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A REMARK ON THE HARNACK INEQUALITY FOR NON-SELF-ADJOINT EVOLUTION EQUATIONS

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Abstract. In this paper we consider a non-self-adjoint evolution equation on a compact Riemannian manifold with boundary. We prove a Harnack inequality for a positive solution satisfying the Neumann boundary condition. In particular, the boundary of the manifold may be nonconvex and this gives a generalization to a theorem of Yau.

1. Introduction

Let (M^n, g) be an n-dimensional compact Riemannian manifold with boundary $\partial M \neq \emptyset$. In this paper, we shall study the equation

$$\frac{\partial u}{\partial t} - \Delta u - \sum_{i=1}^{n} f_i u_i - Vu = 0 \quad \text{in } M, \tag{1}$$

with the boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial M, \tag{2}$$

where Δ is the Laplace operator associated to metric g, f_i and V are smooth functions in $C^2(M) \times C^1((0,\infty))$, and $\frac{\partial}{\partial \nu}$ is the derivative with respect to the unit outward normal vector to the boundary ∂M.

In classical situations, J. Moser established a Harnack inequality locally for positive solutions in [4] and [5]. However, the geometric dependency of the estimates is complicated and sometimes unclear. In a fundamental work [3], Li and Yau derived a version of gradient estimates for the positive solutions to the heat equations on a compact Riemannian manifold. Using those estimates, they deduced a Harnack type inequality and demonstrated how that is applied to establish various upper and lower heat kernel bounds away from the boundary for both the Dirichlet and Neumann boundary conditions. Due to the interior nature of their gradient estimates, in general the heat kernel bounds do not extend to the boundary. However, when the boundary is convex or the manifold is closed, the gradient estimates are valid globally, and so are the corresponding heat kernel bounds. Since many evolution equations in applied mathematics are not self-adjoint and have a convection term, S. T. Yau [8] recently generalized the Li-Yau's parabolic Harnack inequality

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to cover the non-self-adjoint equation (1). In [2], [3], and [8], the equation (1) was studied when M is either a complete noncompact manifold, or M is a compact manifold. When M is a compact manifold, the Neumann condition is imposed on the boundary, and some convexities for the boundary ∂M, functions f_i, $i = 1, \dots, n$, and V are assumed there to obtain good gradient estimates. The purpose of the paper is to develop a Harnack inequality for equations (1) and (2) when ∂M, or functions f_i and V may not satisfy the convexity in [8]. The method employed here to establish the gradient estimate essentially follows from [3], or [8]. However, there are some technical complications due to the nonconvexity of the boundary as the estimates then necessarily involve the second fundamental form of the boundary and a so-called "interior rolling R-ball" condition for the boundary.

To be more specific, we consider a compact manifold (M^n, g) with boundary ∂M satisfying a "interior rolling R-ball". We recall the following definition from [1].

Definition 1.1. ∂M is said to satisfy the "interior rolling R-ball" condition if for each point $p \in \partial M$ there is a geodesic ball $B_q(\dfrac{R}{2})$, centered at $q \in M$ with radius $\frac{R}{2}$, such that $p = B_q(\frac{R}{2}) \cap \partial M$ and $B_q(\frac{R}{2}) \subset M$.

Throughout the paper, let $\{e_1, \dots, e_n\}$ be a locally defined orthonormal frame field of the tangent bundle and $e_n = \nu$ on the boundary ∂M. Also, we make the following assumptions $(*)$ on M and ∂M, unless stated otherwise.

$(*)$ M^n is a compact manifold with boundary ∂M, such that the Ricci curvature of M satisfies $Ric_M \geq -K$ and the second fundamental form elements of ∂M with respect to outward pointing unit normal ν satisfies $II \geq -H$ for some constants K, $H \geq 0$. Further assume that ∂M also satisfies the " interior rolling R-ball condition" with R chosen to be small.

Theorem 1.1. *Let M, ∂M be as in $(*)$, where R satisfies that $-\frac{H}{R} + H \leq 0$, and assume that $f_n = 0$ on the boundary ∂M. Let μ be a positive constant satisfying $2(1+\bar{H})^2 > \mu > (1+\bar{H})^2$, where $\bar{H} = H + \frac{R\theta}{2}$, and θ be a constant such that*

$$-\Big(\frac{2H}{R} + \theta\Big)V + V_\nu + \frac{\mu}{2\theta}\Big[\sum_{i=1}^{n-1} f_{i,\nu}^2 + \sum_{i=1}^{n-1}\Big(\sum_{j=1}^{n-1} h_{ij} f_j\Big)^2\Big] - \frac{\mu}{2}\sum_{i=1}^{n} f_{i,i\nu} < 0, \tag{3}$$

on the boundary, where $f_{i,i\nu} = e_n e_i(f_i)$, $V_\nu = V_n = e_n V$. Suppose that there exists a constant a such that

