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MINIMAL INDEX OF A C^*-CROSSED PRODUCT BY A FINITE GROUP

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ABSTRACT. Let $A \times_\alpha G$ be the C^*-crossed product of a simple unital C^*-algebra A by a finite group G. In this paper we show that the canonical conditional expectation from $A \times_\alpha G$ to A has the minimal index if $A \times_\alpha G$ is simple. It is also proved that if α is an outer action, then the canonical one is the unique conditional expectation of index-finite type from $A \times_\alpha G$ to A, while there are infinitely many conditional expectations when a nontrivial subgroup of G acts innerly on A.

1. INTRODUCTION

In [13], as a generalization of the index for subfactors by Jones ([4]) and Kosaki ([9]), Watatani introduced the notion of index for a pair of C^*-algebras $A \subset B$ with common unit when there exists a faithful conditional expectation $E : B \to A$ of index-finite type. It is shown that the index value $\mathrm{Index}(E)$ is a positive central element in B and the possible range is in the familiar set $\{4\cos^2 \pi/n : n = 3, 4, 5, \cdots\} \cup [4, \infty)$ in the case where $\mathrm{Index}(E)$ is a scalar.

Hiai proved in [3, Theorem 1] that for a pair of factors $N \subset M$ if there is a conditional expectation E from M onto N with Index $(E) < \infty$, then there exists a unique one $E_0 \in \epsilon(M, N)$ (= the set of all faithful normal conditional expectations from M onto N) such that

$$\mathrm{Index}(E_0) \le \mathrm{Index}(E), \quad E \in \epsilon(M, N)$$

and if the relative commutant $N' \cap M$ is non-trivial, then

$$\{\, \mathrm{Index}(E) : E \in \epsilon(M, N)\,\} = [\mathrm{Index}(E_0), \infty).$$

The existence and the characterization, analogous to those in [3], for the conditional expectation on a C^*-algebra having the minimal index were shown in [13, Theorem 2.12.3]. Kajiwara and Watatani ([7]) also defined minimality for Hilbert C^*-bimodules and proved that tensor products of minimal bimodules are also minimal.

In this paper we consider the C^*-crossed product $A \times_\alpha G$ of a simple unital C^*-algebra A by an action α of a finite group G and show in section 3 that the canonical conditional expectation $E : A \times_\alpha G \to A$ defined by $E(\sum_g a_g u_g) = a_e$ is the minimal one when the crossed product $A \times_\alpha G$ is simple, particularly when α

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is an outer action. In section 4 it is proved that the action α is outer if and only if there exists a unique conditional expectation of index-finite type from $A \times_\alpha G$ onto A.

2. Preliminaries

We refer to [13] for definitions and some basic facts on index theory of C^*-algebras. Let A be a C^*-subalgebra of a unital C^*-algebra B. We say that a map $E : B \to A$ is a *conditional expectation* if it is a positive faithful linear map of norm one satisfying

$$E(aba') = aE(b)a', \quad a, a' \in A,\ b \in B.$$

A finite family $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ in $B \times B$ is called a *quasi-basis* for E if

$$\sum_i u_i E(v_i b) = \sum_i E(bu_i)v_i = b \text{ for } \ b \in B.$$

A conditional expectation E is said to be *of index-finite type* if there exists a quasi-basis for E. In this case the *index* of E is defined by

$$\mathrm{Index}(E) = \sum_i u_i v_i.$$

Since it is always possible to find a quasi-basis of the form $\{(u_1, u_1^*), \ldots, (u_n, u_n^*)\}$ whenever E is of index-finite type, the index of E is a positive element. Note that the index value $\mathrm{Index}(E)$ does not depend on the choice of a quasi-basis and it is always contained in the center $Z(B)$ of B. Hence the index is just a scalar whenever B has the trivial center, particularly when B is simple.

We denote the relative commutant $A' \cap B$ of A in B by $C_B(A)$ and the set of all conditional expectations $E : B \to A$ of index-finite type by $\epsilon_0(B, A)$ as in [13].

Theorem 2.1 ([13, Theorem 2.12.3]). *Let $1 \in A \subset B$, where A and B have trivial centers. Suppose $\epsilon_0(B, A) \neq \emptyset$. Then the following hold.*

(1) There exists a unique conditional expectation E_0 such that

$$\mathrm{Index}(E_0) \leq \mathrm{Index}(E), \quad E \in \epsilon_0(B, A).$$

(2) $E = E_0$ if and only if $E|_{C_B(A)}$ is a trace and $\sum_i u_i x u_i^* = c \cdot E(x)$ for $x \in C_B(A)$, where $\{(u_i, u_i^*)\}$ is a quasi-basis of E and c is the index of E.

