# ON THE LIPSCHITZ CLASSIFICATION OF NORMED SPACES, UNIT BALLS, AND SPHERES 

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#### Abstract

For every normed space $Z$, we note its closed unit ball and unit sphere by $B_{Z}$ and $S_{Z}$, respectively. Let $X$ and $Y$ be normed spaces such that $S_{X}$ is Lipschitz homeomorphic to $S_{X \oplus R}$, and $S_{Y}$ is Lipschitz homeomorphic to $S_{Y \oplus R}$.

We prove that the following are equivalent: 1. $X$ is Lipschitz homeomorphic to $Y$. 2. $B_{X}$ is Lipschitz homeomorphic to $B_{Y}$. 3. $S_{X}$ is Lipschitz homeomorphic to $S_{Y}$.

This result holds also in the uniform category, except ( 2 or 3 ) $\Rightarrow 1$ which is known to be false.


## 1. Introduction

Let $f:(M, d) \rightarrow\left(M_{0}, d_{0}\right)$ be a function between metric spaces. If there exists $k \geq 0$ such that $d_{0}(f(x), f(y)) \leq k d(x, y)$ for every $x, y \in M$, then $f$ is called a $k$ Lipschitz function, or just a Lipschitz function. We call $f$ a $k$-biLipschitz function if $f$ is a bijection such that $f$ and $f^{-1}$ are $k$-Lipschitz functions, and in this case we write $M \sim M_{0}$.

For every normed space $Z$, denote its closed unit ball and unit sphere by $B_{Z}$ and $S_{Z}$, respectively, and let the norm on the space $Z \oplus R$ be the sup norm.

Let $X$ and $Y$ be infinite-dimensional Banach spaces of the same density character. It is well known that $X, Y, B_{X}, B_{Y}, S_{X}$, and $S_{Y}$ are mutually homeomorphic (see [1], [6]). Those are nontrivial results which make the topological classification of spaces, balls, and spheres - trivial.

In this paper we prove that the Lipschitz classification of normed spaces $X$ such that $S_{X} \sim S_{X \oplus R}$ is identical to the Lipschitz classification of their closed unit balls and identical to the Lipschitz classification of their spheres.

Let $X$ and $Y$ be normed spaces. Assume that there is a biLipschitz function $f: S_{X} \rightarrow S_{Y}$. Then the homogeneous extension of $f$ to $X$ (i.e., the function $\widehat{f}: X \rightarrow Y$ which is defined by $\left.\widehat{f}=\|x\| f\left(\frac{x}{\|x\|}\right)\right)$ shows that $B_{X} \sim B_{Y}$ and $X \sim Y$.

[^0]In this paper we prove the converse. Namely, we prove:
$B_{X} \sim B_{Y} \Rightarrow S_{X \oplus R} \sim S_{Y \oplus R} \quad$ and
(2) $X \sim Y \Rightarrow S_{X \oplus R} \sim S_{Y \oplus R}$.

The proof of (1) relies on the following theorem which was proved in [5].
Theorem 1.1. Let $X$ be a normed space. Then $B_{X} \sim S_{X \oplus R}$ if and only if $B_{X}$ is Lipschitz homogeneous (i.e., for every $x, y \in B_{X}$ there exists a biLipschitz function $h: B_{X} \rightarrow B_{X}$ such that $\left.h(x)=y\right)$.

Thus, we have:
Theorem 1.2. Let $X$ and $Y$ be normed spaces such that $S_{X} \sim S_{X \oplus R}$ and $S_{Y} \sim$ $S_{Y \oplus R}$. Then, $\quad S_{X} \sim S_{Y} \Longleftrightarrow B_{X} \sim B_{Y} \Longleftrightarrow X \sim Y$.

