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STAR RING HOMOMORPHISMS BETWEEN COMMUTATIVE BANACH ALGEBRAS

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Abstract. We consider a $*$-ring homomorphism from a commutative Banach algebra with an involution to a commutative Banach algebra with a symmetric involution. We give the Gelfand transform of the $*$-ring homomorphism image.

1. Introduction

Definition 1.1. Let A and B be commutative Banach algebras with a $*$-involution and a $\star$-involution, respectively. We say that $\phi : A \to B$ is a $*$-ring homomorphism if the following equalities hold for every $f, g \in A$:

$$\begin{aligned} &\phi(f+g) = \phi(f) + \phi(g), \\ &\phi(fg) = \phi(f)\,\phi(g), \\ &\phi(f^*) = \phi(f)^\star. \end{aligned}$$

Šemrl [4] proved the following theorem on a structure of a $*$-ring homomorphism between two commutative C^*-algebras.

Theorem ([4]). *Let X and Y be compact Hausdorff spaces, $C(X)$ and $C(Y)$ the Banach algebras of all complex-valued continuous functions on X and Y, respectively. If $\phi : C(X) \to C(Y)$ is a $*$-ring homomorphism, then there exist clopen decomposition $\{Y_{-1}, Y_0, Y_1\}$ of Y and a continuous map $\Phi : Y_{-1} \cup Y_1 \to X$ such that the equality*

$$\phi(f)(y) = \begin{cases} \overline{f(\Phi(y))}, & y \in Y_{-1}, \\ 0, & y \in Y_0, \\ f(\Phi(y)), & y \in Y_1, \end{cases}$$

holds for every $f \in C(X)$.

Kaplansky [1] proved that every ring isomorphism between semisimple complex Banach algebras can be decomposed into a linear part, a conjugate-linear part, and a non-continuous part on a finite-dimensional ideal.

We consider a $*$-ring homomorphism from a commutative Banach algebra A with an involution to a non-radical commutative Banach algebra B with a symmetric involution. We prove that the (Jacobson) radical of A is mapped into the (Jacobson)

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radical of B by the $*$-ring homomorphism. If A is non-radical, we show that there exists a decomposition $\{Y_{-1}, Y_0, Y_1\}$ of the maximal ideal space of B such that the Gelfand transform of the $*$-ring homomorphism image of f is 0 on Y_0 and a composition of a non-zero continuous ring homomorphism on $\mathbb{C}$ and the Gelfand transform of f on $Y_{-1} \cup Y_1$ for every $f \in A$.

2. Main results

Let A be a commutative Banach algebra. We say that A is a radical algebra if there is no non-zero complex homomorphism on A. Then we define the radical of A, denoted by $\mathrm{rad}A$, to be A. Unless A is a radical algebra, then we define $\mathrm{rad}A$ to be the intersection of all the maximal regular ideals in A. For convenience, we say that A is non-radical if A is not a radical algebra.

Let B be a commutative Banach algebra with a $\star$-involution. We say that a $\star$-involution is symmetric if the equality

$$\widehat{x^\star} = \overline{\hat{x}}$$

holds for every $x \in B$, where $\hat{\cdot}$ denotes the Gelfand transform, and $\bar{\cdot}$ denotes the complex conjugate.

Theorem 2.1. *Let A be a commutative Banach algebra with a $*$-involution, B a non-radical commutative Banach algebra with a symmetric $\star$-involution, and M_A and M_B the maximal ideal spaces of A and B, respectively. If $\phi : A \to B$ is a $*$-ring homomorphism, then $\phi(\mathrm{rad}A) \subset \mathrm{rad}B$ holds. Therefore*

$$\widehat{\phi(f)} = 0 \quad (f \in A)$$

holds if A is a radical algebra. If A is non-radical, there exist a decomposition $\{Y_{-1}, Y_0, Y_1\}$ of M_B and a continuous map $\Phi : Y_{-1} \cup Y_1 \to M_A$ such that the equality

$$\phi(f)\hat{}(\varphi) = \begin{cases} \overline{\hat{f}(\Phi(\varphi))}, & \varphi \in Y_{-1}, \\ 0, & \varphi \in Y_0, \\ \hat{f}(\Phi(\varphi)), & \varphi \in Y_1, \end{cases}$$

holds for every $f \in A$. Then Y_{-1} and Y_1 are open subsets in M_B. In particular, if A has a unit element, then Y_{-1}, Y_0 and Y_1 are clopen, and Y_{-1} and Y_1 are compact subsets in M_B.

Before we turn to the proof, we show the following lemma.

