PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 7, Pages 1907–1913
S 0002-9939(00)05827-5
Article electronically published on December 4, 2000

ON CLASSIFICATION OF POLARIZED VARIETIES WITH NON-INTEGRAL NEF VALUES

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(Communicated by Ron Donagi)

ABSTRACT. Let M be an n-dimensional normal projective variety with only Gorenstein, terminal, $\mathbb{Q}$-factorial singularities. Let L be an ample line bundle on M. Let τ denote the nef value of (M, L). The classification of (M, L) via the nef value morphism is given for the situations when τ satisfies $n-3 < \tau < n-2$ or $n-4 < \tau < n-3$.

INTRODUCTION

Let M be an n-dimensional normal projective variety with only $\mathbb{Q}$-Gorenstein, terminal singularities and let L be an ample line bundle on M. K_M denotes the canonical divisor, or the canonical sheaf of M. Assume that K_M is not nef. The nef value of (M, L) is a real number defined as $\tau = \min\{r \in \mathbb{R}, K_M + rL \text{ is nef}\}$.

By the Kawamata Rationality Theorem, τ is a rational number and by the Kawamata-Shokurov Base Point Free Theorem, $|m(K_M + \tau L)|$ is base point free for $m \gg 0$, and defines a projective surjective morphism $\varphi\colon M \to X$ onto a normal variety X. φ is called the nef value morphism.

Sommese [S] and Beltrametti and Sommese [BS1] establish and develop the theory of the adjunction theoretic classification of projective varieties. The pairs (M, L) with the nef value $\tau = n-1$, $n-2$, or $n-2 < \tau < n-1$ have been classified in [BS1], [An] and [F2].

This paper studies the pairs (M, L) such that $n-3 < \tau < n-2$ or $n-4 < \tau < n-3$. We give out the adjunction theoretic classification for these polarized varieties.

0. PRELIMINARIES

We work over the complex field $\mathbb{C}$. Throughout this paper, the notions and notations coincide with that in [BS1]. A variety means an irreducible and reduced projective scheme.

We begin by recalling some facts from adjunction theory and Mori theory. We refer for that to [BS1], [KMM] and [Mo].

In this paper, let M be a normal projective variety of dimension $n \geq 2$, and let L be an ample line bundle on M.

Received by the editors September 14, 1998 and, in revised form, November 1, 1999.
2000 *Mathematics Subject Classification.* Primary 14Jxx.
Key words and phrases. Nef value morphism, ample line bundle, projective variety.

(0.1)**Kawamata Rationality Theorem**([KMM, (4-1-1)]). *Let M be a normal variety of dimension n with terminal singularities and let r be the index of M. Let $f\colon M \to \mathcal{S}$ be a projective morphism onto variety $\mathcal{S}$. Let L be an f-ample line bundle on M. If K_M is not f-nef, then $\tau = \min\{r \in \mathbb{R}, K_M + rL \text{ is } f\text{-nef}\}$ is a positive rational number. Furthermore expressing $r\tau = u/v$ with u, v positive coprime integers, we have $u \le r(b+1)$ where $b = \max\{\dim f^{-1}(s), s \in \mathcal{S}\}$.*

(0.2)**Theorem**([BS1, (0.2.3)]). *Let T be the locus of terminal singularities on M. Then $\operatorname{cod}_M T \ge 3$.*

(0.3)**Special varieties**([BS1]). Let M be an n-dimensional normal projective variety with Gorenstein singularities. We say that (M, L) is a *Del Pezzo* variety (resp. a *Mukai* variety) if $K_M \approx -(n-1)L$ (resp. $K_M \approx -(n-2)L$); (M, L) is a *scroll* (resp. a *quadric* variety) if $K_M \approx -(n-1)L$ (resp. $K_M \approx -(n-2)L$); (M, L) is a *scroll* (resp. a *quadric fibration*; resp. a *Del Pezzo fibration*; resp. a *Mukai fibration*) over a normal variety X of dimension m if there exists a surjective morphism with connected fibers $\varphi\colon M \to L$ such that $K_M + (n-m+1)L \approx \varphi^* H$ (resp. $K_M + (n-m)L \approx \varphi^* H$; resp. $K_M + (n-m-1)L \approx \varphi^* H$) for some ample line bundle H on X. Here $\approx$ denotes the linearly equivalence relation.

