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DOUGLAS ALGEBRAS WHICH ADMIT CODIMENSION 1 LINEAR ISOMETRIES

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ABSTRACT. Let B be a Douglas algebra and let B_b be its Bourgain algebra. It is proved that B admits a codimension 1 linear isometry if and only if $B \neq B_b$. This answers the conjecture of Araujo and Font.

1. INTRODUCTION

Let H^∞ be the Banach algebra of bounded analytic functions on the unit disk D. Identifying a function in H^∞ with its boundary function, we view H^∞ as the closed subalgebra of L^∞, the usual Lebesgue space on the unit circle ∂D. A closed subalgebra B with $H^\infty \subset B \subset L^\infty$ is called a Douglas algebra. In [1], Araujo and Font studied codimension 1 linear isometries of Douglas algebras. They noted that H^∞ has the codimension 1 linear isometry; $Tf = zf$, $f \in H^\infty$, where z is the identity function on D, and L^∞ does not have any codimension 1 linear isometries, (see also [2]). They also gave the conjecture that proper Douglas algebras admit no codimension 1 linear isometries. In this paper, we give a characterization of Douglas algebras which admit codimension 1 linear isometries.

First, recall the structure of Douglas algebras. For a Douglas algebra B, we denote by $M(B)$ the set of non-zero multiplicative linear functionals of B. We consider the weak*-topology on $M(B)$. We identify a function in B with its Gelfand transform. Then we may think of $M(B)$ as a compact subset of $M(H^\infty)$. It is known that $M(L^\infty)$ is the Shilov boundary of H^∞. For a subset E of $M(H^\infty)$, we denote by $\overline{E}$ the closure of E in $M(H^\infty)$. For a function f in B, let $Z_B(f) = \{x \in M(B); f(x) = 0\}$. For $x \in M(H^\infty)$, there is a unique probability measure μ_x on $M(L^\infty)$ such that $f(x) = \int_{M(L^\infty)} f\, d\mu_x$ for every $f \in H^\infty$. We denote by $\operatorname{supp}\mu_x$ the closed support set of μ_x.

A function $f \in H^\infty$ is called inner if $|f| = 1$ on $M(L^\infty)$. Let $\{z_n\}_n$ be a sequence in D with $\sum_{n=1}^\infty (1 - |z_n|) < \infty$. Then there is the associated Blaschke product b given by

$$b(z) = \prod_{n=1}^\infty \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z}, \quad z \in D.$$

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A sequence $\{z_n\}_n$ in D is called interpolating if for every bounded sequence of complex numbers $\{a_n\}_n$ there exists $h \in H^\infty$ such that $h(z_n) = a_n$ for every n. A Blaschke product is called interpolating if its zero sequence is interpolating. It is known that a Blaschke product is inner (see Hoffman's book [10]).

The smallest Douglas algebra except H^∞ is $H^\infty + C$, where C is the space of continuous functions on ∂D, and it is known that $M(H^\infty + C) = M(H^\infty) \setminus D$ (see [14]). Chang [4] and Marshall [13] proved that if B is a Douglas algebra, then $B = H^\infty[\overline{b}_\alpha; \alpha \in \Lambda]$, where $\{b_\alpha\}_{\alpha\in\Lambda}$ is a family of some interpolating Blaschke products and $M(B) = \bigcap_{\alpha\in\Lambda}\{x \in M(H^\infty); |b_\alpha(x)| = 1\}$. And they showed that for Douglas algebras A and B, it holds that $A \subset B$ if and only if $M(B) \subset M(A)$. By Chang and Marshall's theorem, we have that $x \in M(B)$ if and only if $B_{|\operatorname{supp}\mu_x} = (H^\infty)_{|\operatorname{supp}\mu_x}$. Put

$$B_{\operatorname{supp}\mu_x} = \{f \in L^\infty; f_{|\operatorname{supp}\mu_x} \in B_{|\operatorname{supp}\mu_x}\}.$$

Then $B_{\operatorname{supp}\mu_x}$ is a Douglas algebra and

$$M(B_{\operatorname{supp}\mu_x}) = M(L^\infty) \cup \{y \in M(B); \operatorname{supp}\mu_y \subset \operatorname{supp}\mu_x\}.$$

A nice reference for Douglas algebras is Garnett's book [6].

