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GENERIC AUTOMORPHISMS OF THE UNIVERSAL PARTIAL ORDER

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Abstract. We show that the countable universal-homogeneous partial order $(P,<)$ has a generic automorphism as defined by the second author, namely that it lies in a comeagre conjugacy class of $\mathrm{Aut}(P,<)$. For this purpose, we work with 'determined' partial finite automorphisms that need not be automorphisms of finite substructures (as in the proofs of similar results for other countable homogeneous structures) but are nevertheless sufficient to characterize the isomorphism type of the union of their orbits.

1. Introduction

The definition of *generic* given in [11] as applied to automorphisms g of a countable first order structure was that g should lie in a comeagre conjugacy class (where the automorphism group of the structure is endowed with the natural topology). A sufficient condition for the existence of generics is that the family $\mathcal{P}$ of finite partial automorphisms of the structure should have the amalgamation property. This property is however false in general, and a weaker condition, that $\mathcal{P}$ should have a cofinal subset closed under conjugacy with the amalgamation property, is also sufficient, and does hold in many cases. Typically we may take the cofinal subset to consist of all partial automorphisms which are automorphisms of finite substructures, and this condition is verified for the structures consisting of a pure (countably infinite) set, and the random graph. This latter case was extended by Hrushovski [7] to mutual generics for the random graph (under the obvious definition of what this should mean) via his 'graph extension lemma', and a similar property of many other structures has been studied by Herwig and Lascar [2, 3, 4].

The purpose of this paper is to show that the sufficient condition can (sometimes) be verified, even without using automorphisms of finite substructures. The two main examples we have in mind are $\mathrm{Aut}(\mathbb{Q},<)$ (where we already knew from [11] that there are generics), and the automorphism group of the countable universal-homogeneous partial ordering $(P,<)$ (where we did not). This latter structure has been considered by Schmerl (in the context of his classification of all the countable homogeneous partial orders [10]) and Glass, McCleary and Rubin [1], in studying

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its automorphism group (principally the verification of its simplicity). The point about these two cases is that we cannot possibly expect to use automorphisms of finite substructures, because all but trivial finite partial automorphisms must have distinct domain and range. For, once an element is moved strictly upwards, or downwards, it must lie in an infinite orbit, so cannot be encompassed by an automorphism of a finite substructure.

Now it is quite easy to show that there are generics in $\mathrm{Aut}(\mathbb{Q},<)$ by giving an explicit description, and this was done for instance in [11]. In the remainder of the introduction we show that this may also be proved indirectly, by showing that there is a cofinal subset of $\mathcal{P}$ (closed under conjugacy) having the amalgamation property. This subset is quite easy to describe, so should serve as a warm-up for the main proof, which does the same job for $(P,<)$.

We call an isomorphism between substructures of a relational structure a *partial automorphism*. If p and q are partial (possibly total) automorphisms, then q is an *extension* of p if for any $a \in \mathrm{dom}\, p$, aq is defined, and equals ap (where actions are written on the right). We write $\mathcal{P}$ for the family of all finite partial automorphisms of the structure, and say that $\mathcal{A} \subseteq \mathcal{P}$ is *cofinal* if any member of $\mathcal{P}$ has an extension in $\mathcal{A}$. If p is a partial automorphism, a *partial orbit* of p is a non-empty set of the form $\{ap^n : n \in \mathbb{Z}, ap^n \text{ defined}\}$ for fixed a.

If p is a partial automorphism of a linearly ordered set $(A,<)$, we say that $a \in A$ has *parity* $+1$ if $a \in \mathrm{dom}\, p \wedge a < ap$ or $a \in \mathrm{range}\, p \wedge ap^{-1} < a$, *parity* 0 if $a \in \mathrm{dom}\, p \wedge ap = a$, and *parity* -1 if $a \in \mathrm{dom}\, p \wedge ap < a$ or $a \in \mathrm{range}\, p \wedge a < ap^{-1}$. This is then well-defined on $\mathrm{dom}\, p \cup \mathrm{range}\, p$.