The only infinite-dimensional Banach spaces $X$ for which it is known that $S_{X} \nsim$ $S_{X \oplus R}$ are "exotic" spaces which were constructed by Gowers and Maurey [2]. However, the classical infinite-dimensional Banach spaces $X$, and in particular the Hilbert spaces, are such that $S_{X} \sim S_{X \oplus R}$.
Open Problem. Let $X$ and $Y$ be normed spaces such that $S_{X \oplus R} \sim S_{Y \oplus R}$. Does it follow that $S_{X} \sim S_{Y}$ ?
The Uniform Category. Let $2 \leq p<q<\infty$. Lindenstrauss [3] proved that $L_{p}$ is not uniformly homeomorphic to $L_{q}$, while the Mazur map [4] shows that $B_{L_{p}}$ is uniformly homeomorphic to $B_{L_{q}}$, and $S_{L_{P}}$ is uniformly homeomorphic to $S_{L_{q}}$. Therefore, the fact that the unit balls or the spheres of two normed spaces are uniformly homeomorphic does not imply that the spaces are also uniformly homeomorphic.

Minor changes in the proofs show that all the remaining implications in theorem 1.2 hold in the uniform category as well, i.e., when Lipschitz functions are replaced by uniformly continuous functions.

## 2. First implication

Theorem 2.1. Let $X$ and $Y$ be normed spaces. If $B_{X} \sim B_{Y}$, then $S_{X \oplus R} \sim S_{Y \oplus R}$.
Proof. Let $f: B_{X} \rightarrow B_{Y}$ be a biLipschitz function. Define the biLipschitz function $\widehat{f}: B_{X \oplus R} \rightarrow B_{Y \oplus R}$ by $\widehat{f}(x, t)=(f(x), t)$. If $f\left[S_{X}\right]=S_{Y}$, then $\widehat{f}\left[S_{X \oplus R}\right]=S_{Y \oplus R}$ and hence $S_{X \oplus R} \sim S_{Y \oplus R}$. Otherwise, there is $x_{0} \in S_{X}$ such that $\left\|f\left(x_{0}\right)\right\|<1$, or there is $y_{0} \in S_{Y}$ such that $\left\|f^{-1}\left(y_{0}\right)\right\|<1$. Without lost of generality, assume that the first case holds.

We prove that for every $x \in B_{X}$ there is a biLipschitz function $h_{x}: B_{X} \rightarrow B_{X}$ such that $h_{x}\left(0_{X}\right)=x$. It follows that $B_{X}$ is Lipschitz homogeneous, and hence $B_{Y}$ is also Lipschitz homogeneous since $B_{X} \sim B_{Y}$. Therefore, by Theorem 1.1 $S_{X \oplus R} \sim B_{X} \sim B_{Y} \sim S_{Y \oplus R}$.

Let $x \in B_{X}$. If $\|x\|<1$, then $h_{x}(z) \stackrel{\text { def }}{=} z+(1-\|z\|) x$ does the job (since $\left.1-\|x\| \leq \frac{\left\|h_{x}\left(z_{1}\right)-h_{x}\left(z_{2}\right)\right\|}{\left\|z_{1}-z_{2}\right\|} \leq 1+\|x\|\right)$. Assume that $\|x\|=1$. Let $x_{1} \in B_{X}$ be such that $\left\|x_{1}\right\|,\left\|f\left(x_{1}\right)\right\|<1$, and let $h_{x_{1}}: B_{X} \rightarrow B_{X}$ and $h_{f\left(x_{0}\right)}, h_{f\left(x_{1}\right)}: B_{Y} \rightarrow B_{Y}$ be as above.

Since the sphere of every normed space is Lipschitz homogeneous (see [5]), there is a biLipschitz function $g: S_{X} \rightarrow S_{X}$ such that $g\left(x_{0}\right)=x$. Let $\widehat{g}$ be the homogeneous extension of $g$. Then the following function is as required:

$$
h_{x} \equiv \widehat{g} \circ f^{-1} \circ h_{f\left(x_{0}\right)} \circ h_{f\left(x_{1}\right)}^{-1} \circ f \circ h_{x_{1}}
$$

Remark 2.2. The proof of the uniform version is similar and relies on the fact that Theorem 1.1 holds also in the uniform category.