(3) If $C_B(A) \neq \mathbb{C}$, then $\{\mathrm{Index}(E) : \ E \in \epsilon_0(B, A)\} = [\mathrm{Index}(E_0), \infty)$.

The second and third assertions together imply that there exists a unique conditional expectation of index-finite type from B onto A if and only if the relative commutant $C_B(A)$ is trivial. $\mathrm{Index}(E_0)$ is called the *minimal index* for a pair $A \subset B$ and denoted by $[B : A]_0$. The following multiplicativity of the minimal index was shown in [6].

Theorem 2.2 ([6, Theorem 3]). *Let $1_C \in A \subset B \subset C$ be unital C^*-algebras with trivial centers. If $E : B \to A$ and $F : C \to B$ are conditional expectations of index finite type, then $E \circ F$ is minimal if and only if E and F are minimal. Moreover,*

$$[C : A]_0 = [C : B]_0 \cdot [B : A]_0.$$

3. Minimal index

Let α be an action of a finite group G on a unital C^*-algebra A by automorphisms and $A \times_\alpha G$ its C^*-crossed product, that is, the universal C^*-algebra generated by A and the unitaries $\{u_g : g \in G\}$ with $\alpha(a) = u_g a u_g^*$ for $g \in G$ and $a \in A$. Then there exists a canonical conditional expectation $E : A \times_\alpha G \to A$ defined by

$$E(\sum_g a_g u_g) = a_e,$$

for $a_g \in A$ and $g \in G$, where e denotes the identity of the group G.

Lemma 3.1. *Under the situation as above, the canonical conditional expectation E is of index-finite type with a quasi-basis $\{(u_g, u_g^*) : g \in G\}$ and $\mathrm{Index}(E) = \sum_{g \in G} u_g u_g^* = |G|$, the order of G.*

Recall that $[M_n(\mathbb{C}) : \mathbb{C}]_0 = n^2$ ([13, Example 2.12.5]). In the following we show that the same value of minimal index is obtained for the pair $(M_n(A), A)$, where $M_n(A)$ is the matrix algebra with entries in a simple C^*-algebra A. To begin with, we prove that $\epsilon_0(M_n(A), A) \neq \emptyset$, where A is regarded as a unital C^*-subalgebra of $M_n(A)$ via the map $a \mapsto \{\delta_{i,j} a\}_{ij}$.

Lemma 3.2. *Let A be a simple unital C^*-algebra. The canonical conditional expectation $E : M_n(A) \to A$ given by $E(\{a_{ij}\}) = 1/n \sum_i a_{ii}$ is of index-finite type and $\mathrm{Index}(E) = n^2$.*

Proof. Let $\{e_{ij}\}$ be the standard matrix units of $M_n(\mathbb{C})$. Let $v_{ij} = n e_{ji}$, $i, j = 1, \ldots, n$. Then $\{(e_{ij}, v_{ij})\}$ forms a quasi-basis for E. In fact, for each $x = \{x_{ij}\} \in M_n(A)$, write $x = \sum_{k,l} x_{kl} e_{kl}$. Then

$$e_{ij} E(v_{ij} x) = e_{ij} E(v_{ij} \sum_{k,l} x_{kl} e_{kl}) = e_{ij} E(n \sum_l x_{il} e_{jl}) = e_{ij}(x_{ij} \cdot 1),$$

so that $\sum_{i,j} e_{ij} E(v_{ij} x) = x$. Similarly one can show that $\sum_{i,j} E(x e_{ij}) v_{ij} = x$, and $\mathrm{Index}(E) = \sum_{i,j} e_{ij} v_{ij} = n^2$ follows easily. $\square$

Lemma 3.3 ([3, Lemma 2.12.2])**.** *For $E, F \in \epsilon_0(B, A)$, $E|_{C_B(A)} = F|_{C_B(A)}$ implies $E = F$.*

Proposition 3.4. *Let A be a simple unital C^*-algebra. Then the map $E : M_n(A) \to A$ defined by $E(\{a_{ij}\}) = 1/n \sum_i a_{ii}$ is the one with minimal index in $\epsilon_0(M_n(A), A)$.*