Theorem 1.1. *There is a cofinal subset of the family of all finite partial automorphisms of* $(\mathbb{Q},<)$ *having the amalgamation property.*

Proof. We let $\mathcal{A}$ comprise all pairs (A,p) such that A is a finite linearly ordered set, p is a partial automorphism of A, and $A = \mathrm{dom}\, p \cup \mathrm{range}\, p$, and such that, if $a < b$ in A have equal parities, and a is maximal in its p-orbit, and b is minimal in its p-orbit, then there is c having different parity and with $a < c < b$. The idea of this condition is that it should be strong enough to stop distinct partial orbits of p 'joining up', that is, for any extension f of p, a and b should lie in distinct f-orbits. The set asserted to exist in the theorem is then the set $\mathcal{A}_1$ of all $(A,p) \in \mathcal{A}$ such that A is a substructure of $(\mathbb{Q},<)$, but it is slightly easier to consider $\mathcal{A}$, since when amalgamating, we can make a free choice of how to relate elements in non-overlapping parts of the two structures being amalgamated, rather than taking isomorphic copies. To see that $\mathcal{A}_1$ is a cofinal subset of $\mathcal{P}$, let $(A,p) \in \mathcal{P}$ be given. Let $\sim$ be the least equivalence relation on A such that $a \sim b$ whenever $a \le b \le ap$ or $a \ge b \ge ap$. Then all $\sim$-classes are intervals of elements all of which have the same parity. Hence we may insert, between any consecutive such intervals of equal parity, one of some different parity.

To verify amalgamation for $\mathcal{A}$, let (A,p), (A_1,p_1), and (A_2,p_2) be such that (A,p) is a restriction of both p_1 and p_2, and assume that $A_1 \cap A_2 = A$. We use induction on $|A_1 - A| + |A_2 - A|$.

Case 1: There is $a \in A$ such that ap is undefined, but ap_1 and ap_2 are both defined. As $a \in \mathrm{dom}\, p \cup \mathrm{range}\, p$, ap^{-1} *is* defined. If $ap^{-1} = a$, then ap is also defined, contrary to assumption. Without loss of generality assume that $ap^{-1} < a$. As p_1, p_2 extend p and are order-preserving, $a < ap_1, ap_2$.

We show that ap_1 and ap_2 lie in the same interval determined by the points of A. First note that $ap_1 \notin A$, and similarly $ap_2 \notin A$. For if $ap_1 \in A$, as $a \notin \operatorname{dom} p$, and p_1 extends p, $ap_1 \notin \operatorname{range} p$. Also, ap_1 has parity $+1$ in (A_1, p_1), hence also in (A, p), and as a, ap_1 are maximal and minimal respectively in their p-orbits, there is a point $b \in A_1$ with $a < b < ap_1$ having different parity in (A_1, p_1). Hence either $bp_1 \le b$ or $b \le bp_1^{-1}$. But if the former applies (that is, $b \in \operatorname{dom} p_1$), $bp_1 \le b < ap_1$, contrary to p_1 order-preserving; and in the latter case we have $bp_1^{-1} < a < b$, which is also impossible.

Now suppose that $ap_1 < a' < ap_2$ where $a' \in A$. If $a' \in \operatorname{range} p$, then $(a')p^{-1} = (a')p_2^{-1} < a < (a')p_1^{-1} = (a')p^{-1}$, which is a contradiction. Hence $a' \in \operatorname{dom} p$. As $a < a'$, $a' < ap_2 < (a')p_2 = a'p$, so a' has parity $+1$, and is the minimal point of its p-orbit. Let $a < b < a'$, where b has different parity. Then $b < a' < ap_2 < bp_2$, contradiction. Similarly we cannot have $ap_2 < a' < ap_1$.

We now let $A' = A \cup \{ap_1\}$ and $p' = p \cup \{(a, ap_1)\}$, where the position of ap_1 in relation to the members of A may be determined either in A_1, or with ap_2 in place of ap_1 in A_2 (which was the point of what we have just shown), and let A_2' and p_2' be obtained from A_2 and p_2 by replacing ap_2 by ap_1. Now (A', p') is a substructure of both (A_1, p_1) and (A_2', p_2'), and $|A_1 - A'| + |A_2' - A'| < |A_1 - A| + |A_2 - A|$, so by the induction hypothesis, (A_1, p_1) and (A_2', p_2') can be amalgamated over (A', p'), and this gives rise to an amalgamation of (A_1, p_1) and (A_2, p_2) over (A, p).

Case 2: There is $a \in A$ such that ap^{-1} is undefined, but ap_1^{-1} and ap_2^{-1} are both defined. This is similar to Case 1.

Case 3: There is $a \in A$ such that one of ap_1 and ap_2 is defined, but not both; suppose the former. Then also ap is not defined, so ap^{-1} is defined and $ap^{-1} \ne a$. Assume $ap^{-1} < a$, so that a has parity $+1$ (in all three structures). Choose $a' \notin A_2$ and let $A_2' = A_2 \cup \{a'\}$, $p_2' = p_2 \cup \{(a, a')\}$ where a' is an immediate successor of $\max(\{a\} \cup \{a''p_2 : a'' < a, a'' \in \operatorname{dom} p_2\})$ in the ordering on A_2'. In other words, we insert a' in the first available interval of A_2. Then $(A_2', p_2') \in \mathcal{A}$. Now we proceed as in Case 1 applied to (A_1, p_1) and (A_2', p_2'). Afterwards, $|A_1 - A| + |A_2 - A|$ has decreased by 1 ($|A_1 - A|$ has gone down by 1, $|A_2 - A|$ has stayed the same), so we may appeal to the induction hypothesis.

