DOUGLAS ALGEBRAS
WHICH ADMIT CODIMENSION 1 LINEAR ISOMETRIES

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Abstract. Let $B$ be a Douglas algebra and let $B_b$ be its Bourgain algebra. It is proved that $B$ admits a codimension 1 linear isometry if and only if $B = B_b$. This answers the conjecture of Araujo and Font.

1. Introduction

Let $H^\infty$ be the Banach algebra of bounded analytic functions on the unit disk $D$. Identifying a function in $H^\infty$ with its boundary function, we view $H^\infty$ as the closed subalgebra of $L^\infty$, the usual Lebesgue space on the unit circle $\partial D$. A closed subalgebra $B$ with $H^\infty \subset B \subset L^\infty$ is called a Douglas algebra. In [1], Araujo and Font studied codimension 1 linear isometries of Douglas algebras. They noted that $H^\infty$ has the codimension 1 linear isometry; $Tf = zf$, $f \in H^\infty$, where $z$ is the identity function on $D$, and $L^\infty$ does not have any codimension 1 linear isometries, (see also [2]). They also gave the conjecture that proper Douglas algebras admit no codimension 1 linear isometries. In this paper, we give a characterization of Douglas algebras which admit codimension 1 linear isometries.

First, recall the structure of Douglas algebras. For a Douglas algebra $B$, we denote by $M(B)$ the set of non-zero multiplicative linear functionals of $B$. We consider the weak*-topology on $M(B)$. We identify a function in $B$ with its Gelfand transform. Then we may think of $M(B)$ as a compact subset of $M(H^\infty)$. It is known that $M(L^\infty)$ is the Shilov boundary of $H^\infty$. For a subset $E$ of $M(H^\infty)$, we denote by $\overline{E}$ the closure of $E$ in $M(H^\infty)$. For a function $f$ in $B$, let $Z_B(f) = \{x \in M(B); f(x) = 0\}$. For $x \in M(H^\infty)$, there is a unique probability measure $\mu_x$ on $M(L^\infty)$ such that $f(x) = \int_{M(L^\infty)} f d\mu_x$ for every $f \in H^\infty$. We denote by $\text{supp}\, \mu_x$ the closed support set of $\mu_x$.

A function $f \in H^\infty$ is called inner if $|f| = 1$ on $M(L^\infty)$. Let $\{z_n\}_n$ be a sequence in $D$ with $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$. Then there is the associated Blaschke product $b$ given by

$$b(z) = \prod_{n=1}^{\infty} \frac{z - z_n}{1 - \overline{z}_n z}, \quad z \in D.$$
A sequence \( \{z_n\}_n \) in \( D \) is called interpolating if for every bounded sequence of complex numbers \( \{a_n\}_n \) there exists \( h \in H^\infty \) such that \( h(z_n) = a_n \) for every \( n \). A Blaschke product is called interpolating if its zero sequence is interpolating. It is known that a Blaschke product is inner (see Hoffman’s book [10]). Chang [4] and Marshall [13] proved that if \( B \) weakly in the following conditions.

By their results, the smallest Douglas algebra except \( H^\infty \) is a Douglas algebra and \( \{b_\alpha\}_{\alpha \in \Lambda} \) is a family of some interpolating Blaschke products and \( M(B) = \bigcap_{\alpha \in \Lambda} \{ x \in M(H^\infty) ; |b_\alpha(x)| = 1 \} \). And they showed that for Douglas algebras \( A \) and \( B \), it holds that \( A \subset B \) if and only if \( M(B) \subset M(A) \). By Chang and Marshall’s theorem, we have that \( x \in M(B) \) if and only if \( B_{\supp \mu_x} = (H^\infty)_{\supp \mu_x} \).

Theorem. Let \( B \) be a Douglas algebra. Then \( B \) admits a codimension 1 linear isometry \( T \) if and only if \( B_0 \neq B \). In this case \( T \) has the following form: \( T f = (ub)(f \circ \varphi) \), \( f \in B \), where \( u \) is an invertible unimodular function in \( B \), \( b \) is an interpolating Blaschke product, and \( \varphi \) is a homeomorphism of \( M(L^\infty) \) satisfying the following conditions.