Proof. For $n \geq 2$, the relative commutant $C_{M_n(A)}(A) \cong M_n(\mathbb{C})$ is nontrivial, so that there are infinitely many conditional expectations. Let τ be the normalized canonical faithful trace from $M_n(\mathbb{C})$ onto $\mathbb{C}$, which is the restriction of E to the relative commutant $C_{M_n(A)}(A)$. If F is the minimal conditional expectation from $M_n(A)$ to A, then $\rho = F|_{C_{M_n(A)}(A)}$ must be a normalized trace on $C_{M_n(A)}(A)$ by Theorem 2.1(2). Since there exists a unique normalized trace on $C_{M_n(A)}(A)$ ($\cong M_n(\mathbb{C})$), we see that $\tau = \rho$, i.e., $E|_{C_{M_n(A)}(A)} = F|_{C_{M_n(A)}(A)}$, and this proves the assertion by Lemma 3.3. $\square$

Let $E : B \to A$ be a conditional expectation. Then $\mathcal{E}_0 = B$ can be viewed as a right pre-Hilbert A-module with an A-valued inner product $\langle x, y \rangle := E(x^* y)$, $x, y \in B$, and its completion $\mathcal{E}$ with respect to the norm induced by the inner product is a right Hilbert A-module. If E is of index-finite type, then $\mathcal{E}_0$ is complete ([7, Lemma 1.11]). It is well known that the set $\mathcal{L}_A(\mathcal{E})$ of all adjointable A-module homomorphisms $T : \mathcal{E} \to \mathcal{E}$ becomes a C^*-algebra in the usual operator norm $\|T\| = \sup\{\|T\xi\| : \|\xi\| = 1\}$. Each element $b \in B$ acts on $\mathcal{E}_0 = B$ by left multiplication $\lambda(b)$, which extends to $\mathcal{E}$ so that the map $\lambda : B \to \mathcal{L}_A(\mathcal{E})$ is injective. For $x \in \mathcal{E}_0$, define $e_A(x) := E(x) \in B = \mathcal{E}_0$. Then e_A extends to $\mathcal{E}$ since $\|e_A\| \leq 1$ and it turns out to be a projection in $\mathcal{L}_A(\mathcal{E})$. The C^*-subalgebra $C^*(B, e_A)$ of $\mathcal{L}_A(\mathcal{E})$ generated by $\{\lambda(x)e_A\lambda(y) : x, y \in B\}$ is called the *C^*-basic construction*. If E is of index-finite type, then $C^*(B, e_A)$ is exactly the whole algebra $\mathcal{L}_A(\mathcal{E})$. Since λ is injective, we simply write x for $\lambda(x)$, $x \in B$.

Let G be a finite group and α be an action of G on a simple unital C^*-algebra A for which the crossed product $B = A \times_\alpha G$ is simple. It is known in [13, p. 106] that if $\{u_g\}_{g \in G}$ is the family of unitaries implementing $\{\alpha_g\}_{g\in G}$, then the element $p = (p_{gh}) \in M_n(A)$, $n = |G|$, $p_{gh} = E(u_g^* u_h)$, is a projection. Furthermore the map

$$\begin{array}{cccl} \pi & : & C^*(B, e_A) & \longrightarrow & p(M_n(A))p \\ & & \pi(xe_Ay) & \longmapsto & (E(u_g^* x)E(yu_h))_{gh} \quad \in M_n(A) \end{array}$$

is an isomorphism. Note that π is surjective since

$$p_{gh} = E(u_g^* u_h) = E(u_{g^{-1}h}) = \delta_{g,h} \cdot 1,$$

and so $p = \sum_{g\in G} e_{gg} = 1$, the identity matrix, where $\{e_{gh}\}$ is the usual matrix unit for $M_n(\mathbb{C})$. Now let $E_B : C^*(B, e_A) \to B$ be the dual conditional expectation of the canonical map $E : B = A \times_\alpha G \to A$ defined by

$$E_B(xe_Ay) = |G|^{-1}xy, \ x, y \in B;$$

then it turns out to be of index-finite type with Index(E_B)=Index(E) ([13, Proposition 2.3.4]). From [13, Proposition 1.7.1] we also see that

$$\mathrm{Index}(E \circ E_B) = \mathrm{Index}(E)\mathrm{Index}(E_B) = n^2,$$

where $E \circ E_B : C^*(B, e_A) \to A$ is the composition of E and E_B. Note that for $x = \sum_{k\in G} x_k u_k$, $y = \sum_{l\in G} y_l u_l \in B$,

$$(E \circ E_B)(xe_Ay) = E(n^{-1}xy) = n^{-1}E(\sum_{k,l} x_k\alpha_k(y_l)u_{kl}) = \sum_{g\in G} x_g\alpha_g(y_{g^{-1}}).$$