(0.4)**Kobayashi-Ochiai Theorem**([F1, (2,2), (2.3)]). *Let M be an n-dimensional normal projective variety and let L be an ample line bundle on M. Then*

(i) $(M, L) \cong (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$ *if* $K_M + (n+1)L \sim O_M$.
(ii) $(M, L) \cong (Q^n, O_{Q^n}(1))$, *$Q^n$ is a hyperquadric in $\mathbb{P}^{n+1}$, if $K_M + nL \sim O_M$. Here $\sim$ denotes the numerically equivalence relation.*

(0.5)**Theorem**([BS1, (2.3)]). *Let M be an n-dimensional irreducible normal projective variety with terminal singularities and let L be an ample line bundle on M. Assume that X is $\mathbb{Q}$-factorial and that $K_M + nL$ is nef and big. Then $K_M + nL$ is ample.*

(0.6)**Theorem**([Ma, (2.1)]). *Let M be an n-dimensional normal projective variety with only terminal singularities. Let L be an ample line bundle on M. Then $K_M + nL$ is nef unless $(M, L) \cong (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$. In particular, $K_M + (n+1)L$ is always nef and is ample unless $(M, L) \cong (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$.*

(0.7)**Theorem**([BS1, (0.8.3)]). *A rational number τ is the nef value of (M, L) if and only if $K_M + \tau L$ is nef but not ample.*

1. Classifications of polarized varieties

The following lemma is needed in the sequel.

(1.1)**Lemma.** *Let M be an n-dimensional normal projective variety with Gorenstein, terminal, $\mathbb{Q}$-factorial singularities, and let L be an ample line bundle on M. Let τ be the nef value of (M, L). By the Kawamata rationality theorem, $r = u/v$ with u, v positive coprime integers. Assume that $n - k < \tau < n - k + 1$ for some positive integer $k < n$. Then*

(i) *$n \le 2k$, and if $n = 2k$ then $(M, L) \cong (\mathbb{P}^n, O_{\mathbb{P}^n}(2))$;*
(ii) *$2 \le v \le \frac{n}{n-k}$ and $\tau = n - k + \frac{i}{v}$ for some positive integer $i < v$ and i, v coprime.*

Proof. From the assumption, we see that $r = u/v$ is not an integer and thus $v \geq 2$. By the Kawamata rationality theorem, we have that $u \leq n+1$, and thus $v(n-k) < u \leq n+1$ or $v(n-k) \leq n$. It follows that $2 \leq v \leq \frac{n}{n-k}$. Moreover, since $v(n-k) < u < v(n-k)+v$, we get that $\tau = n-k+\frac{i}{v}$ for some positive integer $i \leq v$. Note that $u = v(n-k)+i$. If i, v are not coprime, then u, v are not coprime, a contradiction. Hence we find (1.1), (ii).

Let $\varphi\colon M \to X$ be the nef value morphism of (M, L). Suppose $n = 2k$. By the above and the Kawamata rationality theorem, we have

$$u = vk + i \leq \max_{x \in X}\{\dim \varphi^{-1}(x)\} + 1 \leq 2k+1.$$

Since $v \geq 2$ and $i \geq 1$, it follows that $u = 2k+1$ and $v = 2$. Thus $\dim \varphi^{-1}(x) = n$ for some $x \in X$, that is, φ contracts M to a point. We have $2K_M + (n+1)L \approx O_M$. By [BS1, (0.11)] there exists an ample line bundle A on M such that $K_M \approx -(n+1)A$ and $L \approx 2A$. By the Kobayashi-Ochiai Theorem (0.4), $(M, A) \cong (\mathbb{P}^n, P_{\mathbb{P}^n}(1))$ and thus $(M, L) \cong (\mathbb{P}^n, O_{\mathbb{P}^n}(2))$.

The main results of this paper are as follows.

