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POINTWISE FOURIER INVERSION—AN ADDENDUM

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ABSTRACT. In this note we complete a circle of results presented in §5 of an earlier work of the author (J. Fourier Anal. **5** (1999), 449–463), establishing the endpoint case of Proposition 10 of that paper. As a consequence, we have results on pointwise convergence of the Fourier series (summed by spheres) of a function on the 3-dimensional torus with a simple jump across a smooth surface Σ, with no curvature hypotheses on Σ, extending Proposition 7 of that paper.

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the n-dimensional torus with its standard flat metric and Laplace operator Δ, and let χ_R be the characteristic function of $[-R, R]$ (set equal to 1/2 at the endpoints). Given a function f on $\mathbb{T}^n$, we investigate the pointwise convergence of

$$S_R f(x) = \chi_R(\sqrt{-\Delta}) f(x) \tag{1}$$

as $R \to \infty$. Following [BC] and [T], we pick $\beta \in C_0^\infty(\mathbb{R})$ with $\beta(t) = 1$ for $|t| \le a$, 0 for $|t| \ge 2a$ (for some $a > 0$) and write

$$S_R f(x) = S_R^\beta f(x) + T_R^\beta f(x), \tag{2}$$

with

$$S_R^\beta f(x) = \frac{1}{\pi} \int \frac{\sin Rt}{t} \beta(t) \cos t\sqrt{-\Delta} f(x)\, dt \tag{3}$$

and

$$\begin{aligned} T_R^\beta f(x) &= \delta_R(\sqrt{-\Delta}) f(x) = \sum_{\nu \in \mathbb{Z}^n} \delta_R(|\nu|) \hat{f}(\nu) e^{i\nu\cdot x}, \\ \delta_R(\lambda) &= \chi_R(\lambda) - \hat{\beta} * \chi_R(\lambda). \end{aligned} \tag{4}$$

Formula (3) allows for wave equation techniques to be used on the term $S_R^\beta f(x)$; a number of results on this were established in [PT], yielding conditions under which one can say that $S_R^\beta f(x) \to f(x)$ as $R \to \infty$. One then has the task of examining when $T_R^\beta f(x) \to 0$.

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An attack initiated in [BC] starts with an application of Cauchy's inequality to write

$$|T_R^\beta f(x)|^2 \le \Big\{ \sum_\nu |\delta_R(|\nu|)| \cdot |\hat{f}(\nu)|^2 \Big\} \cdot \Big\{ \sum_\nu |\delta_R(|\nu|)| \Big\}. \tag{5}$$

A number of successful estimates on the right side of (5) were produced in [BC], and further estimates were produced in [T]. Here we aim to establish the following.

Theorem 1. *Let Σ be a smooth $(n-1)$-dimensional surface in $\mathbb{T}^n$. Assume*

$$f \in I^{-(n-3)/2}(\mathbb{T}^n, \Sigma). \tag{6}$$

Also assume that there exists $T_0 \in (0-, \infty)$ such that, for $|t| \ge T_0$, $\cos t\sqrt{-\Delta} f(x)$ has no caustics of order $(n-1)/2$, and assume $\beta(t) = 1$ for $|t| \le T_0$. Then

$$\lim_{R\to\infty} \|T_R^\beta f\|_{L^\infty} = 0. \tag{7}$$

Here $I^\mu(\mathbb{T}^n, \Sigma)$ denotes the space of classical conormal distributions with singularity of order μ on Σ. More precisely, it is the linear span of elements of the form Pg, where g is piecewise smooth with a simple jump across Σ and $P \in OPS^\mu(\mathbb{T}^n)$, i.e., P is a classical pseudodifferential operator of order μ. The notion of order of a caustic is as in §10 of [PT] (cf. also Definition 5.7.3 of [D]). Perfect focus caustics are of order $(n-1)/2$, and this is the maximum possible order of a caustic.

Theorem 1 was established in [T], under an additional curvature hypothesis on Σ. Also in [T] there is a result valid for general Σ that is slightly weaker than Theorem 1. Namely, it is shown that (7) holds provided

$$f \in I^\mu(\mathbb{T}^n, \Sigma), \qquad \mu < -\frac{n-3}{2}. \tag{8}$$

Thus Theorem 1 here establishes the endpoint case of that result. A corollary of Theorem 1 is the following result, which extends Proposition 7 of [T]. Its proof follows from Theorem 1 here in the same way that Proposition 7 of [T] follows from Theorem 1 there.