Case 4: There is $a \in A$ such that one of ap_1^{-1} and ap_2^{-1} is defined, but not both. This is similar to Case 3.

Case 5: For all $a \in A$, ap undefined implies that ap_1, ap_2 are both undefined, and ap^{-1} undefined implies that ap_1^{-1}, ap_2^{-1} are both undefined. So here all the partial orbits of p_1 and p_2 are either partial orbits of p, or disjoint from $\operatorname{dom} p \cup \operatorname{range} p = A_1 \cap A_2$. On each interval determined by the points of A we insert the new points of this interval of A_1 to the left of all the new points of this interval of A_2. This results in a partial automorphism that may or may not lie in $\mathcal{A}$. Since $\mathcal{A}$ is cofinal, it can however be extended to a member of $\mathcal{A}$. $\square$

2. The countable universal partial ordering has a generic automorphism

Let $(P, <)$ be the countable universal-homogeneous partial order, G its automorphism group, and $\mathcal{P}$ the family of all finite partial automorphisms of P. We seek a cofinal subset $\mathcal{A}$ of $\mathcal{P}$ having the amalgamation property. The idea is to take $\mathcal{A}$ to be the set of all 'determined' members of $\mathcal{P}$ in the sense we now describe.

The following definition applies more generally (in particular, to the situation discussed in section 1). If R is any relational structure, a *strict extension* of a finite partial automorphism p of R is the restriction of some automorphism of R extending p to the union of its orbits intersecting $\operatorname{dom} p \cup \operatorname{range} p$. Two strict extensions f_1 and f_2 of p are *isomorphic over* p if there is an isomorphism $\theta : \operatorname{dom} f_1 \to \operatorname{dom} f_2$ fixing $\operatorname{dom} p$ pointwise and such that $\theta f_1 = f_2 \theta$ (that is, θ carries the action of f_1 on $\operatorname{dom} f_1$ to the action of f_2 on $\operatorname{dom} f_2$). We say that p is *determined* if any two strict extensions of p are isomorphic over p. We also say that a partial orbit X of $p \in \mathcal{P}$ is *determined* (by p) if for any extensions f_1, f_2 of p to automorphisms with orbits X_1, X_2 extending X, the actions of f_1 on X_1 and f_2 on X_2 are isomorphic over p. Similarly (and this is the important case) we may talk of a pair of partial orbits as being determined.

For $(\mathbb{Q}, <)$, a partial orbit is determined provided it intersects the domain of the function, since on knowing just one value we can tell the parity, which determines the isomorphism type. For a pair of partial orbits to be determined, what is essentially required is that if the maximum of one is less than the minimum of the other, then either they have different parities, or there is another partial orbit in between having different parity, and this was the condition used in defining the family $\mathcal{A}$. These features again appear for $(P, <)$, but in a rather more involved fashion, and there are more cases, even for single orbits.

An orbit X of $f \in G$ may 'spiral' or be an antichain (which we may view as an infinite spiral). If $x \in X$, we let $\operatorname{sp}(x, f) = n > 0$ be least such that x and xf^n are comparable (if any), and if there is no such n, we write $\operatorname{sp}(x, f) = \infty$. If $n = \infty$, or n is finite and $x = xf^n$, we say that X is an orbit of *parity* 0 (which is a *cycle* if n is finite). If $x < xf^n$ or $xf^n < x$, we call $\operatorname{sp}(x, f)$ the 'spiral length' of x in f, and say that X is a *positive* or *negative* spiral respectively. If $\operatorname{sp}(x, f) = \infty$, then X is an infinite antichain, and all xf^i are distinct and incomparable. It is clear that $\operatorname{sp}(x, f)$ and the parity (positive, negative, or zero) of x are independent of the choice of x from X. Usually (and without loss of generality) we may restrict attention to non-negative parities (though it is clear that a generic automorphism must have orbits of all possible kinds).