Lemma 2.2. *Let $\phi : A \to \mathbb{C}$ be a $*$-ring homomorphism on a commutative Banach algebra A with a $*$-involution. Then*

$$\phi = 0 \text{ or } \phi \in M_A \quad \text{or } \overline{\phi} \in M_A$$

holds. Therefore $\phi = 0$ if A is a radical algebra.

Proof. First we consider the case where A has a unit element e. If we define $\phi_e : \mathbb{C} \to \mathbb{C}$ to be

$$\phi_e(\lambda) = \phi(\lambda e) \quad (\lambda \in \mathbb{C}),$$

then ϕ_e is a $*$-ring homomorphism. In fact, it is easy to see that ϕ_e is a ring homomorphism. Since $e = e^*$, the equality

$$\begin{aligned} \phi_e(\bar{\lambda}) &= \phi(\bar{\lambda}e) = \phi((\lambda e)^*) \\ &= \overline{\phi(\lambda e)} = \overline{\phi_e(\lambda)} \end{aligned}$$

holds for every $\lambda \in \mathbb{C}$. In particular,

$$\phi_e(t) = \overline{\phi_e(t)} \quad (t \in \mathbb{R}).$$

That is, $\operatorname{Im}\phi_e = 0$ on $\mathbb{R}$, where $\operatorname{Im}\phi_e$ denotes the imaginary part of ϕ_e. Now, Kestelman [2] proved that if a ring homomorphism $\tau : \mathbb{C} \to \mathbb{C}$ is unbounded, then $\tau(\mathbb{R})$ is dense in $\mathbb{C}$. Therefore ϕ_e must be bounded. If $\phi_e \neq 0$, then $\phi_e(\lambda) = \lambda \quad (\lambda \in \mathbb{C})$ or $\phi_e(\lambda) = \bar{\lambda} \quad (\lambda \in \mathbb{C})$, since $\phi_e(r) = r$ for every rational number r, and since ϕ_e is bounded.

(i) In case $\phi_e = 0$, we have

$$\begin{aligned} \phi(f) &= \phi(e)\phi(f) \\ &= \phi_e(1)\phi(f) \\ &= 0 \end{aligned}$$

for every $f \in A$. Therefore, $\phi = 0$ on A.

(ii) In case $\phi_e(\lambda) = \lambda$ for every $\lambda \in \mathbb{C}$, we have, for every $f \in A$ and every $\lambda \in \mathbb{C}$,

$$\begin{aligned} \phi(\lambda f) &= \phi(\lambda e)\phi(f) \\ &= \phi_e(\lambda)\phi(f) \\ &= \lambda\phi(f). \end{aligned}$$

Therefore, $\phi \in M_A$.

(iii) In case $\phi_e(\lambda) = \bar{\lambda}$ for every $\lambda \in \mathbb{C}$, we see that

$$\overline{\phi}(\lambda f) = \lambda\overline{\phi}(f)$$

for every $f \in A$ and every $\lambda \in \mathbb{C}$, in a way similar to the above. Therefore, $\overline{\phi} \in M_A$.

We have proved the case that A is unital.

In case that A does not have a unit element, put $A_e = \{(f, \lambda) : f \in A, \lambda \in \mathbb{C}\}$ the commutative Banach algebra adjoining a unit element to A. Moreover we can extend the $*$-involution to A_e :

$$(f, \lambda)^* = (f^*, \bar{\lambda}) \quad (f, \lambda) \in A_e.$$

Suppose $\phi \neq 0$. That is, there exists a $g_0 \in A$ such that $\phi(g_0) \neq 0$. Put $\tilde{\phi} : A_e \to \mathbb{C}$ as follows:

$$\tilde{\phi}(f, \lambda) = \phi(f) + \frac{\phi(\lambda g_0)}{\phi(g_0)} \quad (f, \lambda) \in A_e.$$

It is easy to see that $\tilde{\phi}$ is additive and an extension of ϕ. Also

$$\tilde{\phi}((f, \lambda)(g, \mu)) = \tilde{\phi}(f, \lambda)\,\tilde{\phi}(g, \mu)$$

holds for every $(f, \lambda), (g, \mu) \in A_e$, since the equalities

$$\begin{aligned} \frac{\phi(\lambda\mu g_0)}{\phi(g_0)} &= \frac{\phi(\lambda g_0)}{\phi(g_0)}\,\frac{\phi(\mu g_0)}{\phi(g_0)}, \\ \phi(\lambda f) &= \phi(f)\,\frac{\phi(\lambda g_0)}{\phi(g_0)} \end{aligned}$$

hold for every $f \in A, \lambda, \mu \in \mathbb{C}$. Hence, $\tilde{\phi}$ is multiplicative. Moreover, it is easy to see that

$$\tilde{\phi}((f,\lambda)^*) = \overline{\tilde{\phi}(f,\lambda)}$$

holds for every $(f,\lambda) \in A_e$, since

$$\frac{\phi(\lambda g_0)}{\phi(g_0)} = \frac{\phi(\lambda g_0{}^*)}{\phi(g_0{}^*)} \quad (\lambda \in \mathbb{C}).$$