Next, recall the work of Hoffman [11]. For $x, y \in M(H^\infty)$, put

$$\rho(x,y) = \sup\{|f(y)|; f \in H^\infty, f(x) = 0, \|f\|_\infty \le 1\}.$$

For $x \in M(H^\infty)$, let $P(x) = \{y \in M(H^\infty); \rho(x,y) < 1\}$. When $P(x) = \{x\}$, a point x is called trivial. When $P(x) \neq \{x\}$, in this case x is called non-trivial; there is a one-to-one continuous map L_x from D onto $P(x)$ such that $f \circ L_x \in H^\infty$ for every $f \in H^\infty$ and $L_x(0) = x$. Moreover if $f(x) = 0$, we can define the order of zero at x, $\operatorname{ord}(f,x)$, by the usual order of zero of $f \circ L_x$ at $z = 0$. When $f = 0$ on $P(x)$, we put $\operatorname{ord}(f,x) = \infty$.

For a Douglas algebra B, we denote by B_b the set of f in L^∞ such that if $f_n \to 0$ weakly in B, then $\|ff_n + B\| \to 0$. Cima and Timoney [5] studied B_b in a general setting. By their results, B_b is a Douglas algebra and $B \subset B_b$. They called B_b the Bourgain algebra of B. Douglas algebras B satisfying $B \neq B_b$ are characterized in [7]. In this paper, we prove the following theorem.

Theorem. *Let B be a Douglas algebra. Then B admits a codimension 1 linear isometry T if and only if $B_b \neq B$. In this case T has the following form: $Tf = (ub)(f \circ \varphi), f \in B$, where u is an invertible unimodular function in B, b is an interpolating Blaschke product, and φ is a homeomorphism of $M(L^\infty)$ satisfying the following conditions.*

(i) $B \circ \varphi = B$.

(ii) $\{y \in M(B); |b(y)| < 1\} = P(x_0)$ *for some* $x_0 \in M(B)$.

2. Proof of the theorem

First of all, we show a counterexample for Araujo and Font's conjecture. A Blaschke product b with zeros $\{z_n\}_n$ is called sparse (or thin) if

$$\lim_{n\to\infty} \prod_{j:j\neq n} \left|\frac{z_j - z_n}{1 - \overline{z}_n z_j}\right| = 1.$$

Let $x_0 \in Z_{H^\infty + C}(b)$ and $B = H^\infty_{\operatorname{supp}\mu_{x_0}}$. Then $\{y \in M(B); |b(y)| < 1\} = P(x_0)$ and $bB = \{g \in B; g(x_0) = 0\}$ (see [9]). Hence $Tf = bf, f \in B$, is a codimension 1 linear isometry on B.

To prove our theorem, we need two lemmas.

Lemma 1. *Let A and B be Douglas algebras with $B \subsetneq A$. Then $M(B) \setminus M(A)$ is not a closed subset of $M(B)$.*

Proof. Since $B \subsetneq A$, by Chang and Marshall's theorem $M(A) \subset M(B)$ and $M(B) \setminus M(A) \neq \emptyset$. To prove our assertion, suppose not. Then $M(B) \setminus M(A)$ is an open-closed subset of $M(B)$. Hence by Shilov's idempotent theorem, there is $\chi \in B$ such that $\chi = 1$ on $M(B) \setminus M(A)$ and $\chi = 0$ on $M(A)$. Since $M(L^\infty) \subset M(A)$, $\chi = 0$ on $M(L^\infty)$. Thus we get $\chi = 0$ on $M(B)$ and $M(B) \setminus M(A) = \emptyset$. This is a contradiction.

Lemma 2. *Let A and B be Douglas algebras with $B \subsetneq A$. Then*

$$M(A) \cap \overline{M(B) \setminus M(A)}$$

contains uncountably many distinct points.

Proof. By Chang and Marshall's theorem, there is an interpolating Blaschke product q such that

$$\overline{q} \in A \quad \text{and} \quad \overline{q} \notin B. \tag{2.1}$$

Then

$$\emptyset \neq \{y \in M(B); |q(y)| < 1\} \subset M(B) \setminus M(A). \tag{2.2}$$

We have

$$q(M(B)) = D \cup \partial D. \tag{2.3}$$

For, if $\zeta \in D$, then $q_\zeta = (q - \zeta)/(1 - \overline{\zeta} q)$ is an inner function and q_ζ is not invertible in B. Hence there is $x \in M(B)$ such that $q_\zeta(x) = 0$. Thus we get $\zeta \in q(M(B))$.