On the other hand, $\pi(xe_Ay) = (E(u_g^* x)E(yu_h))_{gh} = (\alpha_{g^{-1}}(x_g)y_{h^{-1}})_{gh} \in M_n(A)$. Define a map $F : M_n(A) \to A$ by

$$F((a_{gh})_{gh}) = n^{-1} \sum_{g\in G} \alpha_g(a_{gg}) \in A,$$

where we regard A as a unital C^*-subalgebra of $M_n(A)$ by the map

$$\psi : a \mapsto \sum_{g\in G} \alpha_{g^{-1}}(a)e_{gg}.$$

Then it is easy to see that F is a conditional expectation from $M_n(A)$ onto A. Moreover the above calculation shows that $F \circ \pi = E \circ E_B$, and hence F is of index-finite type and Index(F)= n^2. In fact, $\{(\sqrt{n}e_{gh}, \sqrt{n}e_{hg})\}_{g,h}$ forms a quasi-basis for F since, for $x = (a_{gh})_{gh} \in M_n(A)$,

$$\begin{aligned}\sqrt{n}e_{gh}F(\sqrt{n}e_{hg}(a_{kl})_{kl})) &= ne_{gh}F(e_{hg}\sum a_{kl}e_{kl})\\ &= ne_{gh}F(\sum_l a_{gl}e_{hl})\\ &= e_{gh}\psi(\alpha_h(a_{gh})) = a_{gh}e_{gh},\end{aligned}$$

so $\sum_{g,h}\sqrt{n}e_{gh}F(\sqrt{n}e_{hg}x) = \sum_{gh} a_{gh}e_{gh} = x$. The following lemma slightly generalizes Proposition 3.4.

Lemma 3.5. *Let α be an action of a finite group G on a simple unital C^*-algebra A. Then the conditional expectation $F : M_n(A) \to A$ defined above has the minimal index, where each element $a \in A$ is identified with a diagonal matrix $(\alpha_{g^{-1}}(a))_{gg}$, so that A is a C^*-subalgebra of $M_n(A)$ with the common unit.*

Proof. Let $(a_{gh})_{gh} \in C_{M_n(A)}(A)$. Then we have

$$(a_{gh})_{gh}(\alpha_{g^{-1}}(a))_{gg} = (\alpha_{g^{-1}}(a))_{gg}(a_{gh})_{gh},\ a \in A.$$

A simple calculation shows that $a_{gg} \in Z(A) = \mathbb{C}\cdot 1$ for each $g \in G$ and the relative commutant $C_{M_n(A)}(A)$ contains the diagonal scalar matrices $\sum_{g\in G}\lambda_g e_{gg},\ \lambda_g \in \mathbb{C}$, so $C_{M_n(A)}(A)$ is non-trivial and there exist infinitely many conditional expectations from $M_n(A)$ onto A. But the restriction $F|_{C_{M_n(A)}(A)}$ coincides with the normalized trace by definition. Furthermore, for the quasi-basis $\{(\sqrt{n}e_{gh}, \sqrt{n}e_{hg})\}_{g,h}$ of F and for each $z = (z_{kl})_{k,l} \in C_{M_n(A)}(A)$,

$$\sum_{g,h}(\sqrt{n}e_{gh})z(\sqrt{n}e_{hg}) = \sum_{g,h} nz_{hh}e_{gg} = \sum_g n(\sum_h z_{hh})e_{gg},$$

which is identified with an element $\sum_h nz_{hh} \in A$ $(z_{hh} \in \mathbb{C})$. On the other hand,

$$\sum_h nz_{hh} = n^2\cdot\frac{1}{n}\sum_h z_{hh} = n^2F(z) = \mathrm{Index}(F)F(z).$$

Therefore by Theorem 2.1(2), F is the minimal conditional expectation. □

Recall that if A is simple unital and α is an outer action of a finite group, then the crossed product $A\times_\alpha G$ is always simple ([8]), while the converse is not true in general; Connes shows in [1] that, for each number p and each p-th root of unity γ, there is an automorphism α of the UHF-algebra $\bigotimes M_p$ such that the period of α is pk, k is the order of γ, and α^p is inner, for which the crossed product $\bigotimes M_p \times_\alpha \mathbb{Z}_{pk}$ is simple.