If the orbit X of f containing x is a positive spiral of spiral length n, we may let a_i for $i \geq 1$ be defined by

$$a_i = \begin{cases} 1 & \text{if } x < xf^i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $0 \leq a_i \leq a_{i+n} \leq 1$ and so $(a_i)_{i \geq 1}$ is eventually periodic with period dividing n. We let $w(x, f)$ be the least $m \geq 1$ such that $(a_i)_{i \geq m}$ is periodic. More generally, for another element y we may define b_i for $i \geq 1$ by

$$b_i = \begin{cases} 1 & \text{if } y \leq xf^i, \\ 0 & \text{otherwise.} \end{cases}$$

Since X is a positive spiral, the same argument shows that $(b_i)_{i \geq 1}$ is eventually periodic, and we let $w(y, x, f)$ be the least $m \geq 1$ such that $(b_i)_{i \geq m}$ is periodic. (Thus $w(x, f) = w(x, x, f)$.) Further generalizing, we may define b_i by the same formula, but now for all $i \in \mathbb{Z}$. The same proof shows that $(b_i)_{i \geq 1}$ and $(b_i)_{i \leq 0}$ are both eventually periodic, and we let $w^+(y, x, f) = w(y, x, f)$ be the least $m \geq 1$ such that $(b_i)_{i \geq m}$ is periodic, and $w^-(y, x, f)$ be the greatest $m \leq 0$ such that $(b_i)_{i \leq m}$ is periodic.

The natural partial ordering to consider on $\mathcal{P}$ is just extension. Quite often we want to consider more restricted extensions. We say that q in $\mathcal{P}$ is an *economical* extension of p if it is an extension, and every partial orbit of q contains a partial orbit of p. We sometimes use the terminology of (weak) forcing in set theory to help express what we want (after all, we are talking about *generics*), and we may say that $p \in \mathcal{P}$ *forces* some statement, if it holds no matter which extension in $\mathrm{Aut}(P,<)$ we take.

Lemma 2.1. *Any $p \in \mathcal{P}$ has an economical extension all of whose partial orbits are determined.*

Proof. For this it suffices to extend (economically) to q so that a given partial orbit X is determined, since we may then repeat. Note that it is important that the extensions are economical, as otherwise, as we extend, we could introduce more and more new partial orbits, and this process might never terminate.

Let $x \in X$ and choose an extension f of p in G for which $\mathrm{sp}(x,f)$ is minimal, and, subject to that, in which $|\{i : w(x,f) \le i < w(x,f)+\mathrm{sp}(x,f), x \text{ is comparable with } xf^i\}|$ is maximal, and, subject to that, in which $w(x,f)$ is minimal. Choose a finite restriction q of f extending p so that

$\mathrm{dom}\, q \subseteq (\mathrm{dom}\, p)f^{\mathbb{Z}} = \bigcup_{n\in\mathbb{Z}}(\mathrm{dom}\, p)f^n$,

$y \le z \le yq^n \wedge y, yq^n \in \mathrm{dom}\, q \to z \in \mathrm{dom}\, q$, $(n \in \mathbb{Z})$,

$\mathrm{sp}(x,f)$ finite $\to xq^i$ defined for $0 \le i < \mathrm{sp}(x,f) + w(x,f)$,

q contains all finite cycles of f intersecting its domain.

Suppose then that f_1 and f_2 are extensions of q in G. If $\mathrm{sp}(x,f_1)$ is finite, then by minimality of $\mathrm{sp}(x,f)$, it is finite too. Hence $xq^{\mathrm{sp}(x,f)}$ is defined, so the spiral length of x under q is defined, and equals that for f, f_1, and f_2. Assume the spiral is positive.

If $w(x,f) \le i < w(x,f) + \mathrm{sp}(x,q)$, then xq^i is defined , so $x < xf^i \Leftrightarrow x < xf_1^i$. Therefore

$$\begin{aligned}&|\{i : w(x,f) \le i < w(x,f) + \mathrm{sp}(x,q), x < xf^i\}|\\ &\qquad = |\{i : w(x,f) \le i < w(x,f) + \mathrm{sp}(x,q), x < xf_1^i\}|.\end{aligned}$$

By maximality of $|\{i : w(x,f) \le i < w(x,f) + \mathrm{sp}(x,f), x < xf^i\}|$, and since $|\{i : m \le i < m + \mathrm{sp}(x,q), x < xf_1^i\}|$ is nondecreasing as m increases, and achieves its maximum value at $m = w(x,f_1)$, we deduce that $w(x,f_1) \le w(x,f)$, and that $|\{i : w(x,f_1) \le i < w(x,f_1) + \mathrm{sp}(x,f_1), x < xf_1^i\}|$ is also maximal. By minimality of $w(x,f)$, $w(x,f) \le w(x,f_1)$, and so the two are actually equal. This determines $(Xf^{\mathbb{Z}}, f)$ up to isomorphism.