$$\begin{aligned}
&-2\min\Big\{\big[\mu - (1+\bar{H})^2\big]\sum_{i,j=1}^{n} f_{i,j}x_i x_j,\, (\mu-1)\sum_{i,j=1}^{n} f_{i,j}x_i x_j\Big\}\\
&-2\min\Big\{(1+\bar{H})^2\sum_{i,j=1}^{n} R_{ij}x_i x_j,\, \sum_{i,j=1}^{n} R_{ij}x_i x_j\Big\}\\
&+\mu\sum_{i=1}^{n}\Big(\Delta f_i - \sum_{j=1}^{n} f_{j,ji}\Big)x_i + \mu\sum_{i,j=1}^{n} R_{ij} f_j x_i \leq a
\end{aligned} \tag{4}$$

for all x_i. *Let* $1 > \epsilon > 0$ *be any constant satisfying that* $\epsilon^2 - (1+\bar{H})^2\epsilon + (1+\bar{H}) > 0$, *and assume that there exist nonnegative constants* b *and* γ *so that*

$$\begin{aligned} &2C_4\, b \geq C_2 + C_3 + \frac{C_3^2}{4(\mu-1)C_4}, C_4\, b^2 > (\mu-1)\big(C_1 + a\big), \\ &\textit{and} \quad C_4\, \gamma = \frac{2(\mu-1)}{n}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} C_1 \geq \frac{\mu}{2} \max\Big\{ &-\frac{2(\mu-1)}{\mu}V_t - \sum_{i=1}^n \Big[f_{i,it} - \frac{2(\mu-1)}{\mu} f_i V_i - (\sum_{j=1}^n f_{j,ji} f_i) - \Delta f_{i,i}\Big] \\ &- \frac{2(1+\bar{H})^2}{\mu}\Delta V, 0\Big\} + 8\bar{H}(1+\bar{H})|\nabla V| \\ &+ \Big[\frac{\mu - \epsilon - \epsilon^2}{2(1-\epsilon)} + \frac{(1+\bar{H})^2\epsilon^2 - \epsilon^2\mu - \epsilon^3}{2(1-\epsilon)}\Big] \sum_{i,j=1}^n f_{i,j}^2, \\ C_2 = \frac{4(n-1)\bar{H}(3H+1)}{R} &+ \frac{2\bar{H}}{R^2}, \\ C_3 = \frac{8\bar{H}}{R} + \frac{4\bar{H}}{R} &\sup(\sum_{i=1}^n f_i^2)^{\frac{1}{2}} + \frac{32\bar{H}^2}{\epsilon R^2}, \\ C_4 = \frac{2[\mu - \epsilon^2 - (1+\bar{H})^2\epsilon][\mu - (1+\bar{H})^2]}{n(1-\epsilon)\mu^2(1+\bar{H})^2}&, \end{aligned}$$

for all $x \in M$ *and any* $t \geq 0$. *Then*

$$-\mu(\log u)_t + |\nabla \log u|^2 + \mu \sum_{i=1}^n f_i (\log u)_i + V - \frac{n\gamma}{2t} - \frac{\mu}{2}\sum_{i=1}^n f_{i,i} - b \leq 0.$$

Remarks 1.1. (1) When $\inf_M V \leq 0$, let $w = ue^{-t \inf_M V}$. Then w satisfies the equation

$$\frac{\partial w}{\partial t} = \Delta w + \sum_{i=1}^n f_i w_i + (V - \inf_M V)w,$$

and one draws the conclusion of the theorem with V replaced by $V + (\mu-1)\inf_M V$ in (5).

(2) If $\bar{H} = 0$ as in Theorem 2 of [8], then $C_2 = C_3 = 0$, and we may take $\epsilon = 0$, and obtain that $C_4 = \frac{2(\mu-1)}{n\mu}$. Substituting this into (4), we obtain that $\gamma = \mu$ and we may choose b such that

$$b^2 \geq \frac{n\mu}{2}\big(C_1 + a\big).$$

By setting $\mu = 1$, we recover the result of Theorem 2 in [8].

(3) As in [1], we assume that $-\frac{H}{R} + H \leq 0$, in Theorem 1.1 and if $H > 0$, we further choose $R > 0$ to be sufficiently small such that $\sqrt{|K_R|}\tan(R\sqrt{|K_R|}) \leq \frac{H}{2} + \frac{1}{2}$ and $\frac{H}{\sqrt{|K_R|}}\tan(R\sqrt{|K_R|}) \leq \frac{1}{2}$ when $K_R \neq 0$, and $HR \leq \frac{1}{2}$ when $K_R = 0$, where $K_R = \max\{R_{nana}(x) | x \in \partial M(R),\ 1 \leq a \leq n-1\}$ and $\partial M(R) = \{x \in M \cup \partial M | r(x) \leq R\}$.

2. Proof

In this section, we modify a gradient estimate method as in [1] and [6] to prove our theorem for a positive solution u of (1) satisfying the boundary condition (2).