(1.2)**Theorem.** *Let M be a normal projective variety of dimension $n \geq 5$ with Gorenstein, terminal, $\mathbb{Q}$-factorial singularities. Let L be an ample line bundle on M. Let τ denote the nef value of (M, L) and $\varphi\colon M \to X$ the nef value morphism of (M, L), and let F be any general fiber of φ. Assume that $n-4 < \tau < n-3$. Then (M, L) must be one of the following.*

- (i) $n = 8, \tau = \frac{9}{2}, (M, L) \cong (\mathbb{P}^8, O_{\mathbb{P}^8}(2))$;
- (ii) $n = 7, \tau = \frac{7}{2}$, *either* $(M, L) \cong (Q^7, O_{Q^7}(2))$ *or M is a* $\mathbb{P}^6$*-bundle over a curve under φ and* $(F, L_F) \cong (\mathbb{P}^6, O_{\mathbb{P}^6}(2)$;
- (iii) $n = 6, \tau = \frac{5}{2}, A := K_M + 3L$ *is ample on M. (M, A) is one of the following:*
  - a) *a Del Pezzo variety and* $L \approx 2A$;
  - b) *a quadric fibration over a curve, and* $(F, L_F) \cong (Q^5, O_{Q^5}(2))$;
  - c) *a scroll over a surface, and* $(F, L_F) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(2))$;
  - d) $2K_M + 5L$ *is nef and big, φ contracts disjoint divisors E to smooth points, and each* $(E, L_E) \cong (\mathbb{P}^5, O_{\mathbb{P}^5}(2))$;
- (iii′) $n = 6, \tau = \frac{7}{3}, (M, L) \cong (\mathbb{P}^6, O_{\mathbb{P}^6}(3))$;
- (iv) $n = 5$, *τ is either* $\frac{6}{5}, \frac{5}{4}, \frac{5}{3}, \frac{4}{3}$, *or* $\frac{3}{2}$.

  For $\tau = \frac{6}{5}$, $(M, L) \cong (\mathbb{P}^5, O_{\mathbb{P}^5}(5))$;

  For $\tau = \frac{5}{4}$, $(M, L) \cong (Q^5, O_{Q^5}(4))$, *or (M, A) is a scroll over a curve for some ample line bundle A on M and* $(F, L_F) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(4))$;

  For $\tau = \frac{5}{3}$, $(M, L) \cong (Q^5, O_{Q^5}(3))$, *or (M, A) is a scroll over a curve for some ample line bundle A on M and* $(F, L_F) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(3))$;

  For $\tau = \frac{4}{3}, A := 2K_M + 3L$ *is ample on M, (M, A) is one of the following:*
  - a) *a Del Pezzo variety with* $L \approx 3A$;
  - b) *a quadric fibration over a curve with* $(F, L_F) \cong (Q^4, O_{Q^4}(3))$;
  - c) *a scroll over a surface with* $(F, L_F) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(3))$; *or*
  - d) $3K_M + 4L$ *is nef and big, and φ contracts disjoint divisors E to points and each* $(E, L_E) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(3))$.

  For $\tau = \frac{3}{2}$, $A := K_M + 2L$ *is an ample line bundle on M, (M, A) is either a Mukai variety with $L \approx 2A$, a Del Pezzo fibration over a curve, a quadric fibration over a surface, a scroll over a 3-dimensional variety, or* $2K_M + 3L$ *is* *nef and big, φ contracts disjoint divisors E to curves or points, the structure of each E is as follows.*

a) *if $\varphi(E)$ is a point, then (E, L_E) is $(\mathbb{P}^4, O_{\mathbb{P}^4}(2))$, or $(Q^4, O_{Q^4}(2))$;*

b) *if $\varphi(E)$ is a curve, then $(F, L_F) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(2))$.*

Proof. By Lemma (1.1), $n \leq 8$ and when $n = 8$, $(M, L) \cong (\mathbb{P}^8, O_{\mathbb{P}^8}(2))$. Moreover, from the proof of Lemma (1.1), we have $\tau = \frac{9}{2}$, as in (1.2) (i).

(ii) Let $n = 7$. By Lemma (1.1), $2 \leq v \leq \frac{7}{3}$. Thus $v = 2$ and by Lemma (1.1) again $\tau = 7 - 4 + \frac{1}{2} = \frac{7}{2}$, and so $u = 7$. By Kawamata Rationality Theorem (0.1), we have $7 \leq \max_{x \in X}\{\dim \varphi^{-1}(x)\} + 1 \leq 8$ or $6 \leq \max_{x \in X}\{\dim \varphi^{-1}(x)\} \leq 7$.