A similar argument applies if $\mathrm{sp}(x,f_2)$ is finite.

If $\mathrm{sp}(x,f_1)$ and $\mathrm{sp}(x,f_2)$ are both infinite, then the orbits of f_1 and f_2 containing x are both antichains, so the actions of f_1 and f_2 on these orbits are isomorphic in this case too. □

It may be worth remarking (in case it seems unnecessary to consider such elements) that it is possible for p to force there to be an orbit which is an infinite antichain. For suppose that $p = \{(x, xp), (z, zp)\}$ where $x, y < xp$, $y, zp < z$, and all other pairs are incomparable. See Figure 1. Then by universal-homogeneity, these elements may be taken in P, and p lies in $\mathcal{P}$. Suppose that $f \in \mathrm{Aut}(P,<)$ extends p. Then y is incomparable with yf^n for every $n \ge 1$ (and hence $\{yf^n : n \in \mathbb{Z}\}$ is an infinite antichain). For if $y \le yf^n$, then $y \le yf^n \le zf^n \le zf$, contrary to

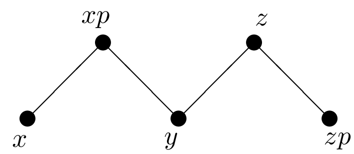

FIGURE 1.

y incomparable with zp, and if $yf^n \le y$, then $yf^n \le xf$, so $yf^{n-1} \le x \le xf^{n-1}$ giving $y \le x$, contradiction.

We shall modify this example below to show why certain configurations of pairs of orbits have to be considered.

Lemma 2.2. *Any $p \in \mathcal{P}$ has an extension which is determined.*

Proof. It suffices to determine all pairs of orbits, essentially because the language of the structure $(P, <)$ is binary. In other words, to tell whether two possible extensions of $p \in \mathcal{P}$ are isomorphic over p, we only need to test two elements at a time, and hence look at two orbits. By Lemma 2.1 we may suppose that all (individual) orbits are determined.

Suppose that X and Y are partial orbits of p, $x \in X, y \in Y$, and let $X(f)$ and $Y(f)$ stand for the corresponding orbits of an extension f of p in $\mathrm{Aut}(P, <)$. Let $A(f) = \{n \in \mathbb{Z} : x \le yf^n\}$ and $B(f) = \{n \in \mathbb{Z} : x \ge yf^n\}$. To specify the isomorphism type of f on $X(f) \cup Y(f)$ it suffices to determine $A(f)$ and $B(f)$.

Case 1: p forces x or y to lie in a spiral, x say. Suppose that the spiral is positive.

Choose an extension f of p in $\mathrm{Aut}(P, <)$ for which

$$|\{i : w^+(y, x, f) \le i < w^+(y, x, f) + \mathrm{sp}(x, f), y \le xf^i\}|$$

is maximal, and let q_1 be a finite restriction of f which is an economical extension of p and such that xq_1^i is defined for $0 \le i < w^+(y, x, f) + \mathrm{sp}(x, f)$. Then q_1 determines the set $\{i \ge 0 : y \le xf^i\}$. Similarly (using $w^-(y, x, f)$ in place of $w^+(y, x, f)$) there is an economical extension q_2 of q_1 which determines $\{i < 0 : y \le xf^i\}$. Thus q_2 determines $A(f)$. Similarly, q_2 has an extension q which also determines $B(f)$.

We remark that if p has an extension f in $\mathrm{Aut}(P, <)$ for which $A(f)$ and $B(f)$ are both non-empty, then $x < yf^i < xf^j$ for some i and j in $\mathbb{Z}$. Hence x lies in a spiral of f, and as p determines all its orbits, this is already forced by p, so that Case 1 applies.

Case 2: For every extension f of p, $A(f) = B(f) = \emptyset$.

Then p already determines the isomorphism type of the pair $(X(f), Y(f))$.

Case 3: Cases 1 and 2 are false. As Case 2 does not hold, there is an extension f of p in $\mathrm{Aut}(P, <)$ such that $A(f) \ne \emptyset$ or $B(f) \ne \emptyset$; suppose the former without loss of generality. By extending p (economically), we assume that it forces this. As Case 1 is false, p also forces $B(f) = \emptyset$. If there is no extension f of p for which $|A(f)| > 1$, then p forces $|A(f)| = 1$, and so determines the isomorphism type of the pair of orbits. So by further economically extending p we suppose that it forces $x < yf^i$ for at least two values of i. (In addition, p forces both x and y to lie in infinite antichains of f, though we do not actually need this fact explicitly.)