(i) \( B \circ \varphi = B \).
(ii) \( \{ y \in M(B) ; |b(y)| < 1 \} = P(x_0) \) for some \( x_0 \in M(B) \).

2. Proof of the theorem

First of all, we show a counterexample for Araujo and Font’s conjecture. A Blaschke product \( b \) with zeros \( \{z_n\}_n \) is called sparse (or thin) if

\[
\lim_{n \to \infty} \prod_{j: j \neq n} \left| \frac{z_j - z_n}{1 - \bar{z_n}z_j} \right| = 1.
\]

Let \( x_0 \in Z_{H^\infty + C}(b) \) and \( B = H^\infty_{\supp \mu_{x_0}} \). Then \( \{ y \in M(B) ; |b(y)| < 1 \} = P(x_0) \) and \( bb = \{ g \in B ; g(x_0) = 0 \} \) (see [5]). Hence \( T f = bf, f \in B \), is a codimension 1 linear isometry on \( B \).
To prove our theorem, we need two lemmas.

**Lemma 1.** Let \( A \) and \( B \) be Douglas algebras with \( B \subset A \). Then \( M(B) \setminus M(A) \) is not a closed subset of \( M(B) \).

**Proof.** Since \( B \subset A \), by Chang and Marshall’s theorem \( M(A) \subset M(B) \) and \( M(B) \setminus M(A) \neq \emptyset \). To prove our assertion, suppose not. Then \( M(B) \setminus M(A) \) is an open-closed subset of \( M(B) \). Hence by Shilov’s idempotent theorem, there is \( \chi \in B \) such that \( \chi = 1 \) on \( M(B) \setminus M(A) \) and \( \chi = 0 \) on \( M(A) \). Since \( M(L^\infty) \subset M(A) \), \( \chi = 0 \) on \( M(L^\infty) \). Thus we get \( \chi = 0 \) on \( M(B) \) and \( M(B) \setminus M(A) = \emptyset \). This is a contradiction.

**Lemma 2.** Let \( A \) and \( B \) be Douglas algebras with \( B \subset A \). Then
\[
M(A) \cap M(B) \setminus M(A)
\]
contains uncountably many distinct points.

**Proof.** By Chang and Marshall’s theorem, there is an interpolating Blaschke product \( q \) such that
\[
(2.1) \quad \overline{q} \in A \quad \text{and} \quad q \notin B.
\]
Then
\[
(2.2) \quad \emptyset \neq \{ y \in M(B); |q(y)| < 1 \} \subset M(B) \setminus M(A).
\]
We have
\[
(2.3) \quad q(M(B)) = D \cup \partial D.
\]
For, if \( \zeta \in D \), then \( q_\zeta = (q - \zeta)/(1 - \overline{q}\zeta) \) is an inner function and \( q_\zeta \) is not invertible in \( B \). Hence there is \( x \in M(B) \) such that \( q_\zeta(x) = 0 \). Thus we get \( \zeta \in q(M(B)) \).

Let
\[
(2.4) \quad \Gamma = \{ y \in M(B); |q(y)| < 1 \} \setminus \{ y \in M(B); |q(y)| < 1 \}.
\]
Then by (2.3), we have \( q(\Gamma) = \partial D \). For each \( \xi \in \partial D \), take \( x_\xi \in \Gamma \) such that \( q(x_\xi) = \xi \). Then \( q = \xi \) on \( \text{supp} \, \mu_{x_\xi} \) and
\[
(2.5) \quad \text{supp} \, \mu_{x_\xi} \cap \text{supp} \, \mu_{x_\eta} = \emptyset \quad \text{if} \, \xi, \eta \in \partial D \text{ and } \xi \neq \eta.
\]
We shall prove that for each \( \xi \in \partial D \) there exists \( y_\xi \in M(B) \) such that
\[
(2.6) \quad \text{supp} \, \mu_{y_\xi} \subset \text{supp} \, \mu_{x_\xi} \quad \text{and} \quad y_\xi \in M(A) \cap M(B) \setminus M(A).
\]
Then by (2.5), points in \( \{ y_\xi; \xi \in \partial D \} \) are distinct and we get our assertion.