Let

$$\Gamma = \overline{\{y \in M(B); |q(y)| < 1\}} \setminus \{y \in M(B); |q(y)| < 1\}. \tag{2.4}$$

Then by (2.3), we have $q(\Gamma) = \partial D$. For each $\xi \in \partial D$, take $x_\xi \in \Gamma$ such that $q(x_\xi) = \xi$. Then $q = \xi$ on $\operatorname{supp} \mu_{x_\xi}$ and

$$\operatorname{supp} \mu_{x_\xi} \cap \operatorname{supp} \mu_{x_\eta} = \emptyset \quad \text{if } \xi, \eta \in \partial D \text{ and } \xi \neq \eta. \tag{2.5}$$

We shall prove that for each $\xi \in \partial D$ there exists $y_\xi \in M(B)$ such that

$$\operatorname{supp} \mu_{y_\xi} \subset \operatorname{supp} \mu_{x_\xi} \quad \text{and} \quad y_\xi \in M(A) \cap \overline{M(B) \setminus M(A)}. \tag{2.6}$$

Then by (2.5), points in $\{y_\xi; \xi \in \partial D\}$ are distinct and we get our assertion.

We shall show the existence of y_ξ satisfying (2.6). If $x_\xi \in M(A)$, by (2.1) we have $|q(x_\xi)| = 1$. Since $x_\xi \in \Gamma$, by (2.4) we have $x_\xi \in \overline{\{y \in M(B); |q(y)| < 1\}}$. Hence by (2.2), $x_\xi \in \overline{M(B) \setminus M(A)}$. Thus $x_\xi \in M(A) \cap \overline{M(B) \setminus M(A)}$. Put $y_\xi = x_\xi$; then (2.6) holds.

Next, suppose that $x_\xi \notin M(A)$. Put $A_1 = A_{\operatorname{supp} \mu_{x_\xi}}$ and $B_1 = B_{\operatorname{supp} \mu_{x_\xi}}$. Then $B_1 \subset A_1$. Since $x_\xi \in M(B)$, $B_1 = (H^\infty)_{\operatorname{supp} \mu_{x_\xi}}$. Since $x_\xi \notin M(B)$, $A_1 \neq (H^\infty)_{\operatorname{supp} \mu_{x_\xi}}$. Hence we have $B_1 \neq A_1$. By Lemma 1, there exists a point y_ξ such that

$$y_\xi \in M(A_1) \cap \overline{M(B_1) \setminus M(A_1)}. \tag{2.7}$$

We have

$$M(B_1) = M(L^\infty) \cup \{y \in M(B); \operatorname{supp} \mu_y \subset \operatorname{supp} \mu_{x_\xi}\},$$

$$M(A_1) = M(L^\infty) \cup \{y \in M(A); \operatorname{supp} \mu_y \subset \operatorname{supp} \mu_{x_\xi}\}.$$

By the above

$$M(B_1) \setminus M(A_1) = \{y \in M(B) \setminus M(A); \operatorname{supp} \mu_y \subset \operatorname{supp} \mu_{x_\xi}\}. \tag{2.8}$$

Hence by (2.7), we have $\operatorname{supp} \mu_{y_\xi} \subset \operatorname{supp} \mu_{x_\xi}$. Since $M(A_1) \subset M(A)$ and by (2.7) and (2.8), we have $y_\xi \in M(A) \cap \overline{M(B) \setminus M(A)}$.

Proof of the theorem. Suppose that $B_b \neq B$. By Chang and Marshall's theorem, there is an interpolating Blaschke product ψ such that $\overline{\psi} \in B_b$ and $\overline{\psi} \notin B$. By [7, Theorem 2], $Z_B(\psi)$ is a finite set. Let $Z_B(\psi) = \{x_1, x_2, \ldots, x_n\}$ such that $x_i \neq x_j$, $i \neq j$. Take an open subset V of $M(H^\infty)$ such that $x_1 \in V$ and $x_j \notin \overline{V}$ for $j = 2, 3, \ldots, n$. Let $\{z_n\}_n$ be the zeros of ψ in D. Let ψ_1 be the subproduct of ψ whose zeros are $\{z_n\}_n \cap V$. By [10, p. 205], $Z_{H^\infty}(\psi) = \overline{\{z_n\}_n}$, so that we have $Z_B(\psi_1) = \{x_1\}$. Let $Tf = \psi_1 f$ for $f \in B$. Then T is a linear isometry on B and $TB \subset \{g \in B; g(x_1) = 0\}$. By [3, 9], for $g \in B$ with $g(x_1) = 0$ there exists $h \in B$ such that $g = \psi_1 h$. Hence $TB = \{g \in B; g(x_1) = 0\}$, so that T is a codimension 1 linear isometry on B.