Suppose $\max_{x \in X}\{\dim \varphi^{-1}(x)\} = 7$; then φ contracts M to a point, and we have $2K_M + 7L \approx O_M$. By [BS1, (0.11)], there exists an ample line bundle A on M such that $K_M \approx -7A$ and $L \approx 2A$. By the Kobayashi-Ochiai Theorem (0.4), we get that $(M, A) \cong (Q^7, O_{Q^7}(1))$. Hence $(M, L) \cong (Q^7, O_{Q^7}(2))$.

Suppose $\max_{x \in X}\{\dim \varphi^{-1}(x)\} = 6$; then $\dim X \geq 1$. We claim that $\dim X = 1$.

Indeed, otherwise, assume that $d := \dim X \geq 2$. First, let $d < n$. Let F be a general fiber of $\varphi\colon M \to X$. Then $\dim F \leq 5$. Since $\tau = \frac{7}{2}$, we have $2K_F + 7L_F \approx O_F$. By [BS1, (0.11)], there exists an ample line bundle A on F such that $K_F + 7A \approx O_F$ and $7 \leq \dim F + 1 \leq 6$, a contradiction. Second, let $d = n$. Then $2K_M + 7L$ is nef and big. Write $K_M + 7(K_M + 4L) = 4(2K_M + 7L)$. Then $A := K_M + 4L$ is an ample line bundle on M, and $K_M + 7A$ is nef and big but not ample. However, by Theorem (0.5), $K_M + 7A$ is ample. We get a contradiction.

Now, we have that $\operatorname{cod}_M \operatorname{Sing}(M) \geq 3 > \dim X$ by Theorem (0.2) and that $u = \max_{x \in X}\{\dim \varphi^{-1}(x)\} + 1 = 7$. By [BS2, (1.4)], M is a $\mathbb{P}^6$-bundle over a curve X under φ. Finally, for any general fiber F of φ, we have $2K_F + 7L_F \approx O_F$. Note that $F \cong \mathbb{P}^6$ and $K_M \cong O_{\mathbb{P}^6}(-7)$. We get that $(F, L_F) \cong (\mathbb{P}^6, O_{\mathbb{P}^6}(2)$, as in (1.2) (ii).

(iii) Let $n = 6$. By Lemma (1.1), $2 \leq v \leq \frac{6}{6-4} = 3$, so $v = 2$ or $v = 3$.

Let $v = 2$ and $\tau = \frac{u}{v} = 6 - 4 + \frac{1}{2} = \frac{5}{2}$. Write $K_M + 5(K_M + 3L) = 3(2K_M + 5L)$. Then $A := K_M + 3L$ is an ample line bundle on M.

First, assume that $\varphi\colon M \to X$ has lower dimensional image. If $\dim X = 0$, then $K_M + 5A \approx O_M$ and $L \approx 2A$. By definition (0.3), (M, A) is a Del Pezzo variety.

Let $\dim X \geq 1$, and let F be any general fiber of φ. Since $2K_M + 5L_M \approx \varphi^* H$ for some ample line bundle H on X by [KMM, (3-2-1)], $K_M + 5A \approx \varphi^*(3H)$.

If $\dim X = 1$, (M, A) is a quadric fibration over X under φ by definition. Moreover, $K_F + 5A_F \approx O_F$. Since $\dim F = 5$, $(F, A_F) \cong (Q^5, O_{Q^5}(1))$ by (0.4) [F1]. Note that $K_F \cong O_{Q^5}(-t)$ and $2K_F + 5L_F \cong O_F$. We get $(F, L_F) \cong (Q^5, O_{Q^5}(2))$.

If $\dim X = 2$, (M, A) is a scroll over X under φ since $K_M + 5A \approx \varphi^*(3H)$. Note that $\dim F = 4$ and $K_F + 5A_F \approx O_F$. We have $(F, A_F) \cong (P^4, O_{\mathbb{P}^4}(1))$. Since $K_F \cong O_{\mathbb{P}^4}(-5)$ and $2K_F + 5L_F \cong O_F$, $L_F \cong O_{\mathbb{P}^4}(2)$ and so $(F, L_F) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(2))$.

If $\dim X \geq 3$, then $\dim F \leq 3$. Since $K_F + 5A_F \approx O_F$, $K_F + 5A_F$ is ample on F by Theorem (0.6). We get a contradiction. Therefore, $\dim X \leq 2$.