Theorem 3.6. *Let A be a simple unital C^*-algebra and α be an action of a finite group G on A for which the crossed product $A\times_\alpha G$ is simple. Then the canonical conditional expectation $E : A\times_\alpha G \to A$ defined by $E(\sum a_g u_g) = a_e$ is minimal.*

Proof. We have seen that the composition $E\circ E_B$ is a conditional expectation of index-finite type from $C^*(B, e_A)$ onto A, and $E\circ E_B = F\circ\pi$, where $B = A\times_\alpha G$,π : $C^*(B, e_A) \to M_n(A)$ is an isomorphism and F is the expectation discussed in the above lemma. Since F has the minimal index, by Theorem 2.2, E and E_B must be the minimal ones, respectively. □

4. Outer actions and uniqueness of conditional expectations

As noted in the paragraph following Theorem 2.1, if $\epsilon_0(B,A)\neq\emptyset$ there exists a unique conditional expectation from B onto A if and only if the relative commutant $C_B(A)$ is trivial. In fact, if $E,F\in\epsilon_0(B,A)$ and $\{(u_i,v_i)\,:\,i=1,\dots,n\}$ is a quasi-basis for E, then there exists a unique invertible element $h\in C_B(A)$ with $E(h)=1$ such that $F(x)=E(xh)$ for each $x\in B$. Furthermore, in this case $\{(u_i,h^{-1}v_i)\}$ is a quasi-basis for F $(=E_h)$ (see [13]).

Remark 4.1. If α_g is an inner automorphism for each $g\in G$, then it is known ([2, IIC]) that the map

$$au_g\mapsto av_g\otimes\lambda_g : A\times_\alpha G\to A\otimes C^*(G),\ \ a\in A, g\in G,$$

is an isomorphism, where $\{\lambda_g\}$ are the unitaries generating $C^*(G)$ and $\{v_g\}$ are unitaries in A with $\alpha_g(a)=v_gav_g^*,\ a\in A$. Thus the relative commutant $C_{A\times_\alpha G}(A)$ is isomorphic to $Z(A)\otimes C^*(G)$, so nontrivial if $|G|>1$. Therefore there are infinitely many conditional expectations from the crossed product $A\times_\alpha G$ onto A in case α is an inner action.

If G has a nontrivial subgroup $H=\{h\in G\,:\,\alpha_h \text{ is an inner automorphism of } A\}$, then the restriction map $E_H : A\times_\alpha G\to A\times_\alpha H$ given by $E_H(\sum_{g\in G}a_gu_g)=\sum_{h\in H}a_hu_h$ is a conditional expectation by [10, Proposition 3.1]. Furthermore E_H is of index-finite type by [10, Theorem 3.4] since G is a finite group. As discussed in the preceding paragraph there are infinitely many conditional expectations from $A\times_\alpha H(\cong A\otimes C^*(H))$ onto A and their compositions with E_H give infinitely many conditional expectations of index-finite type from $A\times_\alpha G$ onto A.

Now, we consider outer actions of finite groups. The condition that the crossed product $A\times_\alpha G$ is simple is necessary for outerness of the action α in general. In the following we prove an equivalent condition for outerness of α. If H is a subgroup of G, then we simply denote the action α restricted to H by α again.

Proposition 4.2. *Let α be an action of a finite group G on a simple unital C^*-algebra A. Then α is an outer action if and only if for each cyclic subgroup H of G the crossed product $A\times_\alpha H$ is simple.*

Proof. The simplicity of the crossed product for each subgroup when α is outer follows from [8]. For the converse suppose α_h is an inner automorphism for some $h\in G\setminus\{e\}$. Then $A\times_\alpha H\cong A\otimes C^*(H)$ by Remark 4.1, where H is the cyclic subgroup of G generated by h and so cannot be simple because $A\otimes C^*(H)\cong A\otimes C(\hat H)\cong\bigoplus A$ ($|H|$-times). □