Proof. We define a function on M by

$$\rho(x) = (1 + \eta(\frac{r(x)}{R}))^2,$$

where $r(x)$ denotes the distance from $x \in M$ to ∂M and $\eta(r)$ is a nonnegative smooth function defined on $[0, \infty)$ such that

$$\begin{cases} \eta(r) \leq H + \frac{R\theta}{2} & \text{if } r \in [0, \frac{1}{2}), \\ \eta(r) = H + \frac{R\theta}{2} & \text{if } r \in [1, \infty) \end{cases} \tag{6}$$

with $\eta(0) = 0$, $0 \leq \eta'(r) \leq 2(H + \frac{R\theta}{2})$, $\eta'(0) = H + \frac{R\theta}{2}$, $\eta''(r) \geq -2(H + \frac{R\theta}{2})$. By applying Warner's Rauch comparison theorem (cf. Theorem 3.2 in [7]), one concludes that there is no focal points for the Jacobi fields associated to the boundary for $r(x) \leq R$ when R is chosen to satisfy the condition (3) in Remark 1.1. Hence, $r(x)$ is differentiable provided that $r(x) \leq R$. This implies that the function $\rho(x) = (1 + \eta(\frac{r(x)}{R}))^2$ is smooth in $\bar{M}$.

Let $\varphi = -\log u$. Then

$$\varphi_t = \Delta\varphi - |\nabla\varphi|^2 + \sum_{i=1}^n f_i\varphi_i - V. \tag{7}$$

Consider

$$\psi = \mu\varphi_t + \rho|\nabla\varphi|^2 - \mu\sum_{i=1}^n f_i\varphi_i + \rho V - (\frac{n\gamma}{2t} + \frac{\mu}{2}\sum_{i=1}^n f_{i,i} + b), \tag{8}$$

where b is a constant and will be chosen later.

Direct computations give us

$$\psi_t = \mu\varphi_{tt} + \rho|\nabla\varphi|_t^2 - \mu(\sum_{i=1}^n f_i\varphi_i)_t + \rho V_t + \frac{n\gamma}{2t^2} - \frac{\mu}{2}\sum_{i=1}^n f_{i,it},$$

$$\begin{aligned} 2\langle\nabla\varphi, \nabla\psi\rangle = {} & \mu|\nabla\varphi|_t^2 + 2(|\nabla\varphi|^2 + V)\sum_{i=1}^n \rho_i\varphi_i + 2\rho\sum_{i=1}^n \varphi_i(|\nabla\varphi|_i^2 + V_i) \\ & - 2\mu\sum_{i=1}^n \varphi_i(\sum_{j=1}^n f_j\varphi_j)_i - \mu\sum_{i,j=1}^n f_{i,ij}\varphi_j, \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^n f_i\psi_i = {} & \mu\sum_{i=1}^n f_i\varphi_{ti} + (\sum_{i=1}^n f_i\rho_i)(|\nabla\varphi|^2 + V) + \rho\sum_{i=1}^n f_i(|\nabla\varphi|_i^2 + V_i) \\ & - \mu\sum_{i=1}^n f_i(\sum_{j=1}^n f_j\varphi_j)_i - \frac{\mu}{2}\sum_{i=1,j}^n f_{j,ji}f_i, \end{aligned}$$

and

$$\begin{aligned}\Delta\psi &= \mu(\Delta\varphi)_t + 2\rho[\sum_{i,j=1}^{n}\varphi_{ij}^2 + \sum_{i=1}^{n}\varphi_j(\Delta\varphi)_i + \sum_{i,j=1}^{n}R_{ij}\varphi_i\varphi_j] + 2\sum_{i=1}^{n}\rho_i|\nabla\varphi|_i^2\\ &\quad + (\Delta\rho)|\nabla\varphi|^2 - \mu\sum_{i,j=1}^{n}R_{ij}f_j\varphi_i - 2\mu\sum_{i,j=1}^{n}f_{i,j}\varphi_{ij} - \mu\sum_{i=1}^{n}f_i(\Delta\varphi)_i\\ &\quad - \mu\sum_{i=1}^{n}(\Delta f_i)\varphi_i + (\Delta\rho)V + 2\sum_{i=1}^{n}\rho_i V_i + \rho\Delta V - \frac{\mu}{2}\sum_{i=1}^{n}\Delta f_{i,i},\end{aligned}$$

where R_{ij} is the Ricci curvature of the manifold (which is zero if defined on Euclidean space). Using (7), we have

$$(\varphi_t - \Delta\varphi)_t = -(|\nabla\varphi|^2 - \sum_{i=1}^{n}f_i\varphi_i + V)_t$$

and we have

(9)