Our object now is to extend p to q so as to determine completely the behaviour of the set $A(f)$. In fact we can show that $A(f)$ can be forced to be eventually

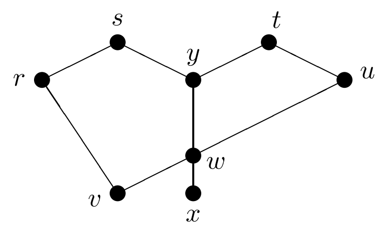

FIGURE 2.

periodic on the left, and on the right, which will suffice. More precisely, we show that there are q extending p, and positive integers N_1, N_2, n_1, and n_2, such that for any two extensions f_1 and f_2 of q in $\mathrm{Aut}(P,<)$, if $i, j \geq N_1$ with $i \equiv j \bmod n_1$, then $i \in A(f_1) \Leftrightarrow i \in A(f_2) \Leftrightarrow j \in A(f_2) \Leftrightarrow j \in A(f_1)$, and if $i, j \leq -N_2$ with $i \equiv j \bmod n_2$, then $i \in A(f_1) \Leftrightarrow i \in A(f_2) \Leftrightarrow j \in A(f_2) \Leftrightarrow j \in A(f_1)$. This clearly suffices. Moreover, it is enough to consider the behaviour on the right, since the argument about the behaviour on the left will be essentially the same.

Before going on, let us show, by means of an example, why we need to consider the behaviour 'on the left' and 'right' separately. Let $A^+(f) = A(f) \cap \mathbb{N}$ and $A^-(f) = A(f) - \mathbb{N}$. Consider the partial ordering illustrated in Figure 2, in which r is incomparable with each of t, u, w, x, y; s is incomparable with each of t and u; and y and u are incomparable, and so are v and x. By universal-homogeneity, this may be taken as a substructure of P, and $p = \{(r,s),(t,u),(v,w)\}$ lies in $\mathcal{P}$. Suppose that f is an automorphism extending p.

As in the previous example, $\{yf^n : n \in \mathbb{Z}\}$ is an antichain. If $x \leq yf^{-n}$ for $n > 0$, then $x \leq (yf^{-1})f^{-(n-1)} \leq rf^{-(n-1)} \leq r$, contrary to r incomparable with x. On the other hand, if $n \geq 0$, $x \leq vf \leq vf^{n+1} \leq yf^n$. So $A(f) = \{n \in \mathbb{Z} : x \leq yf^n\} = \mathbb{N}$. The point of this example is that the behaviour of $A(f)$ on left and right can be forced to be different (though each is eventually periodic), so that we must treat the right and left directions separately. We can also easily ensure that x lies in an infinite antichain if so desired, using a similar 'trick' for x instead of y.

If there is a bound on the size of $|A^+(f)|$ for extensions f of p, let n be the maximum value of such $|A^+(f)|$, and extend p to q so that q satisfies $x \leq yq^i$ for each $i \in A^+(f)$ where $A^+(f)$ is some suitable set of this maximal size. Then q determines the value of $A^+(f)$. So we suppose that there is no bound, and show that we can still extend so that the value of $A^+(f)$ is determined (though now it will be infinite (and periodic)).

Let us now relabel x and y if necessary so that for some $n > 0$, $x < y, yp^n$ where $xp^n, yp^n \notin \mathrm{dom}\, p$.

Consider elements $\{a_i : 0 \leq i \leq n\}$, $\{b_i : 0 \leq i \leq n\}$ not lying in $\mathrm{dom}\, p \cup \mathrm{range}\, p$ so that

a_i, a_j are incomparable for $i \neq j$, except that $a_0 < a_n$,

b_i, b_j are incomparable for $i \neq j$, except that $b_n < b_0$,

$a_i < yp^i < b_i$,

if $xp^i < yp^j$, then $xp^i < a_j$,

if $z, zp^r \in \mathrm{dom}\, p \cup \mathrm{range}\, p$ and $z \leq xp^i < yp^j$ with $0 \leq i \leq m$, $0 \leq j, j+r \leq n$, then $zp^r < a_{j+r}$.

The partial ordering on $\operatorname{dom} p \cup \operatorname{range} p \cup \{a_i : 0 \leq i \leq n\} \cup \{b_i : 0 \leq i \leq n\}$ is taken to be the transitive closure of this list (together with that on $\operatorname{dom} p \cup \operatorname{range} p$). The fact that this *is* a partial ordering follows from $yp^i \not\leq xp^j$ for each i, j. So by universal-homogeneity of P, we may suppose that each a_i, b_i lies in P, and that $\operatorname{dom} p \cup \operatorname{range} p \cup \{a_i : 0 \leq i \leq n\} \cup \{b_i : 0 \leq i \leq n\}$ is a substructure of P. The extension q of p is given by

$$q = p \cup \{(a_i, a_{i+1}) : 0 \leq i < n\} \cup \{(b_i, b_{i+1}) : 0 \leq i < n\}.$$

We show that $q \in \mathcal{P}$, that is, that it is a partial automorphism. Suppose that $z < t$ in dom q. It suffices to show that $zq < tq$ for (z, t) lying in the list of pairs generating the partial ordering given above.

