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AN INEQUALITY BETWEEN DIRICHLET AND NEUMANN EIGENVALUES IN A CENTRALLY SYMMETRIC DOMAIN

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Abstract. We prove an inequality between Dirichlet and Neumann eigenvalues of the Laplacian in a centrally symmetric Euclidean domain.

David Jerison and Nikolai Nadirashvili conjectured in [1] (Conjecture 8.6) that, for a centrally symmetric *convex* domain, λ_2, the lowest nonzero Neumann eigenvalue of the Laplacian for an eigenfunction that is even with respect to the central symmetry $x \mapsto -x$ is smaller than μ_2, the lowest Dirichlet eigenvalue for an eigenfunction that is odd with respect this symmetry. Notice that this inequality does not follow from the general inequalities between the Dirichlet and the Neumann eigenvalues: it may happen that μ_2 is the second eigenvalue of the Dirichlet Laplacian, and λ_2 is, say, 100th eigenvalue of the Neumann Laplacian.

It turned out that this conjecture, even in somewhat stronger form, has a rather simple proof. In this short paper, I present the proof. The technique is very similar to the one used in [2].

Theorem. *Let Ω be a centrally symmetric domain in $\mathbb{R}^n$ with C^2-boundary $\Gamma = \partial\Omega$ of non-negative mean curvature. Let λ_2 be the lowest non-zero eigenvalue for the Neumann Laplacian in Ω for an even eigenfunction with respect to the central symmetry, and let μ_2 be the lowest eigenvalue of the Dirichlet Laplacian in Ω for an odd eigenfunction with respect to this symmetry. Then*

$$\lambda_2 < \mu_2.$$

Proof. Let $u(x)$ be an odd eigenfunction of the Dirichlet Laplacian that corresponds to the eigenvalue μ_2. For a unit vector $\omega \in S^{n-1}$, we define the function

$$v_\omega(x) = \nabla u \cdot \omega.$$

Clearly, $v_\omega(x)$ is an even function, and

$$\int_\Omega v_\omega(x) = 0.$$

The statement of the theorem would follow from:

there exists $\omega \in S^{n-1}$ such that

$$\int_\Omega |\nabla v_\omega|^2 dx < \mu_2 \int_\Omega |v_\omega|^2 dx. \tag{1}$$

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The function $v_\omega(x)$ satisfies the equation

$$\Delta v_\omega(x) + \mu_2 v_\omega(x) = 0,$$

so

$$\int_\Omega |\nabla v_\omega|^2 dx = \mu_2 \int_\Omega |v_\omega|^2 dx + \int_\Gamma v_\omega(x) \frac{\partial v_\omega(x)}{\partial \nu} dA \tag{2}$$

where $\partial/\partial\nu$ is differentiation in the outward normal direction, and dA is the surface element on Γ. Our goal is to show that

$$\int_{S^{n-1}} d\omega \int_\Gamma v_\omega(x) \frac{\partial v_\omega(x)}{\partial \nu} dA < 0. \tag{3}$$

Here $d\omega$ is the standard measure on S^{n-1} normalized in such a way that the total area of the sphere equals 1. Clearly, (3) implies that

$$\int_\Gamma v_\omega(x) \frac{\partial v_\omega(x)}{\partial \nu} dA < 0$$

for some value of ω, and then (1) would follow from (2).

To prove (3), we will explicitly compute the integral on the left. First, on Γ, $\nabla u = (\partial u/\partial \nu)\nu$; so

$$v_\omega(x) = \frac{\partial u(x)}{\partial \nu}(\nu \cdot \omega), \quad x \in \Gamma. \tag{4}$$

Here $\nu(x)$ is the outward normal unit vector to Γ at x.