$$\begin{aligned}&\psi_t + 2\langle\nabla\varphi,\nabla\psi\rangle - \sum_{i=1}^{n}f_i\psi_i - \Delta\psi\\ &= -\mu[|\nabla\varphi|^2 - \sum_{i=1}^{n}f_i\varphi_i + V]_t + 2\rho\sum_{i=1}^{n}\varphi_i\varphi_{ti} - \mu(\sum_{i=1}^{n}f_i\varphi_i)_t + \rho V_t + \frac{n\gamma}{2t^2} + \mu|\nabla\varphi|_t^2\\ &\quad - \frac{\mu}{2}\sum_{i=1}^{n}f_{i,it} + 2\sum_{i=1}^{n}\rho_i\varphi_i(|\nabla\varphi|^2 + V) + 2\rho\sum_{i=1}^{n}\varphi_i(|\nabla\varphi|_i^2 + V_i)\\ &\quad - 2\mu\sum_{i=1}^{n}\varphi_i(\sum_{j=1}^{n}f_j\varphi_j)_i - \mu\sum_{i,j=1}^{n}f_{j,ji}\varphi_i - \mu\sum_{i=1}^{n}f_i\varphi_{ti} - (\sum_{i=1}^{n}f_i\rho_i)(|\nabla\varphi|^2 + V)\\ &\quad - \rho\sum_{i=1}^{n}f_i(|\nabla\varphi|_i^2 + V_i) + \mu\sum_{i=1}^{n}f_i(\sum_{j=1}^{n}f_j\varphi_j)_i + \frac{\mu}{2}\sum_{i=1,j}^{n}f_{j,ji}f_i - 2\rho[\sum_{i,j=1}^{n}\varphi_{ij}^2\\ &\quad + \sum_{i=1}^{n}\varphi_i(\varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^{n}f_j\varphi_j + V)_i + \sum_{i,j=1}^{n}R_{ij}\varphi_i\varphi_j] - 2\sum_{i=1}^{n}\rho_i|\nabla\varphi|_i^2\\ &\quad - (\Delta\rho)|\nabla\varphi|^2 + \mu\sum_{i=1}^{n}(\Delta f_i)\varphi_i + 2\mu\sum_{i,j=1}^{n}f_{i,j}\varphi_{ij} + \mu\sum_{i=1}^{n}f_i(\varphi_t + |\nabla\varphi|^2\\ &\quad - \sum_{j=1}^{n}f_j\varphi_j + V)_i + \mu\sum_{i,j=1}^{n}R_{ij}f_j\varphi_i - (\Delta\rho)V - 2\sum_{i=1}^{n}\rho_i V_i - \rho\Delta V + \frac{\mu}{2}\sum_{i=1}^{n}\Delta f_{i,i}\\ &= -\mu V_t + \rho V_t + \frac{n\gamma}{2t^2} - \frac{\mu}{2}\sum_{i=1}^{n}f_{i,it} + 2\sum_{i=1}^{n}\rho_i\varphi_i(|\nabla\varphi|^2 + V) - 2\mu\sum_{i=1}^{n}\varphi_i(\sum_{j=1}^{n}f_j\varphi_j)_i\\ &\quad - \mu\sum_{i,j=1}^{n}f_{j,ji}\varphi_i - (\sum_{i=1}^{n}f_i\rho_i)(|\nabla\varphi|^2 + V) - \rho\sum_{i=1}^{n}f_i(|\nabla\varphi|_i^2 + V_i) + \frac{\mu}{2}\sum_{i=1,j}^{n}f_{j,ji}f_i\end{aligned}$$

$$
\begin{aligned}
&-2\rho[\sum_{i,j=1}^{n}\varphi_{ij}^2-\sum_{i=1}^{n}\varphi_i(\sum_{j=1}^{n}f_j\varphi_j)_i+\sum_{i,j=1}^{n}R_{ij}\varphi_i\varphi_j]-2\sum_{i=1}^{n}\rho_i|\nabla\varphi|_i^2\\
&-(\Delta\rho)(|\nabla\varphi|^2+V)+\mu\sum_{i=1}^{n}(\Delta f_i)\varphi_i+2\mu\sum_{i,j=1}^{n}f_{i,j}\varphi_{ij}+\mu\sum_{i=1}^{n}f_i(|\nabla\varphi|^2+V)_i\\
&+\mu\sum_{i,j=1}^{n}R_{ij}f_j\varphi_i-2\sum_{i=1}^{n}\rho_iV_i-\rho\Delta V+\frac{\mu}{2}\sum_{i=1}^{n}\Delta f_{i,i}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-2\mu\sum_{i=1}^{n}\varphi_i(\sum_{j=1}^{n}f_j\varphi_j)_i-\rho\sum_{i=1}^{n}f_i|\nabla\varphi|_i^2+2\rho\sum_{i=1}^{n}\varphi_i(\sum_{j=1}^{n}f_j\varphi_j)_i\\
&\quad+\mu\sum_{i=1}^{n}f_i|\nabla\varphi|_i^2=2(\rho-\mu)\sum_{i,j=1}^{n}f_{i,j}\varphi_i\varphi_j.
\end{aligned}
$$

Substituting this into (9) and grouping terms with a factor $(|\nabla\varphi|^2+V)$ and terms without a factor φ_i, or a factor φ_{ij} together, respectively, we have

$$
\begin{aligned}
&\psi_t+2\langle\nabla\varphi,\nabla\psi\rangle-\sum_{i=1}^{n}f_i\psi_i-\Delta\psi\\
&=-(\mu-\rho)V_t+\frac{n\gamma}{2t^2}-\frac{\mu}{2}\sum_{i=1}^{n}f_{i,it}+(\mu-\rho)\sum_{i=1}^{n}f_iV_i\\
&\quad+\frac{\mu}{2}\sum_{i=1,j}^{n}f_{j,ji}f_i-2\sum_{i=1}^{n}\rho_iV_i\\
&\quad-\rho\Delta V+\frac{\mu}{2}\sum_{i=1}^{n}\Delta f_{i,i}+[2\sum_{i=1}^{n}\rho_i\varphi_i-\sum_{i=1}^{n}f_i\rho_i-(\Delta\rho)](|\nabla\varphi|^2+V)\\
&\quad-2(\mu-\rho)\sum_{i,j=1}^{n}f_{i,j}\varphi_i\varphi_j-2\rho\sum_{i,j=1}^{n}R_{ij}\varphi_i\varphi_j+\mu\sum_{i=1}^{n}(\Delta f_i-\sum_{j=1}^{n}f_{j,ji})\varphi_i\\
&\quad+\mu\sum_{i,j=1}^{n}R_{ij}f_j\varphi_i-2\rho\sum_{i,j=1}^{n}\varphi_{ij}^2-2\sum_{i=1}^{n}\rho_i|\nabla\varphi|_i^2+2\mu\sum_{i,j=1}^{n}f_{i,j}\varphi_{ij}.
\end{aligned}
\tag{10}
$$