## 3. SECOND IMPLICATION

Theorem 3.1. Let $X$ and $Y$ be normed spaces. If $X \sim Y$, then $S_{X \oplus R} \sim S_{Y \oplus R}$.
Proof. Let $g: X \rightarrow Y$ be a $k$-biLipschitz function. Without lost of generality, assume that $k \geq 2$ and $g\left(0_{X}\right)=0_{Y}$.

First, we need some definitions. For every normed space $Z$ and every integer $n \geq 0$, define:

$$
\begin{aligned}
K_{Z} & =\{(z, t) \in Z \oplus R \mid\|z\| \leq t \leq 1\} \\
E_{Z}^{n} & =\left\{(z, t) \in \partial K_{Z} \mid 2^{-(n+1)} \leq t \leq 2^{-n}\right\}
\end{aligned}
$$

Clearly, $\partial K_{Z} \sim S_{Z \oplus R}$ and $\partial K_{Z} \backslash\left\{\left(0_{Z}, 0\right)\right\}=\bigcup_{n=0}^{\infty} E_{Z}^{n}$.
Define the homeomorphism $h_{Z}: \partial K_{Z} \backslash\left\{\left(0_{Z}, 0\right)\right\} \rightarrow Z$ by: if $(z, t) \in E_{Z}^{n}$, then

$$
h_{Z}(z, t)=\left(k^{n}+\frac{2^{-n}-t}{2^{-(n+1)}}\left(k^{n+1}-k^{n}\right)\right) \frac{z}{t} .
$$

Then, for every $n \geq 1$ and every $\left(z_{1}, t_{1}\right),\left(z_{2}, t_{2}\right) \in E_{Z}^{n}$ :

$$
\begin{gather*}
k^{n} \leq\left\|h_{Z}\left(z_{1}, t_{1}\right)\right\|,\left\|h_{Z}\left(z_{2}, t_{2}\right)\right\| \leq k^{n+1}  \tag{1}\\
\left\|h_{Z}\left(z_{1}, t_{1}\right)-h_{Z}\left(z_{2}, t_{2}\right)\right\| \leq 3(2 k)^{n+1}\left\|\left(z_{1}, t_{1}\right)-\left(z_{2}, t_{2}\right)\right\|  \tag{2}\\
3\left\|h_{Z}\left(z_{1}, t_{1}\right)-h_{Z}\left(z_{2}, t_{2}\right)\right\| \geq(2 k)^{n}\left\|\left(z_{1}, t_{1}\right)-\left(z_{2}, t_{2}\right)\right\| \tag{3}
\end{gather*}
$$

Clearly, (1) holds. To see (2) and (3), let $\alpha \stackrel{\text { def }}{=}\left\|h_{Z}\left(z_{1}, t_{1}\right)-h_{Z}\left(z_{2}, t_{2}\right)\right\|$ and assume, without lost of generality, that $t_{1} \geq t_{2}$.

Then,

$$
\begin{aligned}
\alpha & \leq\left\|h_{Z}\left(z_{1}, t_{1}\right)-h_{Z}\left(\frac{t_{2}}{t_{1}} z_{1}, t_{2}\right)\right\|+\left\|h_{Z}\left(\frac{t_{2}}{t_{1}} z_{1}, t_{2}\right)-h_{Z}\left(z_{2}, t_{2}\right)\right\| \\
& =\frac{k^{n+1}-k^{n}}{2^{-(n+1)}}\left|t_{1}-t_{2}\right|+\left(k^{n}+\frac{2^{-n}-t_{2}}{2^{-(n+1)}}\left(k^{n+1}-k^{n}\right)\right) \frac{1}{t_{1}}\left\|z_{1}-\frac{t_{1}}{t_{2}} z_{2}\right\| \\
& \leq(2 k)^{n+1}\left(\left|t_{1}-t_{2}\right|+\left\|z_{1}-\frac{t_{1}}{t_{2}} z_{2}\right\|\right) \quad\left(\text { since } t_{1}, t_{2} \geq 2^{-(n+1)}\right) \\
& \leq(2 k)^{n+1}\left(\left|t_{1}-t_{2}\right|+\left\|z_{1}-z_{2}\right\|+\left\|z_{2}-\frac{t_{1}}{t_{2}} z_{2}\right\|\right) \\
& \leq 3(2 k)^{n+1}\left\|\left(z_{1}, t_{1}\right)-\left(z_{2}, t_{2}\right)\right\|
\end{aligned}
$$