Suppose that B admits a codimension 1 linear isometry T. Then by the work of Araujo and Font [1], there exist a homeomorphism φ of $M(L^\infty)$ and a unimodular function ψ on $M(L^\infty)$ such that

$$(Tf)(x) = \psi(x) f(\varphi(x)) \quad \text{for all } x \in M(L^\infty) \text{ and } f \in B. \tag{2.9}$$

Since $1 \in B$, we have

$$\psi \in B. \tag{2.10}$$

First, we prove that

$$B \circ \varphi \subset B. \tag{2.11}$$

To prove this, suppose not. Let A be the Douglas algebra generated by B and $B \circ \varphi$. Then $B \subsetneq A$. By (2.9) and (2.10), we have $\psi A \subset B$. Hence by [15, Theorem 1],

$$M(B) \setminus M(A) \subset Z_B(\psi). \tag{2.12}$$

By Lemma 2, $M(A) \cap \overline{(M(B) \setminus M(A))}$ is an uncountable set. Let $y \in M(A) \cap \overline{(M(B) \setminus M(A))}$ and $f \in B$. Since $\psi, f \circ \varphi \in A$ and $y \in M(A)$, we have $(Tf)(y) = (\psi f \circ \varphi)(y) = \psi(y)(f \circ \varphi)(y)$. Since $y \in \overline{M(B) \setminus M(A)}$, by (2.12) $\psi(y) = 0$. Hence $(Tf)(y) = 0$, so that

$$M(A) \cap \overline{(M(B) \setminus M(A))} \subset Z_A(Tf) \subset Z_B(Tf) \quad \text{for every } f \in B.$$

By [1, Proposition 3.1], $\bigcap\{Z_B(Tf); f \in B\}$ is a finite set. Hence by the above, $M(A) \cap \overline{(M(B) \setminus M(A))}$ is a finite set. This is a contradiction. Thus we obtain (2.11).

Next, we prove that

$$\overline{\psi} \notin B. \tag{2.13}$$

To prove this, suppose not. Then $\overline{\psi} \in B$ and ψ is invertible in B. Put $T_0 f = \overline{\psi} Tf$ for $f \in B$. Then T_0 is a codimension 1 linear isometry on B. By (2.9), $T_0 B = B \circ \varphi$

and there exists $\lambda \in B$ such that $B = B \circ \varphi + \mathbf{C}\lambda$ and $\lambda \notin B \circ \varphi$, where $\mathbf{C}$ is the set of complex numbers. Then we have

$$B \circ \varphi^{-1} = B + \mathbf{C}\lambda \circ \varphi^{-1} \quad \text{and} \quad \lambda \circ \varphi^{-1} \notin B.$$

Since φ is a homeomorphism of $M(L^\infty)$, $B \circ \varphi^{-1}$ is a closed subalgebra of L^∞ and $H^\infty \subset B$. Then by the above, $B \circ \varphi^{-1}$ is a Douglas algebra which contains B properly. And B is a linear subspace of $B \circ \varphi^{-1}$ of codimension 1. By Chang and Marshall's theorem, there exists an interpolating Blaschke product b such that $\overline{b} \in B \circ \varphi^{-1}$ and $\overline{b} \notin B$. Since $\overline{b}^n \in B \circ \varphi^{-1} \setminus B$ and $\{\overline{b}^n; n = 1, 2, \dots\}$ is linearly independent, B is not a linear subspace of $B \circ \varphi^{-1}$ of codimension 1. This is a contradiction. Thus we get (2.13).

By (2.13), there exists $x_0 \in M(B)$ such that $\psi(x_0) = 0$. By (2.9), (2.10), and (2.11), $Tf = 0$ on $Z_B(\psi)$ for every $f \in B$. Then we have

$$Z_B(\psi) = \{x_0\}. \tag{2.14}$$

For, if $y \in Z_B(\psi)$ and $y \neq x_0$, there exist $f, g \in B$ such that $f(y) = 1$, $f(x_0) = 0$, and $g(y) = 0$, $g(x_0) = 1$. Then $f, g \notin TB$ and f, g are linear independent. Hence T is not a codimension 1 linear isometry.

Also we have

$$TB = \{f \in B; f(x_0) = 0\} \tag{2.15}$$

since $TB \subset \{f \in B; f(x_0) = 0\}$ and TB is a linear subspace of B of codimension 1.