Let α be an action of a finite abelian group G on a simple unital C^*-algebra A. Recall that the *dual action* $\hat\alpha$ of the dual group $\hat G$ of G on the C^*-crossed product $A\times_\alpha G$ is defined by

$$\hat\alpha_\sigma(\sum_g a_gu_g)=\sum_g a_g\langle\sigma,g\rangle u_g,\ \ a_g\in A,\ \ g\in G,\ \ \sigma\in\hat G,$$

where $\langle\sigma,g\rangle$ is the evaluation of σ at g. If α is an outer action, then the crossed product $A\times_\alpha G$ is simple and furthermore by Takai duality we see that the double crossed product $(A\times_\alpha G)\times_{\hat\alpha}\hat G\cong M_n(A),\ \ n=|G|$, is also simple. So the dual action $\hat\alpha$ of $\hat G$ gives the simple crossed product $B\times_{\hat\alpha}\hat G$, where $B=A\times_\alpha G$. From [12, Theorem 8.10.10, 8.10.12] it follows that for a cyclic group $\mathbb{Z}_p$ of prime order an action β of $\mathbb{Z}_p$ on a simple unital C^*-algebra B is outer if and only if the crossed product $B \times_\beta \mathbb{Z}_p$ is simple. More generally, this is true for a finite abelian group G whose order is the product of distinct primes ([11, Theorem 5]) though this is not true in general as discussed before Theorem 3.6. Therefore if α is an outer action of $\mathbb{Z}_p$ on a simple unital C^*-algebra A, then its dual $\beta = \hat{\alpha}$ of $\hat{\mathbb{Z}}_p$ ($\cong \mathbb{Z}_p$) should act outerly on the crossed product $B = A \times_\alpha \mathbb{Z}_p$ since B is simple and its crossed product $B \times_\beta \hat{\mathbb{Z}}_p$ is isomorphic to the simple C^*-algebra $M_p(A)$. More generally, it follows from the proof of Theorem 4.4 below that the dual action of an outer action by a finite abelian group is always outer.

Lemma 4.3 ([12, Proposition 8.10.13])**.** *Let β be an action of a finite abelian group on a simple unital C^*-algebra B. Then $\beta_g \neq id$ is an outer automorphism if and only if $C_B(B^\beta) = \mathbb{C} \cdot 1$.*

If α is an outer action of a finite group G on a simple unital C^*-algebra A, then the conditional expectation

$$E_\alpha : A \to A^\alpha, \quad E_\alpha(a) = \frac{1}{|G|} \sum_g \alpha_g(a)$$

is of index-finite type by [13, Proposition 2.8.6] since $A \times_\alpha G$ is simple. Moreover, by the above lemma if G is abelian, there is no conditional expectation of index-finite type from A onto A^α other than E_α since $C_A(A^\alpha) = \mathbb{C} \cdot 1$. On the contrary if $G = K \times H$ is a finite abelian group, K acts outerly and H acts trivially on A, then $C_A(A^\alpha) = C_A(A^{\alpha|_K}) = \mathbb{C} \cdot 1$. Thus there exists a unique conditional expectation of index-finite type from A onto A^α, the canonical one, even though the action α is not outer.

While outerness of the action is strictly stronger than the uniqueness of conditional expectation from A onto A^α as we have seen above, in the following we prove that the outerness of the action α is equivalent to the uniqueness of the conditional expectation of index-finite type from $A \times_\alpha G$ onto A.

Theorem 4.4. *Let α be an action of a finite group G on a simple unital C^*-algebra A, and $E : A \times_\alpha G \to A$ be the canonical conditional expectation. Then α is an outer action if and only if $\epsilon_0(A \times_\alpha G, A) = \{E\}$.*

Proof. If α is not outer, then there are infinitely many conditional expectations by Remark 4.1.

For the converse suppose that α is an outer action. We first assume that the group G is abelian. It is enough to show that the relative commutant $C_{A\times_\alpha G}(A)$ is trivial. Let $B := A \times_\alpha G$. If $x = \sum_{g\in G} x_g u_g \in C_B(A)$, then, for any $a \in A$, we have

$$xa = \sum x_g u_g a = \sum x_g \alpha_g(a) u_g = \sum a x_g u_g = ax.$$

Thus for each $g \in G$, $x_g \alpha_g(a) = a x_g$, $a \in A$. Then $x_g a = a x_g$, $a \in A^\alpha$, and hence $x_g \in C_A(A^\alpha)$. Note that the relative commutant $C_A(A^\alpha)$ is trivial by Lemma 4.3. Hence $x_g \in \mathbb{C} \cdot 1$, $g \in G$. The outerness of α also implies that, for each $g \neq e$, there exists an element $a \in A$ with $a \neq \alpha_g(a)$, and then from the identity $x_g \alpha_g(a) = a x_g$, $x_g \in \mathbb{C} \cdot 1$, it follows that $x_g = 0$. Therefore $C_B(A) = \mathbb{C} \cdot 1$.