Second, assume that $\varphi\colon M \to X$ is birational. Then $K_M + 5A$ is nef and big but not ample. By Theorem (0.7) the nef value of (M, A) is 5. By [An, Theorem 1], φ contracts disjoint divisors $E \cong \mathbb{P}^5$ to smooth points. Moreover, $L_E(E) \cong O_{\mathbb{P}^5}(-1)$ and $A_E \cong O_{\mathbb{P}^5}(1)$. Note that $A_E \approx K_M|_E + 3L_E$ and $K_M|_E + O_E(E) \cong K_E \cong O_{\mathbb{P}^5}(-6)$. We find that $L_E \cong O_{\mathbb{P}^5}(2)$. Therefore, $(E, L_E) \cong (\mathbb{P}^5, O_{\mathbb{P}^5}(2))$.

(iii′) Let $n = 6$ and $v = 3$. Note that $2 < \tau = \frac{u}{3} < 3$ and $u \leq 7$. We have $\tau = \frac{7}{3}$. Then $u = 7 \leq \max_{x \in X}\{\dim \varphi^{-1}(x)\} + 1 \leq 7$ by (0.1), and so

$\max_{x\in X}\{\dim\varphi^{-1}(x)\} = 6$. It follows that φ contracts M to a point. Thus $3K_M + 7L \approx O_M$. Let $A = 2K_M + 5L$, then $K_M + 7A = 5(3K_M + 7L) \approx O_M$. Thus, by (0.4), $(M, A) \cong (\mathbb{P}^6, O_{\mathbb{P}^6}(1))$. Since $K_M \approx O_{\mathbb{P}^6}(-7)$, we find that $(M, L) \cong (\mathbb{P}^6, O_{\mathbb{P}^6}(3))$.

(iv) Let $n = 5$. By Lemma (1.1) and the Kawamata Rationality Theorem (0.1), we have that $2 \le v \le \frac{5}{5-4} = 5$, $1 < \tau = 1 + \frac{i}{v} < 2$, and $u \le \max_{x\in X}\{\dim\varphi^{-1}(x)\} + 1 \le 6$. It follows that τ is either $\frac{6}{5}$, $\frac{5}{4}$, $\frac{5}{3}$, $\frac{4}{3}$, or $\frac{3}{2}$.

For $\tau = \frac{6}{5}$, we have $\max_{x\in X}\{\dim\varphi^{-1}(x)\} + 1 = 6$ and so φ contracts M to a point. Thus $5K_M + 6L \approx O_M$. Write $K_M + 6(4K_M + 5L) = 5(5K_M + 6L)$. Then $A := 4K_M + 5L$ is an ample line bundle on M, and $K_M + 6A \approx O_M$. By (0.4) [F1] we have $(M, A) \cong (\mathbb{P}^5, O_{\mathbb{P}^5}(1))$. Clearly, $L \cong O_{\mathbb{P}^5}(5)$ and hence $(M, L) \cong (\mathbb{P}^5, O_{\mathbb{P}^5}(5))$.

For $\tau = \frac{5}{4}$, there exists an ample line bundle H such that $4K_M + 5L \approx \varphi^* H$. Write $K_M + 5(3K_M + 4L) = 4(4K_M + 5L)$. Then $A := 3K_M + 4L$ is an ample line bundle on M and $K_M + 5A \approx \varphi^*(4H)$.

First, assume that φ has lower dimensional image. If X is a point, then $K_M + 5A \approx O_M$ and thus $(M, A) \cong (Q^5, O_{Q^5}(1)$ by (0.4) [F1]. From $A = 3K_M + 4L$, we find that $L \cong O_{Q^5}(4)$. Therefore, $(M, L) \cong (Q^5, O_{Q^5}(4))$.

If $\dim X = 1$, then (M, A) is a scroll over X under φ since $K_M + 5A \approx \varphi^*(4H)$. Note that $K_F + 5A_F \approx O_F$. We have $(F, A_F) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(1))$. Clearly, $L_F \cong O_{\mathbb{P}^4}(4)$.

We claim that $\dim X \le 1$. Indeed, otherwise, let $\dim X \ge 2$; then $\dim F \le 3$ and $K_F + 5A_F \approx O_F$. But, by Theorem (0.6), $K_F + 5A_F$ is ample on F, a contradiction.