Therefore, $\tilde{\phi} : A_e \to \mathbb{C}$ is a $*$-ring homomorphism. Then $\tilde{\phi}$ satisfies only one of the following, by the result proved above:

$$\tilde{\phi} = 0 \text{ or } \tilde{\phi} \in M_{A_e} \text{ or } \overline{\tilde{\phi}} \in M_{A_e}.$$

Since $\tilde{\phi}$ is an extension of ϕ, and $\phi \neq 0$, we see that $\phi \in M_A$ or $\overline{\phi} \in M_A$. In particular, $\phi = 0$ if A is a radical algebra. This completes the proof. □

Proof of Theorem 2.1. For every $\varphi \in M_B$, put $\phi_\varphi : A \to \mathbb{C}$ as follows:

$$\phi_\varphi(f) = \phi(f)\hat{}(\varphi) \quad (f \in A).$$

Then we see that ϕ_φ is a $*$-ring homomorphism. Therefore, for every $\varphi \in M_B, \phi_\varphi = 0$ or $\phi_\varphi \in M_A$ or $\overline{\phi_\varphi} \in M_A$, by Lemma 2.2. Therefore, $\phi_\varphi = 0$ for every $\varphi \in M_B$, if A is a radical algebra. By the definition of $\mathrm{rad}A$,

$$\phi(\mathrm{rad}A) = \phi(A) \subset \mathrm{rad}B$$

holds, if A is a radical algebra. If A is non-radical, we define Y_{-1}, Y_0 and Y_1 as follows:

$$\begin{aligned} Y_{-1} &= \{\varphi \in M_B : \overline{\phi_\varphi} \in M_A\}, \\ Y_0 &= \{\varphi \in M_B : \phi_\varphi = 0\}, \\ Y_1 &= \{\varphi \in M_B : \phi_\varphi \in M_A\}. \end{aligned}$$

Then it is easy to see that $\{Y_{-1}, Y_0, Y_1\}$ is a decomposition of M_B, and that Y_{-1} and Y_1 are open subsets in M_B. Finally, we define $\Phi : Y_{-1} \cup Y_1 \to M_A$ as follows:

$$\Phi(\varphi) = \begin{cases} \overline{\phi_\varphi}, & (\varphi \in Y_{-1}), \\ \phi_\varphi, & (\varphi \in Y_1). \end{cases}$$

Then we see that Φ is continuous on $Y_{-1} \cup Y_1$. For every $f \in A$, the equality

$$\phi(f)\hat{}(\varphi) = \begin{cases} \overline{\hat{f}(\Phi(\varphi))}, & (\varphi \in Y_{-1}), \\ 0, & (\varphi \in Y_0), \\ \hat{f}(\Phi(\varphi)), & (\varphi \in Y_1), \end{cases}$$

holds, by the definition of Φ. Therefore,

$$\phi(\mathrm{rad}A) \subset \mathrm{rad}B$$

holds, even if A is non-radical.

In particular, if A has the unit element e, for every $\varphi \in M_B$, we have $\phi(ie)\hat{}(\varphi) = -i$ or $\phi(ie)\hat{}(\varphi) = 0$ or $\phi(ie)\hat{}(\varphi) = i$, by the formula of $\phi(f)\hat{}(\varphi)$ above. Consequently, the equalities

$$\begin{aligned} Y_{-1} &= \{\varphi \in M_B : \phi(ie)\hat{}(\varphi) = -i\}, \\ Y_0 &= \{\varphi \in M_B : \phi(ie)\hat{}(\varphi) = 0\}, \\ Y_1 &= \{\varphi \in M_B : \phi(ie)\hat{}(\varphi) = i\} \end{aligned}$$

hold for the decomposition $\{Y_{-1}, Y_0, Y_1\}$ of M_B. Since $\phi(ie)\hat{}$ is continuous on M_B, Y_{-1}, Y_0 and Y_1 are all clopen subsets in M_B. Hence, Y_{-1}, Y_0 and Y_1 are compact if M_B is compact. If M_B is locally compact, $Y_{-1} \cup Y_1 = \{\varphi \in M_B : |\phi(ie)\hat{}(\varphi)| = 1\}$ is a compact subset in M_B, since $\widehat{B} \subset C_0(M_B)$ the algebra of all complex-valued continuous functions on M_B which vanish at infinity. Therefore, Y_{-1} and Y_1 are compact. $\square$

If ϕ is a $*$-homomorphism between C^*-algebras A and B, then ϕ is norm decreasing [3, Theorem 1.5.7], where $*$-homomorphism is a linear $*$-ring homomorphism. We consider a $*$-ring homomorphism from a commutative Banach algebra with an involution to a commutative Banach algebra with a symmetric involution. If the Gelfand transform on B is an isometry, the following result holds.