Since

$$
\begin{aligned}
&-2\min\big\{\big[\mu-(1+\bar{H})^2\big]\sum_{i,j=1}^{n}f_{i,j}x_ix_j,\big(\mu-1\big)\sum_{i,j=1}^{n}f_{i,j}x_ix_j\big\}\\
&-2\min\big\{(1+\bar{H})^2\sum_{i,j=1}^{n}R_{ij}x_ix_j,\sum_{i,j=1}^{n}R_{ij}x_ix_j\big\}\\
&+\mu\sum_{i=1}^{n}(\Delta f_i-\sum_{j=1}^{n}f_{j,ji})x_i+\mu\sum_{i=1}^{n}(\Delta f_i-\sum_{j=1}^{n}f_{j,ji})x_i+\mu\sum_{i,j=1}^{n}R_{ij}f_jx_i\le a
\end{aligned}
$$

for all x_i, we have

$$\begin{aligned}
&\psi_t + 2\langle\nabla\varphi, \nabla\psi\rangle - \sum_{i=1}^n f_i\psi_i - \Delta\psi \\
\le& -(\mu-\rho)V_t + \frac{n\gamma}{2t^2} - \frac{\mu}{2}\sum_{i=1}^n f_{i,it} + (\mu-\rho)\sum_{i=1}^n f_iV_i \\
&+ \frac{\mu}{2}\sum_{i=1,j}^n f_{j,ji}f_i - 2\sum_{i=1}^n \rho_iV_i \\
&- \rho\Delta V + \frac{\mu}{2}\sum_{i=1}^n \Delta f_{i,i} - a + \Big(2\sum_{i=1}^n \rho_i\varphi_i - \sum_{i=1}^n f_i\rho_i - \Delta\rho\Big)(|\nabla\varphi|^2 + V) \\
&- 2\rho\sum_{i,j=1}^n \varphi_{ij}^2 - 2\sum_{i=1}^n \rho_i|\nabla\varphi|_i^2 + 2\mu\sum_{i,j=1}^n f_{i,j}\varphi_{ij}.
\end{aligned} \tag{11}$$

Define

$$\bar\rho = \frac{\mu+\epsilon^2-\rho\epsilon}{1-\epsilon}, \tag{12}$$

then we have

$$\frac{\mu+\epsilon^2-(1+\bar H)^2\epsilon}{1-\epsilon} \le \bar\rho \le \frac{\mu+\epsilon^2-\epsilon}{1-\epsilon}. \tag{13}$$

Using (13) and the inequality $2xy \le \frac{1}{2\epsilon}x^2 + 2\epsilon y^2$ for any $\epsilon > 0$, it is easy to see that

$$\begin{aligned}
&-2\rho\sum_{i,j=1}^n \varphi_{ij}^2 - 4\sum_{i,j=1}^n \varphi_{ij}\rho_j\varphi_i + 2\mu\sum_{i,j=1}^n f_{i,j}\varphi_{ij} \\
=& -2\rho\sum_{i,j=1}^n \varphi_{ij}^2 - 4\sum_{i,j=1}^n \varphi_{ij}\rho_j\varphi_i + 2(\mu-\bar\rho)\sum_{i,j=1}^n f_{i,j}\varphi_{ij} + 2\bar\rho\sum_{i,j=1}^n f_{i,j}\varphi_{ij} \\
\le& -2\rho\sum_{i,j=1}^n \varphi_{ij}^2 + \frac{2|\nabla\rho|^2}{\epsilon}(|\nabla\varphi|^2+V) + 2\epsilon\sum_{i,j=1}^n \varphi_{ij}^2 + \frac{2(\mu-\bar\rho)}{\epsilon}\sum_{i,j=1}^n \varphi_{ij}^2 \\
&+ \frac{(\mu-\bar\rho)\epsilon}{2}\sum_{i,j=1}^n f_{i,j}^2 + 2\bar\rho\sum_{i,j=1}^n f_{i,j}\varphi_{ij} \\
=& \frac{2|\nabla\rho|^2}{\epsilon}(|\nabla\varphi|^2+V) - 2\bar\rho\sum_{i,j=1}^n \varphi_{ij}^2 + 2\bar\rho\sum_{i,j=1}^n f_{i,j}\varphi_{ij} + \frac{(\mu-\bar\rho)\epsilon}{2}\sum_{i,j=1}^n f_{i,j}^2 \\
=& \frac{2|\nabla\rho|^2}{\epsilon}(|\nabla\varphi|^2+V) - 2\bar\rho\sum_{i,j=1}^n [\varphi_{ij} - \frac{1}{2}f_{i,j}]^2 + [\frac{\bar\rho}{2} + \frac{(\mu-\bar\rho)\epsilon}{2}]\sum_{i,j=1}^n f_{i,j}^2 \\
\le& \frac{2|\nabla\rho|^2}{\epsilon}(|\nabla\varphi|^2+V) - \frac{2\bar\rho}{n}[\Delta\varphi - \frac{1}{2}\sum_{i=1}^n f_{i,i}]^2 + [\frac{\bar\rho}{2} + \frac{(\mu-\bar\rho)\epsilon}{2}]\sum_{i,j=1}^n f_{i,j}^2.
\end{aligned} \tag{14}$$