Second, assume that $\varphi\colon M \to X$ is birational. Then $K_M + 5A$ is nef and big but not ample. But, by Theorem (0.5), $K_M + 5A$ is ample. We get a contradiction. This shows that φ cannot be birational.

Similarly, for $\tau = \frac{5}{3}$, we have that $(M, L) \cong (Q^5, O_{Q^5}(3))$ or (M, A) is a scroll over a curve under φ, and $(F, L_F) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(3))$, where $A := K_M + 2L$ is an ample line bundle on M.

For $\tau = \frac{4}{3}$, $K_M + 4(2K_M + 3L) = 3(3K_M + 4L) \approx \varphi^*(3H)$ for some ample line bundle H on X. $A := 2K_M + 3L$ is an ample line bundle on M. When φ has lower dimensional image, from $K_M + 4A \approx \varphi^*(3H)$, we have that (M, A) is either a Del Pezzo variety with $L \approx 3A$, a quadric fibration over a curve with $(F, L_F) \cong (Q^4, Q_{Q^4}(3))$, or a scroll over a surface with $(F, L_F) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(3))$. When φ is birational, then $K_M + 4A$ is nef and big but not ample, by [An, Theorem 1] φ contracts disjoint divisors E to points. Moreover, $(E, A_E) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(1))$ and $O_E(E) \cong O_{\mathbb{P}^4}(-1)$. Thus $(E, L_E) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(3))$.

For $\tau = \frac{3}{2}$, write $K_M + 3(K_M + 2L) = 2(2K_M + 3L)$. Then $A := K_M + 2L$ is ample on M. Since $2K_M + 3L \approx \varphi^* H$ for some line bundle H on X, $K_M + 3A \approx \varphi^*(2H)$. When φ has lower dimensional image, we get that (M, A) is either a Mukai variety with $L \approx 2L$, a Del Pezzo fibration over a curve, a quadric fibration over a surface, or a scroll over a 3-dimensional variety.

When φ is birational, then $K_M + 3A$ is nef and big but not ample. Let E be the exceptional locus of φ. By [An, Theorem 3] and its proof, we get that φ contracts disjoint divisors E to curves or points. Moreover, if $\varphi(E)$ is a point, then (E, L_E) is $(\mathbb{P}^4, O_{\mathbb{P}^4}(2))$, or $(Q^4, O_{Q^4}(2))$. If $\varphi(E)$ is a curve, then $(F, L_F) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(2))$.

(1.3)**Theorem.** *Let M be a normal projective variety of dimension $n \ge 4$ with Gorenstein, terminal, $\mathbb{Q}$-factorial singularities. Let L be an ample line bundle on M. Let τ be the nef value of (M, L) and $\varphi\colon M \to X$ the nef value morphism of*

(M, L). Let F be any general fiber of φ. Assume that $n-3<\tau<n-2$. Then (M, L) must satisfy one of the following:

(i) $n=6$, $\tau=\frac{7}{2}$, $(M, L) \cong (\mathbb{P}^6, O_{\mathbb{P}^6}(2))$;

(ii) $n=5$, $\tau=\frac{5}{2}$, $A := K_M+3L$ *is ample on* M, *either* $(M, L) \cong (Q^5, O_{Q^5}(2))$ *or* (M, A) *is a scroll over a curve under* φ *and* $(F, L_F) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(2))$;

(iii) $n=4$, $\tau=\frac{5}{3}, \frac{5}{4}, \frac{4}{3}$, *or* $\frac{3}{2}$.
For $\tau=\frac{5}{3}$, $(M, L) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(3))$;
For $\tau=\frac{5}{4}$, $(M, L) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(4))$;
For $\tau=\frac{4}{3}$, $A := 2K_M+3L$ *is ample on* M, *either* $(M, L) \cong (Q^4, O_{Q^4}(3))$ *or* (M, A) *is a scroll over a curve and* $(F, L_F) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(3))$.
For $\tau=\frac{3}{2}$, $A := K_M+2L$ *is ample on* M, (M, A) *is one of the following:*
  a) *A Del Pezzo variety;*
  b) *a quadric fibration over a curve;*
  c) *a scroll over a surface; or*
  d) $2K_M+3L$ *is nef and big,* φ *contracts disjoint divisors* E *to smooth points, and* $(E, L_E) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(2))$.