Corollary 2. *Let $\Sigma \subset \mathbb{T}^3$ be a smooth surface, and suppose the Lagrange flow off $N^*\Sigma \backslash 0$ has no caustics of order 1. Let f be a piecewise smooth function on $\mathbb{T}^3$, with a simple jump across Σ. If $x \in \Sigma$, set $f(x)$ equal to the mean value of its limits from the two sides. Then we have pointwise Fourier inversion:*

$$\lim_{R\to\infty} S_R f(x) = f(x), \quad \forall x \in \mathbb{T}^3. \tag{9}$$

The convergence holds locally uniformly on $\mathbb{T}^3 \backslash \Sigma$, and the Gibbs phenomenon is manifested near Σ.

We turn to the proof of Theorem 1, which uses the arguments developed in §5 of [T], with an additional twist. As in [T] we make use of a one-parameter family of cutoffs. Take a function $\gamma \in C_0^\infty(\mathbb{R})$, equal to 1 on $(-1,1)$ and 0 outside $(-2,2)$, and set

$$\gamma_\varepsilon(t) = \gamma(\varepsilon t), \qquad S_R^\varepsilon = S_R^{\gamma_\varepsilon}. \tag{10}$$

We have a formula for $T_R^\varepsilon = S_R - S_R^\varepsilon$ similar to (4), and an estimate similar to (5), with $\delta_R(t)$ replaced by $\delta_R^\varepsilon(\lambda) = \chi_R(\lambda) - \hat{\gamma}_\varepsilon * \chi_R(\lambda)$. Note that $\delta_R^\varepsilon(\lambda)$ is a sum of two bump functions, concentrated near $\lambda = \pm R$, with "width" proportional to ε. We have the estimate

$$\sum_{\nu\in\mathbb{Z}^n} |\delta_R^\varepsilon(|\nu|)| \le C\varepsilon R^{n-1} + CR^{n-1-\alpha_n}, \tag{11}$$

for some $\alpha_n > 0$, as a consequence of the lattice point estimate (5.12) of [T]. On the other hand, as shown in Propositions 8–9 of [T],

$$\begin{aligned} f \in I^\mu(\mathbb{T}^n, \Sigma) \Rightarrow & \sum_{k\le|\nu|\le k+1} |\hat{f}(\nu)|^2 \le Ck^{2\mu-2} \\ & \Rightarrow \sum_{\nu\in\mathbb{Z}^n} |\delta_R^\varepsilon(|\nu|)| \cdot |\hat{f}(\nu)|^2 \le CR^{2\mu-2}, \end{aligned} \tag{12}$$

for $k, R \ge 1$, with C independent of $\varepsilon \in (0, 1]$. We apply this with $\mu = -(n-3)/2$. Thus, by the analogue of (5), we have

$$\|S_R f - S_R^\varepsilon f\|_{L^\infty}^2 \le C\varepsilon + CR^{-\alpha_n}. \tag{13}$$

Next, under the hypotheses on β made in Theorem 1, as in (5.20) of [T], we have

$$\|S_R^\beta f - S_R^\varepsilon f\|_{L^\infty} \le C_{\varepsilon\beta} R^{\kappa-(n-1)/2}, \tag{14}$$

for $\varepsilon < 1/T_0$, $R \ge 1$, with $\kappa = \kappa(\varepsilon, \beta) < (n-1)/2$. Hence

$$\limsup_{R\to\infty} \|S_R f - S_R^\beta f\|_{L^\infty} \le C\varepsilon^{1/2}, \quad \forall \varepsilon > 0. \tag{15}$$

This proves Theorem 1.

Remark 1. We have phrased our results in terms of Fourier series on the standard torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, but they also work on $\mathbb{R}^n/\Gamma$ for general lattices $\Gamma \subset \mathbb{R}^n$.

Remark 2. In [T] we used the fact that, if Σ has nowhere vanishing Gauss curvature, then

$$f \in I^\mu(\mathbb{T}^n, \Sigma) \Rightarrow \sum_{k\le|\nu|\le k+\varepsilon} |\hat{f}(\nu)|^2 \le C\varepsilon k^{2\mu-2} + Ck^{2\mu-2-\alpha_n}. \tag{16}$$

We have finessed this point here, but it is of independent interest to know in what generality such estimates hold. These estimates are related to certain nonisotropic lattice point estimates; we plan to discuss them in a future paper.

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