$$
\begin{aligned}
3 \alpha & \geq\left\|h_{Z}\left(z_{1}, t_{1}\right)-h_{Z}\left(z_{2}, t_{2}\right)\right\|+2\left|\left\|h_{Z}\left(z_{1}, t_{1}\right)\right\|-\left\|h_{Z}\left(z_{2}, t_{2}\right)\right\|\right| \\
& \geq\left\|h_{Z}\left(z_{1}, t_{1}\right)-\right\| h_{Z}\left(z_{1}, t_{1}\right)\left\|\frac{h_{Z}\left(z_{2}, t_{2}\right)}{\left\|h_{Z}\left(z_{2}, t_{2}\right)\right\|}\right\|+\left|\left\|h_{Z}\left(z_{1}, t_{1}\right)\right\|-\left\|h_{Z}\left(z_{2}, t_{2}\right)\right\|\right| \\
& =\left(k^{n}+\frac{2^{-n}-t_{1}}{2^{-(n+1)}}\left(k^{n+1}-k^{n}\right)\right) \frac{1}{t_{1}}\left\|z_{1}-\frac{t_{1}}{t_{2}} z_{2}\right\|+\frac{k^{n+1}-k^{n}}{2^{-(n+1)}}\left|t_{1}-t_{2}\right| \\
& \geq(2 k)^{n}\left(\left\|z_{1}-\frac{t_{1}}{t_{2}} z_{2}\right\|+2\left|t_{1}-t_{2}\right|\right) \quad\left(\text { since } t_{1}, t_{2} \leq 2^{-n}\right) \\
& =(2 k)^{n}\left(\left\|z_{1}-\frac{t_{1}}{t_{2}} z_{2}\right\|+\left\|\frac{t_{1}}{t_{2}} z_{2}-z_{2}\right\|+\left|t_{1}-t_{2}\right|\right) \\
& \geq(2 k)^{n}\left\|\left(z_{1}, t_{1}\right)-\left(z_{2}, t_{2}\right)\right\| .
\end{aligned}
$$

Now, define the following homeomorphism :

$$
f \equiv h_{Y}^{-1} \circ g \circ h_{X}: \partial K_{X} \backslash\left\{\left(0_{X}, 0\right)\right\} \rightarrow \partial K_{Y} \backslash\left\{\left(0_{Y}, 0\right)\right\}
$$

We shall show that $f$ is Lipschitz, and in the same way $f^{-1}$ is Lipschitz. Therefore, $S_{X \oplus R} \sim \partial K_{X} \sim \partial K_{Y} \sim S_{Y \oplus R}$.

Clearly, it is sufficient to show that there is $l \geq 1$ such that for every $n \geq 0,\left.f\right|_{E_{X}^{n}}$ is $l$-Lipschitz. It is not difficult to check that $\left.f\right|_{E_{X}^{0}}$ is Lipschitz. So, let $n \geq 1$ and let $\left(x_{0}, t_{0}\right),\left(x_{3}, t_{3}\right) \in E_{X}^{n}$.

Let $y_{0} \stackrel{\text { def }}{=}\left(g \circ h_{X}\right)\left(x_{0}, t_{0}\right)$ and $y_{3} \stackrel{\text { def }}{=}\left(g \circ h_{X}\right)\left(x_{3}, t_{3}\right)$. Without lost of generality, assume that $\left\|y_{0}\right\| \leq\left\|y_{3}\right\|$. Since $g\left(0_{X}\right)=0_{Y}$ and $g$ is $k$-biLipschitz, we get by (1) that $k^{n-1} \leq\left\|y_{0}\right\| \leq\left\|y_{3}\right\| \leq k^{n+2}$. Therefore, there are $y_{1}, y_{2} \in\left[y_{0}, y_{3}\right]$ such that for every $0 \leq i \leq 2$, either $k^{n+i-1} \leq\left\|y_{i}\right\| \leq\left\|y_{i+1}\right\| \leq k^{n+i}$ or $y_{i}=y_{i+1}$.

