|\,|\lambda|dt$. Hence

$$|g'(\lambda) - g'(0)| \leq 2.81\|g\|_B \int_0^1 \frac{|\lambda|dt}{(1-t^2|\lambda|^2)^2} = \int_0^{|\lambda|} \frac{ds}{(1-s^2)^2}.$$

An easy calculation gives

$$\int_0^{|\lambda|} \frac{ds}{(1-s^2)^2} = \frac{1}{4}[\frac{2|\lambda|}{1-|\lambda|^2} + \ln\frac{1+|\lambda|}{1-|\lambda|}].$$

So we get

$$(1-|\lambda|^2)|g'(\lambda) - g'(0)| \leq 2.81\|g\|_B|\lambda| = 2.81\|f\|_B|\lambda|.$$

Thus if $|\lambda| \leq 1/2$, we have

$$|(1-|\lambda|^2)|g'(\lambda)| - |g'(0)|| \leq |\lambda|^2\|f\|_B + 2.81|\lambda|\|f\|_B \leq 3.31|\lambda|\|f\|_B.$$

Case 2. If $|\lambda| > 1/2$, then $2|\lambda| > 1$. In this case,

$$|(1-|\lambda|^2)|g'(\lambda)| - |g'(0)||$$

$$\leq \max\{(1-|\lambda|^2)|g'(\lambda)|, |g'(0)|\} \leq \|g\|_B = \|f\|_B \leq 2|\lambda|\|f\|_B.$$

Combining the above two cases we have

$$|(1-|\lambda|^2)|g'(\lambda)| - |g'(0)|| \leq 3.31|\lambda|\|f\|_B,$$

for $|\lambda| < 1$. Hence we conclude that

$$|(1-|z|^2)|f'(z)| - (1-|w|^2)|f'(w)|| = |(1-|\lambda|^2)|g'(\lambda)| - |g'(0)||$$

$$\leq 3.31|\lambda|\|f\|_B = 3.31|\rho(z,w)\|f\|_B,$$

noting that $\rho(z,w) = |\lambda|$. This completes the proof. □

Theorem 2. *If for some constants* $0 < r < 1/4$, *and* $\epsilon > 0$, *for each* $w \in D$, *there is a point* $z_w \in D$ *such that*

$$\rho(\varphi(z_w), w) < r \textit{ and } |\tau_\varphi(z_w)| > \epsilon,$$

then $C_\varphi : B \to B$ *is bounded below.*

Proof. Let $\lambda = \varphi(0)$. Then $\varphi = \varphi_\lambda \circ \varphi_\lambda \circ \varphi$. Let $\psi = \varphi_\lambda \circ \varphi$. Thus $\psi(0) = 0$, and $C_\varphi = C_\psi C_{\varphi_\lambda}$. Since φ_λ is a Möbius transform, C_{φ_λ} is an isometry on B. So we need only to prove that C_ψ is bounded below on the Bloch space. Moreover ψ still satisfies the conditions of the theorem.

In order to prove that C_ψ is bounded below on the Bloch space it suffices to prove

$$\|C_\psi f\|_\beta \geq k$$

for some constant $k > 0$ and all $f \in B$ with $\|f\|_\beta = 1$. To do this, let f be a function in the Bloch space with norm 1. For each $z \in D$, we have

$$(1-|z|^2)|(C_\psi f)'(z)|$$

$$= (1-|\psi(z)|^2)|f'(\psi(z))|\tau_\psi(z)|.$$

Since $\|f\|_\beta = 1$, there is a point $w \in D$ such that

$$(1-|w|^2)|f'(w)| \geq (1-(1/4-r)/2)(1-|f(0)|).$$

By Theorem 1, we have

$$|(1-|w|^2)|f'(w)| - (1-|z|^2)|f'(z)|| \leq 4\rho(z,w)(1-|f(0)|).$$

Thus whenever $\rho(\psi(z_w), w) < r < 1/4$, we have that

$$(1-|\psi(z_w)|^2)|f'(\psi(z_w))| \geq (1-|w|^2)|f'(w)| - 4r(1-|f(0)|)$$

$$\geq [(1-(1/4-r)/2) - 4r](1-|f(0)|) = [7(1-4r)/8](1-|f(0)|).$$

So

$$\|C_\psi f\|_\beta \geq |f(\psi(0))| + (1-|z|^2)|(C_\psi f)'(z)|$$

$$\geq |f(0)| + (1-|\psi(z)|^2)|f'(\psi(z))||\tau_\psi(z)|,$$

for all $z \in D$. In particular,

$$\|C_\psi f\|_\beta \geq |f(0)| + (1-|\psi(z_w)|^2)|f'(\psi(z_w))||\tau_\psi(z_w)|$$

$$\geq |f(0)| + [7(1-4r)\epsilon/8](1-|f(0)|) \geq [7(1-4r)\epsilon/8].$$

Let $k = [7(1-4r)\epsilon/8]$. We have proved that

$$\|C_\psi f\|_\beta \geq k$$

whenever $\|f\|_\beta = 1$. This completes the proof. □

Addendum

As an expected corollary to Theorem 1 we get a converse to the Rochberg-Semmes result on interpolation by derivatives of Bloch functions. An easy exposition of it is found in Proposition 1 of [2] along with more references to similar work.

Corollary 1. *If a sequence $\{z_n\} \subseteq D$ satisfies the property that the map $S : B \to \ell^\infty$ defined by $S(f) = \{(1-|z_n|^2)f'(z_n)\}$ is onto, then $\{z_n\}$ is separated in the pseudo-hyperbolic metric.*

Proof. For a fixed n choose $f_n \epsilon B$ with $\|f_n\|_B \leq c$, $(1-|z_n|^2)f_n'(z_n) = 1$ and $(1-|z_m|^2)f_n'(z_m) = 0$ whenever $m \neq n$. Then by Theorem 1, $1 \leq 4c\rho(z_n, z_m) \forall n \neq m$. □

We are grateful to Richard Rochberg for pointing out the connection.

Final remark

The preliminary investigations in this paper have led to stronger results including a set of necessary and sufficient conditions and their geometric interpretation. They will appear elsewhere.

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Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115
E-mail address: `ghatage@math.csuohio.edu`

Department of Mathematics, Chongqing University, Chongqing, People's Republic of China

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240
E-mail address: `zheng@math.vanderbilt.edu`