Corollary 2.3. *In addition to the assumptions in Theorem 2.1, if $\|b\|_B = \|\hat{b}\|_\infty$ holds for every $b \in B$, then ϕ is norm decreasing.*

Proof. For every $f \in A$

$$\|\phi(f)\|_B = \|\widehat{\phi(f)}\|_\infty \le \|\hat{f}\|_\infty \le \|f\|_A$$

holds, where $\|\cdot\|_A$ and $\|\cdot\|_B$ denote the norms on A and B, respectively. $\square$

Corollary 2.4. *In addition to the assumptions in Theorem 2.1, if ϕ is surjective, then Φ is defined on M_B into M_A and injective.*

Proof. First, we show that A must be non-radical. Suppose not. Then

$$B = \mathrm{rad} B$$

holds, since ϕ is surjective. This is a contradiction. Hence, A is non-radical.

Moreover, $M_B = Y_{-1} \cup Y_1$ holds. In fact, assume to the contrary that there exists a $\varphi^{'} \in M_B$ such that $\phi_{\varphi'} = 0$. Since ϕ is surjective, $\varphi^{'} = 0$ on B. We arrived at a contradiction. Hence, Φ is defined on M_B into M_A.

Suppose $\varphi_1 \ne \varphi_2$ ($\varphi_1, \varphi_2 \in M_B$). We show that there exists a $x \in B$ with $x = x^\star$ such that $\varphi_1(x) \ne \varphi_2(x)$. In fact, by the hypothesis, there exists a $y \in B$ such that $\varphi_1(y) \ne \varphi_2(y)$. We can write $y = u + iv$ for some $u, v \in B$ with $u = u^\star$ and $v = v^\star$. Then

$$\varphi_j(y) = \varphi_j(u) + i\varphi_j(v) \quad (j = 1, 2).$$

Since $\star$-involution is symmetric, $\overline{\varphi_j(u)} = \varphi_j(u), \overline{\varphi_j(v)} = \varphi_j(v)$ holds for $j = 1, 2$. Hence, $\varphi_1(u) \ne \varphi_2(u)$ or $\varphi_1(v) \ne \varphi_2(v)$ holds. Therefore, we proved that there exists a $x \in B$ with $x = x^\star$ such that $\varphi_1(x) \ne \varphi_2(x)$. Since ϕ is surjective, there exists an $f \in A$ such that $\phi(f) = x$. Then $\phi(f)\hat{}$ is real valued on M_B. Therefore,

$$\hat{x}(\varphi_j) = \hat{f}(\Phi(\varphi_j)) \quad (j = 1, 2)$$

holds, by the Gelfand transform formula of $\phi(f)$ in Theorem 2.1. Hence, $\Phi(\varphi_1) \ne \Phi(\varphi_2)$. This completes the proof. $\square$

Example 2.1. In Theorem 2.1, Y_{-1} and Y_1 need not be closed subsets in M_A, unless A is unital. In fact, put

$$A = \{f \in C([\,0,1\,]) : f(\tfrac{1}{3}) = 0 = f(\tfrac{2}{3})\},$$
$$B = C([\,0,1\,]),$$

$$\phi(f)(x) = \begin{cases} \overline{f(x)} & (x \in [0, \frac{1}{3})) \\ 0 & (x \in [\frac{1}{3}, \frac{2}{3}\,]\,) \\ f(x) & (x \in (\frac{2}{3}, 1]\,) \end{cases} \qquad (f \in A).$$

Then, A and B are commutative Banach algebras with respect to the pointwise operations and the supremum norm, and $\phi : A \to B$ is a $*$-ring homomorphism, where involutions on A and B are both complex conjugates. Then the decomposition $\{Y_{-1}, Y_0, Y_1\}$ of $M_B = [0,1]$ is as follows:

$$Y_{-1} = [0, \frac{1}{3}), Y_0 = [\,\frac{1}{3}, \frac{2}{3}\,], Y_1 = (\frac{2}{3}, 1].$$

Hence Y_{-1} and Y_1 are not closed subsets in $[0,1]$.

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