Proof. By Lemma (1.1) we have $n \leq 6$.

(i) Let $n=6$. Then $v=2$, $\tau=6-3+\frac{1}{2}=\frac{7}{2}$, and $(M, L) \cong (\mathbb{P}^6, O_{\mathbb{P}^6}(2))$.

(ii) Let $n=5$. Then $2 \leq v \leq \frac{t}{5-2}=\frac{5}{2}$ and thus $v=2$, $\tau=5-3+\frac{1}{2}=\frac{5}{2}$. Write $K_M+5(K_M+3L)+3(2K_M+5L)$. Then $A := K_M+3L$ is an ample line bundle on M. There exists an ample line bundle H on X such that $K_M+5A \approx \varphi^*(3H)$.

First, assume that φ has lower dimensional image. Let F be any general fiber of φ. If X is a point, then $(M, A) \cong (Q^5, O_{Q^5}(1))$ by (0.4). Thus $(M, L) \cong (Q^5, O_{Q^5}(2))$.

If $\dim X=1$, then (M, A) is a scroll over a curve under φ by definition, and $(F, L_F) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(2))$.

If $\dim X \geq 2$, then $\dim F \leq 3$ and $K_F+5A_F \approx O_F$. But by (0.6), K_F+5A_F is ample on F. We get a contradiction.

Second, assume that φ is birational. Then K_M+5A is nef and big but not ample. But by (0.5), K_M+5A is ample, a contradiction. Thus, φ cannot be birational.

(iii) Let $n=4$. Then $1<\tau=\frac{u}{v}<2$ and $u \leq 5$. Using Lemma (1.1), we find that τ is either $\frac{5}{3}, \frac{5}{4}, \frac{4}{3}$, or $\frac{3}{2}$.

For $\tau=\frac{5}{3}$, write $K_M+5(K_M+2L)=2(3K_M+5L)$ and $A := K_M+2L$ is ample on M. There exists an ample line bundle H on X such that $K_M+5A \approx \varphi^*(2H)$.

First, assume that φ has lower dimensional image. If X is a point, then $(M, A) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(1))$. Clearly, $L \cong O_{\mathbb{P}^4}(3)$. Hence $(M, L) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(3))$.

If $\dim X \geq 1$, then $\dim F \leq 3$ and $K_F+5A_F \approx O_F$. But by (0.6), K_F+5A_F is ample on F. We get a contradiction.

Second, assume that φ is birational. Then K_M+5A is nef and big but not ample. But by (0.5), K_M+5A is ample, a contradiction. Thus, φ cannot be birational.

Similarly, for $\tau=\frac{5}{4}$, we have $(M, L) \cong (\mathbb{P}^4, O_{\mathbb{P}^4}(4))$.

For $\tau=\frac{4}{3}$, $K_M+4(2K_M+3L)=3(3K_M+4L) \approx \varphi^*(3H)$ for some ample line bundle H on X. $A := 2k_M+3L$ is an ample line bundle on M. By the same way as above, we get that either $(M, L) \cong (Q^4, O_{Q^4}(3))$ or (M, A) is a scroll over a curve and $(F, L_F) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(3))$.

For $\tau=\frac{3}{2}$, write $K_M+3(K_M+2L)=2(2K_M+3L)$. Then $A := K_M+2L$ is ample on M and $K_M+3A \approx \varphi^*(2H)$ for some line bundle Y on X. When φ

has lower dimensional image, we get that (M, A) is either a Del Pezzo variety with $L \approx 2L$, a quadric fibration over a curve with $(F, L_F) \cong (Q^3, O_{Q^3}(2))$, or a scroll over a surface with $(F, L_F) \cong (\mathbb{P}^2, O_{\mathbb{P}^2}(2))$.

When φ is birational, K_M+3A is nef and big but not ample. By [An, Theorem 1], φ contracts disjoint divisors E to smooth points. Moreover, $O_E(E) \cong O_{\mathbb{P}^3}(-1))$ and $(E, A_E) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(1))$. Hence $(E, L_E) \cong (\mathbb{P}^3, O_{\mathbb{P}^3}(2))$. The proof is completed.

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