Substituting (12)–(14) into (11), we have

(15)

$$\begin{aligned}
&\psi_t + 2\langle \nabla\varphi, \nabla\psi\rangle - \sum_{i=1}^n f_i\psi_i - \Delta\psi \\
\le& -(\mu-\rho)V_t + \frac{n\gamma}{2t^2} - \frac{\mu}{2}\sum_{i=1}^n f_{i,it} + (\mu-\rho)\sum_{i=1}^n f_iV_i \\
&+ \frac{\mu}{2}\sum_{i=1,j}^n f_{j,ji}f_i - 2\sum_{i=1}^n \rho_iV_i - \rho\Delta V + \frac{\mu}{2}\sum_{i=1}^n \Delta f_{i,i} + a + [\frac{\bar\rho}{2} + \frac{(\mu-\bar\rho)\epsilon}{2}]\sum_{i,j=1}^n f_{i,j}^2 \\
&+ \Big[2\sum_{i=1}^n \rho_i\varphi_i - \sum_{i=1}^n f_i\rho_i - \Delta\rho + \frac{2|\nabla\rho|^2}{\epsilon}\Big](|\nabla\varphi|^2+V) - \frac{2\bar\rho}{n}[\Delta\varphi - \frac{1}{2}\sum_{i=1}^n f_{i,i}]^2.
\end{aligned}$$

If ψ achieves its maximum on the boundary, then $\frac{\partial\psi}{\partial\nu} \ge 0$ at that point. However, for R satisfying that $\frac{H}{R} + H \le 0$, we have

$$\begin{aligned}
0 \le \frac{\partial\psi}{\partial\nu} =& \mu\varphi_{\nu t} + \frac{\partial\rho}{\partial\nu}|\nabla\varphi|^2 + \rho\frac{\partial}{\partial\nu}|\nabla\varphi|^2 - \mu\sum_{i=1}^{n-1}(f_{i,\nu}\varphi_i + \sum_{j=1}^{n-1} h_{ij}f_j\varphi_i) \\
&+ \frac{\partial}{\partial\nu}(\rho V) - \frac{\mu}{2}\frac{\partial}{\partial\nu}(\sum_{i=1}^n f_{i,i}) \\
\le& -(\frac{2H}{R}+\theta)|\nabla\varphi|^2 + 2H|\nabla\varphi|^2 + \mu\{\big(\sum_{i=1}^{n-1} f_{i,\nu}^2\big)^{\frac{1}{2}} + \big[\sum_{i=1}^{n-1}\big(\sum_{j=1}^{n-1} h_{ij}f_j\big)^2\big]^{\frac{1}{2}}\}|\nabla\varphi| \\
&- \left(\frac{2H}{R}+\theta\right)V + V_\nu - \frac{\mu}{2}\sum_{i=1}^n f_{i,i\nu} \\
\le& -\frac{2H}{R}|\nabla\varphi|^2 + 2H|\nabla\varphi|^2 + \frac{\mu}{2\theta}\big[\sum_{i=1}^{n-1} f_{i,\nu}^2 + \sum_{i=1}^{n-1}\big(\sum_{j=1}^{n-1} h_{ij}f_j\big)^2\big] \\
&- \left(\frac{2H}{R}+\theta\right)V + V_\nu - \frac{\mu}{2}\sum_{i=1}^n f_{i,i\nu} < 0
\end{aligned}$$

which is a contradiction. Assume that at $t_0 > 0$, ψ becomes zero at some point p in the interior of the manifold and $\psi < 0$ for $t < t_0$. Then $\psi_t \ge 0$, $\nabla\psi = 0$, and $\Delta\psi \le 0$ at this point, and substituting these into (15), we have

$$\begin{aligned}
0 \le& -(\mu-\rho)V_t + \frac{n\gamma}{2t_0^2} - \frac{\mu}{2}\sum_{i=1}^n f_{i,it} + (\mu-\rho)\sum_{i=1}^n f_iV_i + \frac{\mu}{2}\sum_{i=1,j}^n f_{j,ji}f_i \\
&- 2\sum_{i=1}^n \rho_iV_i - \rho\Delta V + \frac{\mu}{2}\sum_{i=1}^n \Delta f_{i,i} + a + [\frac{\bar\rho}{2} + \frac{(\mu-\bar\rho)\epsilon}{2}]\sum_{i,j=1}^n f_{i,j}^2 \\
&+ \Big[2\sum_{i=1}^n \rho_i\varphi_i - (\sum_{i=1}^n f_i\rho_i) - \Delta\rho + \frac{2|\nabla\rho|^2}{\epsilon}\Big](|\nabla\varphi|^2+V) \\
&- \frac{2\bar\rho}{n}\Big(|\nabla\varphi|^2 + V - \sum_{i=1}^n f_i\varphi_i + \varphi_t - \frac{1}{2}\sum_{i=1}^n f_{i,i}\Big)^2.
\end{aligned} \tag{16}$$