Then we have

$$\begin{aligned} \psi B &\subset \{f \in B; f(x_0) = 0\} && \text{by (2.14)} \\ &= \psi B \circ \varphi && \text{by (2.9) and (2.15)} \\ &\subset \psi B && \text{by (2.11).} \end{aligned}$$

Thus we get

$$B = B \circ \varphi, \tag{2.16}$$

$$\psi B = \{f \in B; f(x_0) = 0\}. \tag{2.17}$$

We prove that $B \neq B_b$. By (2.13), it is sufficient to prove $\overline{\psi} \in B_b$. To show this, let $\{f_n\}_n \subset B$ such that $f_n \to 0$ weakly in B. Then $f_n(x_0) \to 0$. By (2.17), $f_n - f_n(x_0) \in \psi B$. Then there exists $g_n \in B$ such that $f_n - f_n(x_0) = \psi g_n$. Hence we have $\|f_n\overline{\psi} - B\| \leq \|f_n\overline{\psi} - g_n\| \leq |f_n(x_0)| \to 0$ as $n \to \infty$. Therefore $\overline{\psi} \in B_b$.

Now we complete the proof. Condition (i) follows from (2.16). Since $x_0 \in M(B)$, $B_{|\operatorname{supp}\mu_{x_0}} = (H^\infty)_{|\operatorname{supp}\mu_{x_0}}$. By (2.10), there exists $h \in H^\infty$ such that $h_{|\operatorname{supp}\mu_{x_0}} = \psi_{|\operatorname{supp}\mu_{x_0}}$. If $\operatorname{ord}(h, x_0) \geq 2$, there is a factorization $h = h_1 h_2$ such that $h_1(x_0) = h_2(x_0) = 0$ and $h_1, h_2 \in H^\infty$ (see [11, Section 5]). Since $h_i \in B$, by (2.17) we have $h_i = \psi g_i$ for some $g_i \in B$ for $i = 1, 2$. Then $h = \psi^2 g_1 g_2$. Hence $\psi_{|\operatorname{supp}\mu_{x_0}} = (\psi^2 g_1 g_2)_{|\operatorname{supp}\mu_{x_0}}$. Since $|\psi| = 1$ on $M(L^\infty)$, $1 = (\psi g_1 g_2)_{|\operatorname{supp}\mu_{x_0}}$. Since $(\psi g_1 g_2)(x_0) = 0$, we have a contradiction. Hence $\operatorname{ord}(h, x_0) = 1$ and x_0 is a nontrivial point. Therefore by (2.14) and [12, Corollary 4.5], there are an interpolating Blaschke product b and an invertible unimodular function u in B such that $\psi = ub$ and $Z_B(b) = \{x_0\}$. Then by [8, Theorem 3.2], $\{y \in M(B); |b(y)| < 1\} = P(x_0)$. This completes the proof.

References

[1] J. Araujo and J. J. Font, *Codimension 1 linear isometries on function algebras*, Proc. Amer. Math. Soc. **127**(1999), 2273-2281. MR **99j:**46059

[2] J. Araujo and J. J. Font, *Linear isometries between subspaces of continuous functions*, Trans. Amer. Math. Soc. **349**(1997), 413-428. MR **97d:**46026

[3] S. Axler and P. Gorkin, *Division in Douglas algebras*, Michigan Math. J. **31**(1984), 89-94. MR **85h:**46075

[4] S.-Y. Chang, *A characterization of Douglas algebras*, Acta Math. **137**(1976), 81-89. MR **55:**1074a

[5] J. Cima and R. Timoney, *The Dunford Pettis property for certain planner uniform algebras*, Michigan Math. J. **34**(1987), 99-104. MR **88e:**46023

[6] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981. MR **83g:**30037

[7] P. Gorkin, K. Izuchi, and R. Mortini, *Bourgain algebras of Douglas algebras*, Canad. J. Math. **44**(1992), 797-804. MR **94c:**46104

[8] P. Gorkin, H. -M. Lingenberg, and R. Mortini, *Homeomorphic disks in the spectrum of H^∞*, Indiana Univ. Math. J. **39**(1990), 961-983. MR **92b:**46082

[9] C. Guillory, K. Izuchi and D. Sarason, *Interpolating Blaschke products and division in Douglas algebras*, Proc. Roy. Irish Acad. Sect. A **84**(1984), 1-7. MR **86j:**46054

[10] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, N.J., 1962. MR **24:**A2844

[11] K. Hoffman, *Bounded analytic functions and Gleason parts*, Michigan Math. J. **40**(1993), 53-75.

[12] K. Izuchi, *Interpolating Blaschke products and factorization theorems*, J. London Math. Soc. (2) **50**(1994), 547-567. MR **95k:**46086

[13] D. Marshall, *Subalgebras of L^∞ containing H^∞*, Acta Math. **137**(1976), 91-98. MR **55:**10746

[14] D. Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc. **79**(1973), 286-299. MR **48:**2777

[15] R. Younis, *Division in Douglas algebras and some applications*, Arch. Math. **45**(1985), 555-560. MR **87b:**46059

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