For an arbitrary finite group G let F be a conditional expectation in $\epsilon_0(A \times_\alpha G, A)$. To show $F = E$ it is enough to prove that their restrictions to each subalgebra $A \times_\alpha H$ coincide, where H is a cyclic subgroup of G. Note that $\alpha|_H$ is outer.

Then $C_{A\times_\alpha H}(A)=\mathbb{C}\cdot 1$ and the restrictions $F|_{A\times_\alpha H},\ E|_{A\times_\alpha H}$ are of index-finite type by [13, Proposition 2.10.2]. But $\epsilon_0(A\times_\alpha H, A)=\{E|_{A\times_\alpha H}\}$ by the above argument in the case of an abelian group, so that $F|_{A\times_\alpha H}=E|_{A\times_\alpha H}$. $\square$

Remarks 4.5. (a) The theorem can be shown directly using the following fact: Let α be an action of a finite group G on a simple unital C^*-algebra A. Then the following three conditions are equivalent: (1) α is outer. (2) For $g\neq e,\ \alpha_g$ is free in the sense of Kallman ([5]), that is, $ab=\alpha_g(b)a$ for all $b\in A$ implies $a=0$. (3) The relative commutant $C_{A\times_\alpha G}(A)$ is trivial. The proof is similar to the case where A is a factor.

(b) If G is abelian in Theorem 4.4, then it is easy to see that the fixed point algebra $B^{\hat\alpha}$ for the dual action $\hat\alpha$ is A. Thus if α is an outer action, then $C_B(B^{\hat\alpha})=\mathbb{C}\cdot 1$. Since each automorphism $\hat\alpha_\sigma$ ($\sigma\neq\hat e$, the identity of $\hat G$) acts nontrivially on B by the definition of dual action, it follows from Lemma 4.3 that $\hat\alpha$ is also outer.

References

[1] A. Connes,*Periodic automorphism of the hyperfinite factor of type II_1*, Acta Sci. math. 39(1977), 39–66. MR **56:**6411

[2] A. Connes,*Noncommutative geometry*, Academic Press, 1994. MR **95j:**46063

[3] F. Hiai,*Minimizing indices of conditional expectations onto a subfactor*, Publ. RIMS, Kyoto Univ. 24(1988), 673–678. MR **90a:**46157

[4] V. Jones,*Index for subfactors*, Invent. Math. 72(1983), 1–25. MR **84d:**46097

[5] R. R. Kallman,*A generalization of free action*, Duke Math. J. 36(1969), 781–789. MR **41:**838

[6] S. Kawakami and Y. Watatani,*The multiplicativity of the minimal index of simple C^*-algebras*, Proc. Amer. Math. Soc. 123(1995), 2809–2813. MR **95k:**46093

[7] T. Kajiwara and Y. Watatani,*Jones index theory by Hilbert C^*-bimodules and K-theory*, Tran. Amer. Math. Soc. to appear. CMP 98:13

[8] A. Kishimoto,*Outer automorphisms and related crossed products of simple C^*-algebras*, Comm. Math. Phys. 81(1981), 429–435. MR **83c:**46061

[9] H. Kosaki,*Extensions of Jones' theory on index to arbitrary factors*, J. Func. Anal. 66(1986), 123–140. MR **87g:**46093

[10] M. Khoshkam,*Hilbert C^*-modules and conditional expectations on crossed products*, J. Austral. Math. Soc. (Series A) 61(1996), 106–118. MR **97i:**46100

[11] D. Olesen, G. K. Pedersen, E. Stormer, *Compact abelian groups of automorphisms of simple C^*-algebras*, Invent. math. 39 (1977), 55–64. MR **55:**13250; Appendix MR **55:**13251

[12] G. K. Pedersen,*C^*-algebras and their automorphism groups*, Academic Press, 1979. MR **81e:**46037

[13] Y. Watatani,*Index for C^*-subalgebras*, Memoirs of the Amer. Math. Soc. 83 (1990). MR **90i:**46104

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