Claim 1.

$$\Big(|\nabla\varphi|^2+V-\sum_{i=1}^n f_i\varphi_i+\varphi_t-\frac{1}{2}\sum_{i=1}^n f_{i,i}\Big)^2 \geq \beta^2\Big[\rho(|\nabla\varphi|^2+V)-\sum_{i=1}^n f_i\varphi_i+\varphi_t-\frac{1}{2}\sum_{i=1}^n f_{i,i}\Big]^2, \tag{17}$$

where $\beta>0$ and

$$\beta^2=\frac{\mu-(1+\bar{H})^2}{(\mu-1)(1+\bar{H})^2}\leq 1. \tag{18}$$

Let $w=|\nabla\varphi|^2+V$ and $z=\sum_{i=1}^n f_i\varphi_i+\frac{1}{2}\sum_{i=1}^n f_{i,i}-\varphi_t$. Since $\psi=0$ at (p,t_0), the equation (8) implies that

$$\rho(|\nabla\varphi|^2+V)-\mu\sum_{i=1}^n f_i\varphi_i+\mu\varphi_t-(\frac{n\gamma}{2t_0}+\frac{\mu}{2}\sum_{i=1}^n f_{i,i}+b)=0. \tag{19}$$

In other words,

$$z=\sum_{i=1}^n f_i\varphi_i-\varphi_t+\frac{1}{2}\sum_{i=1}^n f_{i,i}<\frac{\rho}{\mu}(|\nabla\varphi|^2+V)=\frac{\rho}{\mu}w. \tag{20}$$

Therefore,

$$\begin{aligned}[w-z]^2-\beta^2[\rho w-z]^2&=\{[w-z]+\beta[\rho w-z]\}\times\{[w-z]-\beta[\rho w-z]\}\\&=[(1+\rho\beta)w-(1+\beta)z]\times[(1-\rho\beta)w-(1-\beta)z].\end{aligned}$$

Using (20), one easily checks that the above expression is nonnegative and the claim is verified. Let C_1 be a nonnegative constant defined by the inequality

$$\begin{aligned}C_1\geq\frac{\mu}{2}\max\Big\{&-\frac{2(\mu-1)}{\mu}V_t-\sum_{i=1}^n\Big[f_{i,it}-\frac{2(\mu-1)}{\mu}f_iV_i\\&-(\sum_{j=1}^n f_{j,ji}f_i)-\Delta f_{i,i}\Big]-\frac{2(1+\bar{H})^2}{\mu}\Delta V,0\Big\}+8\bar{H}(1+\bar{H})|\nabla V|\\&+\Big[\frac{\mu-\epsilon-\epsilon^2}{2(1-\epsilon)}+\frac{(1+\bar{H})^2\epsilon^2-\epsilon^2\mu-\epsilon^3}{2(1-\epsilon)}\Big]\sum_{i,j=1}^n f_{i,j}^2.\end{aligned} \tag{21}$$

As in [1], we choose R according to condition (3) in Remark 1.1 and apply a comparison theorem in [7] to obtain that

$$\Delta r\geq-(n-1)\,\frac{H+\sqrt{|K_R|}\tan(R\sqrt{|K_R|})}{1-\frac{H}{\sqrt{|K_R|}}\tan(R\sqrt{|K_R|})}\geq-(n-1)(3H+1) \tag{22}$$

when $|K_R|\neq 0$. When $K_R=0$, we also obtain the comparison (22) by letting $\sqrt{|K_R|}$ go to 0. Also, we obtain that

$$\frac{\Delta\rho}{\rho}\geq-\Big\{\frac{4(n-1)\bar{H}(3H+1)}{R}+\frac{4\bar{H}}{R^2}\Big\}=-C_2, \tag{23}$$

$$\begin{aligned}\frac{2|\nabla\rho|}{\rho^{\frac{3}{2}}}+\frac{|\nabla\rho|}{\rho}(\sum_{i=1}^{n}f_i^2)^{\frac{1}{2}}+\frac{2|\nabla\rho|^2}{\epsilon\rho}\\ \leq\frac{8\bar{H}}{R}+\frac{4\bar{H}}{R}\sup(\sum_{i=1}^{n}f_i^2)^{\frac{1}{2}}+\frac{32\bar{H}^2}{\epsilon R^2}=C_3.\end{aligned}\tag{24}$$