(i) $z = a_j$, $t = yp^j$ where $j < n$: then $zq = a_{j+1} < yp^{j+1} = tq$.

(ii) $z = yp^j$, $t = b_j$, is similar.

(iii) $z = xp^i$, $t = a_j$ where $xp^i < yp^j$: since z and t lie in dom p, $i, j < n$, and so $xp^{i+1} < yp^{j+1}$. Hence $zq = xp^{i+1} < a_{j+1} = tq$.

(iv) $zp^{-r}, z \in \operatorname{dom} p \cup \operatorname{range} p$ and $zp^{-r} \leq xp^i < yp^j$ with $0 \leq i \leq m$, $0 \leq j - r, j \leq n$, $t = a_{j+r}$: as $z \in \operatorname{dom} p$, $(zp^{-r})p^{r+1}$ is defined, and as $t \in \operatorname{dom} p$, $j + r < n$. So $zp^{-r} \leq xp^i < yp^j$, and $0 \leq i \leq m$, $0 \leq j, j + r + 1 \leq n$. Hence $zq = (zp^{-r})p^{r+1} < a_{j+r+1} = tq$.

The fact that $zq < tq$ implies $z < t$ follows by a similar argument.

Now let f be any extension of q to an automorphism of $(P, <)$. If $x < yp^i$ where $0 \leq i < n$ and $k \geq 0$, then $x < a_i = a_0 f^i \leq a_0 f^{kn+i}$ (since $a_0 < a_n = a_0 q^n$) $= a_i f^{kn} < (yp^i) f^{kn} = y f^{kn+i}$. Conversely, if $x < y f^{kn+i}$, then $x < (yp^i) f^{kn} < b_i f^{kn} = b_0 f^{kn+i} \leq b_0 f^i$ (since $b_0 > b_n = b_0 q^n$) $= b_i$, so $x < yp^i$.

This shows that q forces $A^+(f)$ to equal $\{kn+i : x < yp^i, k \geq 0\}$, so it determines the value of $A^+(f)$ (which is thus periodic on the right).

Since in this proof (unlike Lemma 2.1) we had to introduce new partial orbits to 'freeze' the desired behaviour of $A(f)$, that is, the extensions were not always 'economical', we have to justify termination of the procedure. The point is that all new partial orbits introduced were spirals (that is having finite spiral length). So we begin by listing all pairs of partial orbits that are determined as infinite antichains, and determine these, which may involve addition of extra spirals. We then have to argue that we can further extend to determine all pairs of partial orbits for which at least one is a spiral, without addition of extra partial orbits. But this requires a fixed finite number of applications of Case 1, which was accomplished entirely using economical extensions. $\square$

Theorem 2.3. *The family of determined partial automorphisms of* $(P, <)$ *has the amalgamation property.*

Proof. As in the proof of Theorem 1.1, we actually work with the family of isomorphic copies of determined partial automorphisms of $(P, <)$, that is, the family $\mathcal{A}$ of all $(A, <, p)$ such that $(A, <)$ is a finite partially ordered set, p is a partial automorphism of $(A, <)$, $A = \operatorname{dom} p \cup \operatorname{range} p$, and for some determined partial automorphism q of $(P, <)$, $(A, <, p) \cong (\operatorname{dom} q \cup \operatorname{range} q, <, q)$, under the induced partial ordering.

Let $(A, <, p), (A_1, <, p_1), (A_2, <, p_2) \in \mathcal{A}$ be such that $(A_1, <, p_1)$ and $(A_2, <, p_2)$ are extensions of $(A, <, p)$. By universal-homogeneity of $(P, <)$ we may suppose that $(A, <), (A_i, <)$ are substructures of $(P, <)$. Let f_i be an automorphism extending p_i, and X_i be the union of the orbits of f_i which intersect A. Since p is determined, there is an isomorphism θ from $f_1|X_1$ to $f_2|X_2$ fixing A pointwise. Thus all points of $A_1 - X_1$ and $A_2 - X_2$ lie in orbits of f_1 and f_2 respectively which are disjoint from all partial orbits of p. By taking copies if necessary (which do not now have to be substructures of $(P,<)$), we assume that $A_2 - X_2$ is disjoint from $A_1 \cup X_1$. In addition, by replacing X_2 by its image under θ^{-1}, we assume that $X_1 = X_2$, and θ is the identity. Then $(A_1 \cup X_1) \cap (A_2 \cup X_1) = X_1$.