Then,

$$
\begin{align*}
\left\|\left(x_{0}, t_{0}\right)-\left(x_{3}, t_{3}\right)\right\| & \geq \frac{(2 k)^{-(n+1)}}{3}\left\|h_{X}\left(x_{0}, t_{0}\right)-h_{X}\left(x_{3}, t_{3}\right)\right\|  \tag{2}\\
& \geq \frac{(2 k)^{-(n+1)}}{3 k}\left\|y_{0}-y_{3}\right\| \quad(g \text { is } k \text {-biLip. }) \\
& =\frac{(2 k)^{-(n+1)}}{3 k} \sum_{i=0}^{2}\left\|y_{i}-y_{i+1}\right\| \\
& \geq \frac{(2 k)^{-(n+1)}}{3 k} \sum_{i=0}^{2} \frac{(2 k)^{n+i-1}}{3}\left\|h_{Y}^{-1}\left(y_{i}\right)-h_{Y}^{-1}\left(y_{i+1}\right)\right\| \quad(\text { by }(3))  \tag{3}\\
& \geq \frac{(2 k)^{-2}}{9 k} \sum_{i=0}^{2}\left\|h_{Y}^{-1}\left(y_{i}\right)-h_{Y}^{-1}\left(y_{i+1}\right)\right\| \\
& \geq \frac{(2 k)^{-2}}{9 k}\left\|h_{Y}^{-1}\left(y_{0}\right)-h_{Y}^{-1}\left(y_{3}\right)\right\| \\
& =\frac{1}{36 k^{3}}\left\|f\left(x_{0}, t_{0}\right)-f\left(x_{3}, t_{3}\right)\right\| .
\end{align*}
$$

Remark 3.2. The proof of the uniform version is similar, except that $k$ is not given and we have to define $k \geq 2$ such that for every $n \geq 1$ and every $x \in X$, if $k^{n} \leq\|x\| \leq k^{n+1}$, then $k^{n-1} \leq\|g(x)\| \leq k^{n+2}$.

Since $g$ and $g^{-1}$ are uniformly continuous, there is $\delta>0$ such that for every $x_{1}, x_{2} \in X$ and every $y_{1}, y_{2} \in Y:$
if $\left\|x_{1}-x_{2}\right\|,\left\|y_{1}-y_{2}\right\|<\delta$ then $\left\|g\left(x_{1}\right)-g\left(x_{2}\right)\right\|,\left\|g^{-1}\left(y_{1}\right)-g^{-1}\left(y_{2}\right)\right\|<1$.
We show that $k \stackrel{\text { def }}{=} \frac{2}{\delta}+2$ is as required.
Let $n \geq 1$ and let $x \in X$ be such that $k^{n} \leq\|x\| \leq k^{n+1}$. Let $m \stackrel{\text { def }}{=}\left[\frac{\|x\|}{\delta}\right]$ and for every $0 \leq i \leq 2 m$ let $x_{i} \stackrel{\text { def }}{=} \frac{i \delta x}{2\|x\|}$, also let $x_{2 m+1} \stackrel{\text { def }}{=} x$. Then, $\left\|x_{i}-x_{i+1}\right\|<\delta$ for every $0 \leq i \leq 2 m$. Therefore,

$$
\|g(x)\| \leq \sum_{i=0}^{2 m}\left\|g\left(x_{i}\right)-g\left(x_{i+1}\right)\right\|<2 m+1 \leq \frac{2\|x\|}{\delta}+1 \leq \frac{2 k^{n+1}}{\delta}+1<k^{n+2}
$$

Similarly, if $\|g(x)\| \leq k^{n-1}$, then

$$
\|x\|=\left\|g^{-1}(g(x))\right\|<\frac{2\|g(x)\|}{\delta}+1 \leq \frac{2 k^{n-1}}{\delta}+1<k^{n}
$$

Since $\|x\| \geq k^{n}$, it must be that $\|g(x)\|>k^{n-1}$.

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