Therefore, we have the following inequality:

$$\begin{aligned}&\left\{2\sum_{i=1}^{n}\rho_i\varphi_i-\sum_{i=1}^{n}f_i\rho_i-\Delta\rho+\frac{2|\nabla\rho|^2}{\epsilon}\right\}(|\nabla\varphi|^2+V)\\ &\quad\leq\frac{2|\nabla\rho|}{\rho^{\frac{3}{2}}}\Big[\rho(|\nabla\varphi|^2+V)\Big]^{\frac{3}{2}}+\left\{\frac{|\nabla\rho|}{\rho}\sup(\sum_{i=1}^{n}f_i^2)^{\frac{1}{2}}\right.\\ &\qquad\qquad\left.-\frac{\Delta\rho}{\rho}+\frac{2|\nabla\rho|^2}{\epsilon\rho}\right\}\rho(|\nabla\varphi|^2+V)\\ &\quad\leq C_3\Big[\rho(|\nabla\varphi|^2+V)\Big]^{\frac{3}{2}}+[C_2+C_3]\rho(|\nabla\varphi|^2+V).\end{aligned}\tag{25}$$

Substituting (17), (18), and (21)–(25) into (16), we have

$$\begin{aligned}0\leq C_1+\frac{n\gamma}{2t_0^2}+a+C_3\Big[\rho(|\nabla\varphi|^2+V)\Big]^{\frac{3}{2}}+(C_2+C_3)\rho(|\nabla\varphi|^2+V)\\ -\frac{2\bar{\rho}}{n}\frac{[\mu-(1+\bar{H})^2]}{(\mu-1)(1+\bar{H})^2}\Big[\rho(|\nabla\varphi|^2+V)-\sum_{i=1}^{n}f_i\varphi_i+\varphi_t-\frac{1}{2}\sum_{i=1}^{n}f_{i,i}\Big]^2.\end{aligned}\tag{26}$$

Let $y=\rho(|\nabla\varphi|^2+V)$ and $z=\sum_{i=1}^{n}f_i\varphi_i+\frac{1}{2}\sum_{i=1}^{n}f_{i,i}-\varphi_t$. Then (19) implies that

$$\begin{aligned}(y-z)^2&=\left[\frac{1}{\mu}(y-\mu z)+\frac{\mu-1}{\mu}y\right]^2\\ &=\frac{1}{\mu^2}(y-\mu z)^2+(\frac{\mu-1}{\mu})^2y^2+\frac{2(\mu-1)}{\mu^2}y(y-\mu z)\\ &\geq\frac{1}{\mu^2}(\psi+\frac{n\gamma}{2t_0}+b)^2+(\frac{\mu-1}{\mu})^2y^2+\frac{2b(\mu-1)}{\mu^2}y\\ &=\frac{1}{\mu^2}(\frac{n\gamma}{2t_0}+b)^2+(\frac{\mu-1}{\mu})^2y^2+\frac{2b(\mu-1)}{\mu^2}y\end{aligned}\tag{27}$$

since $\psi(p,t_0)=0$, $y\geq 0$, and $y-\mu z\geq 0$. Letting

$$C_4=\frac{2[\mu-\epsilon^2-(1+\bar{H})^2\epsilon][\mu-(1+\bar{H})^2]}{n(1-\epsilon)\mu^2(1+\bar{H})^2},\tag{28}$$

and combining (13), (26) and (27), we get

$$\begin{aligned}0\leq C_1+a+\frac{n\gamma}{2t_0^2}-(\mu-1)C_4y^2+C_3y^{\frac{3}{2}}\\ +(C_2+C_3-2bC_4)y-\frac{C_4}{(\mu-1)}(\frac{n\gamma}{2t_0}+b)^2.\end{aligned}\tag{29}$$

Consider $-Ay^2+By^{\frac{3}{2}}+Cy$, where A,B,C are positive. Clearly

$$-Ay^2+By^{\frac{3}{2}}+Cy=-Ay^2+By^{\frac{3}{2}}-\frac{B^2}{4A}y+(C+\frac{B^2}{4A})y\leq(C+\frac{B^2}{4A})y.\tag{30}$$

Applying (30) to (29) with

$$A = (\mu - 1)C_4, \qquad B = C_3, \quad \text{and} \quad C = C_2 + C_3 - 2bC_4,$$

we conclude from (29) that

$$\begin{aligned}
0 &\le C_1 + a + \frac{n\gamma}{2t_0^2} + \big(C_2 + C_3 + \frac{C_3^2}{4(\mu-1)C_4} - 2bC_4\big)y \\
&\quad - \frac{C_4}{(\mu-1)}\big(\frac{n\gamma}{2t_0} + b\big)^2 \\
&\le C_1 + a + \frac{n\gamma}{2t_0^2} + \big(C_2 + C_3 + \frac{C_3^2}{4(\mu-1)C_4} - 2bC_4\big)y \\
&\quad - \frac{C_4}{(\mu-1)}\big(\frac{n^2\gamma^2}{4t_0^2} + b^2\big).
\end{aligned} \tag{31}$$

If we choose b and γ to be the constants as in (5), then we have

$$0 \le C_1 + a + \frac{n\gamma}{2t_0^2} - \frac{C_4}{(\mu-1)}\big(\frac{n^2\gamma^2}{4t_0^2} + b^2\big) < 0,$$

which is a contradiction. Hence, $\psi \le 0$ in $M \times [0,\infty)$. This completes the proof of Theorem 1.1.

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