In a neighborhood of Γ, we introduce local coordinates $x = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$, in such a way that y_n is the distance from a point x to Γ taken with the positive sign if the point lies outside Ω, and taken with the negative sign otherwise; y' are the local coordinates on Γ of the point that is the closest to x. In this coordinate system, the Euclidean metric tensor equals

$$g = \begin{pmatrix} g'(y', y_n) & 0 \\ 0 & 1 \end{pmatrix}$$

where g' is a metric tensor on Γ that depends on y_n. In the y-coordinates, the Laplacian has the form

$$\Delta = \Delta' + \frac{\partial^2}{\partial y_n^2} + \frac{1}{2} \frac{\partial \log g'}{\partial y_n} \frac{\partial}{\partial y_n}$$

where Δ' is the "tangent" Laplacian, and g' is the determinant of the covariant tensor $g'(y', y_n)$. The function $u(x)$ is an eigenfunction of the Dirichlet Laplacian, so one has

$$\frac{\partial^2 u}{\partial y_n^2} = -\frac{1}{2} \frac{\partial \log g'}{\partial y_n} \frac{\partial u}{\partial y_n}, \quad y_n = 0.$$

It is a well-known fact and it is a result of an easy computation that

$$\frac{\partial \log g'}{\partial y_n} = 2H(x)$$

where $H(x)$ is the mean curvature of Γ, the sum of its principal curvatures. Therefore,

$$\frac{\partial^2 u(y', 0)}{\partial y_n^2} = -H(y') \frac{\partial u(y', 0)}{\partial y_n}. \tag{5}$$

Now, we proceed to computing $\partial v_\omega(x)/\partial\nu$ on Γ. Note that $\partial/\partial\nu = \partial/\partial y_n$. Let ∇' be the gradient with respect the y' variables; $\nabla' u$ is a vector field in $\mathbb{R}^n$ perpendicular to ν. Then

$$v_\omega(y) = \nabla' u(y', y_n)\cdot\omega + \frac{\partial u}{\partial y_n}\nu\cdot\omega,$$

and

$$\frac{\partial v_\omega}{\partial y_n} = \frac{\partial}{\partial y_n}(\nabla' u(y', y_n))\cdot\omega + \frac{\partial^2 u}{\partial y_n^2}\nu\cdot\omega.$$

To evaluate $\partial v_\omega/\partial y_n$ on Γ (when $y_n = 0$) we use (5):

$$\frac{\partial v_\omega}{\partial\nu} = \frac{\partial}{\partial y_n}(\nabla' u(y', 0))\cdot\omega - H(y')\frac{\partial u}{\partial\nu}(\nu\cdot\omega).$$

The function $u(x)$ vanishes on Γ; therefore

$$\frac{\partial}{\partial y_n}(\nabla' u(y', 0)) = \nabla'\left(\frac{\partial u}{\partial y_n}(y', 0)\right),$$

and, finally,

$$\frac{\partial v_\omega}{\partial\nu} = \nabla'\left(\frac{\partial u}{\partial\nu}\right)\cdot\omega - H(y')\frac{\partial u}{\partial\nu}(\nu\cdot\omega). \tag{6}$$

From (4) and (6), one obtains

$$\begin{aligned}\int_\Gamma v_\omega\frac{\partial v_\omega}{\partial\nu}dA = &-\int_\Gamma H(y')\left(\frac{\partial u}{\partial\nu}\right)^2(\nu\cdot\omega)^2 dA\\ &+\int_\Gamma\frac{\partial u}{\partial\nu}\left(\nabla'\left(\frac{\partial u}{\partial\nu}\right)\cdot\omega\right)(\nu\cdot\omega)dA.\end{aligned} \tag{7}$$

The second term on the right in (7) can be rewritten as

$$\frac{1}{2}\int_\Gamma\left(\nabla'\left(\frac{\partial u}{\partial\nu}\right)^2\cdot\omega\right)(\nu\cdot\omega)dA.$$

Now, we integrate both sides of (7) with respect to ω. One computes that

$$\int_{S^{n-1}}(\nu(x)\cdot\omega)^2 d\omega = \frac{1}{n}$$

and

$$\int_{S^{n-1}}\left(\nabla'\left(\frac{\partial u}{\partial\nu}\right)^2\cdot\omega\right)(\nu\cdot\omega)d\omega = \frac{1}{n}\left(\nabla'\left(\frac{\partial u}{\partial\nu}\right)^2\cdot\nu\right) = 0.$$

Finally, one gets

$$\int_{S^{n-1}}d\omega\int_\Gamma v_\omega\frac{\partial v_\omega}{\partial\nu}dA = -\frac{1}{n}\int_\Gamma H(x)\left(\frac{\partial u}{\partial\nu}\right)^2 dA < 0.$$

$\square$

References

[1] D.Jerison, N.Nadirashvili, *The "Hot spots" conjecture for domains with two axes of symmetry*, Preprint.

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