Let $<_1$ and $<_2$ be the partial orderings on $A_1 \cup X_1$ and $A_2 \cup X_1$ respectively, and let $\prec$ be the transitive closure of $<_1 \cup <_2$. Then $\prec$ partially orders $A_1 \cup X_1 \cup A_2$ since $<_1$ and $<_2$ agree on the intersection X_1 of $A_1 \cup X_1$ and $A_2 \cup X_1$. For the same reason, $(A_1 \cup X_1, <_1)$ and $(A_2 \cup X_1, <_2)$ are both substructures of $(A_1 \cup X_1 \cup A_2, \prec)$.

Let $g = f_1|(A_1 \cup X_1) \cup f_2|(A_2 \cup X_1)$. We show that g is a partial automorphism. Let $x \prec y$ in $\operatorname{dom} g$, with the object of showing that $xg \prec yg$. If x and y lie in the same one of $A_1 \cup X_1$ and $A_2 \cup X_1$, this is immediate since f_1 and f_2 are automorphisms. Otherwise let $x = x_0 <_{i_0} x_1 <_{i_1} x_2 <_{i_2} \ldots <_{i_{n-1}} x_n = y$ for minimal n (≥ 2). Then the i_j must alternate, and hence for $0 < j < n$, $x_j \in (A_1 \cup X_1) \cap (A_2 \cup X_1) = X_1$. By minimality of n, $n = 2$. Suppose that $i_0 = 1$ and $i_1 = 2$ (the case $i_0 = 2$ and $i_1 = 1$ being similar). Thus $x <_1 x_1 <_2 y$, with $x_1 \in X_1$. Hence $xg = xf_1 <_1 x_1 f_1 = x_1 f_2 <_2 yf_2 = yg$, as required. The proof that $xg \prec yg \Rightarrow x \prec y$ is similar.

Now $A_1 \cup A_2$, together with the restrictions of $\prec$ and g, forms a common extension $(B, <, q)$ of $(A_1, <, p_1)$ and $(A_2, <, p_2)$. The reason we have to use the larger structure $A_1 \cup X_1 \cup A_2$ as intermediate when defining $\prec$ is to ensure that g is defined on x_1. (If we just worked with $A_1 \cup A_2$, then we could have $x <_1 x_1 <_2 y$ for which $x, y \in \operatorname{dom} g$ but $x_1 \notin \operatorname{dom} g$, and then we could not deduce $xg \prec yg$. This is a similar point to (iv) towards the end of the proof of the previous lemma.) Finally, by Lemma 2.2, $(B, <, q)$ can be extended to a determined partial automorphism. □

Corollary 2.4. *There is a generic automorphism of the countable universal-homogeneous partial ordering.*

Proof. This follows from Theorem 2.1 of [11], since it is clear that the family of determined partial automorphisms of $(P,<)$ is closed under conjugacy, and Lemma 2.2 told us that the family is cofinal. □

3. Further questions

The existence of a generic automorphism may seem rather a technical matter, and certainly one would like to see the result of this paper applied and extended. We have not yet done so, but the principal goals to aim at are clear. First, one should establish the existence of (arbitrarily long) mutually generic sequences of automorphisms, which would be the key step in verifying the small index property for $(P,<)$ (see [5]). For $(\mathbb{Q},<)$, it has been remarked by Hodkinson [6] that there can be no pair of mutually generic automorphisms, but the more complicated structure of $(P,<)$ (in particular, the fact that generics possess orbits which are infinite antichains) suggests that we cannot immediately rule them out in this case. Failing this, one might try to verify the small index property by other methods, or interpret $(P,<)$ in its automorphism group more directly. This would place this work in the context of more general results about such interpretations. See [8, 9] for example.

Another possible use of generics would be to streamline the proof of the simplicity of $\operatorname{Aut}(P,<)$ given in [1] (where 16 conjugates were in general required to express one non-identity element as a product of conjugates of another). The correct minimum number is probably 3 or 4, and one could establish sufficiency of 4 by showing that for any non-identity g_1 and g_2, there is h such that $g_1h^{-1}g_2h$ is generic. Very likely one can take h to be generic 'over' (g_1, g_2) (but to show that this is possible is at least as hard as finding mutually generic pairs).

The most promising aspect of our work is perhaps that it shows that one can sometimes work with partial automorphisms which are not automorphisms of substructures, and it should be possible to use this idea in other contexts.

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