We shall show the existence of \( y_\xi \) satisfying (2.6). If \( x_\xi \in M(A) \), by (2.1) we have \( |q(x_\xi)| = 1 \). Since \( x_\xi \in \Gamma \), by (2.4) we have \( x_\xi \in \{ y \in M(B); |q(y)| < 1 \} \). Hence by (2.2), \( x_\xi \in M(B) \setminus M(A) \). Thus \( x_\xi \in M(A) \cap M(B) \setminus M(A) \). Put \( y_\xi = x_\xi \); then (2.6) holds.

Next, suppose that \( x_\xi \notin M(A) \). Put \( A_1 = A_{\text{supp} \, \mu_{x_\xi}} \) and \( B_1 = B_{\text{supp} \, \mu_{x_\xi}} \). Then \( B_1 \subset A_1 \). Since \( x_\xi \in M(B) \), \( B_1 = (H^\infty)_{\text{supp} \, \mu_{x_\xi}} \). Since \( x_\xi \notin M(B) \), \( A_1 \neq (H^\infty)_{\text{supp} \, \mu_{x_\xi}} \). Hence we have \( B_1 \neq A_1 \). By Lemma 1, there exists a point \( y_\xi \) such that
\[
(2.7) \quad y_\xi \in M(A_1) \cap M(B_1) \setminus M(A_1).
\]
We have
\[ M(B_1) = M(L^\infty) \cup \{y \in M(B); \text{supp } \mu_y \subset \text{supp } \mu_{x_1}\}, \]
\[ M(A_1) = M(L^\infty) \cup \{y \in M(A); \text{supp } \mu_y \subset \text{supp } \mu_{x_1}\}. \]

By the above
\[ (2.8) \quad M(B_1) \setminus M(A_1) = \{y \in M(B) \setminus M(A); \text{supp } \mu_y \subset \text{supp } \mu_{x_1}\}. \]

Hence by (2.7), we have \(\text{supp } \mu_{y_k} \subset \text{supp } \mu_{x_1}\). Since \(M(A_1) \subset M(A)\) and by (2.7) and (2.8), we have \(y_k \in M(A) \cap M(B) \setminus M(A)\).

**Proof of the theorem.** Suppose that \(B_h \neq B\). By Chang and Marshall’s theorem, there is an interpolating Blaschke product \(\psi\) such that \(\overline{\psi} \in B_h\) and \(\overline{\psi} \notin B\). By [7, Theorem 2], \(Z_B(\psi)\) is a finite set. Let \(Z_B(\psi) = \{x_1, x_2, \ldots, x_n\}\) such that \(x_i \neq x_j, i \neq j\). Take an open subset \(V\) of \(M(H^\infty)\) such that \(x_1 \in V\) and \(x_j \notin V\) for \(j = 2, 3, \ldots, n\). Let \(\{z_n\}_n\) be the zeros of \(\psi\) in \(D\). Let \(\psi_1\) be the subproduct of \(\psi\) whose zeros are \(\{z_n\}_n \cap V\). By [10, p. 205], \(Z_{H^\infty}(\psi) = \overline{\{z_n\}_n}\), so that we have \(Z_B(\psi_1) = \{x_1\}\). Let \(T_f = \psi_1 f\) for \(f \in B\). Then \(T\) is a linear isometry on \(B\) and \(TB \subset \{g \in B; g(x_1) = 0\}\). By [3, 9], for \(g \in B\) with \(g(x_1) = 0\) there exists \(h \in B\) such that \(g = \psi_1 h\). Hence \(TB = \{g \in B; g(x_1) = 0\}\), so that \(T\) is a codimension 1 linear isometry on \(B\).

Suppose that \(B\) admits a codimension 1 linear isometry \(T\). Then by the work of Araujo and Font [1], there exist a homeomorphism \(\varphi\) of \(M(L^\infty)\) and a unimodular function \(\psi\) on \(M(L^\infty)\) such that
\[ (2.9) \quad (Tf)(x) = \psi(x)f(\varphi(x)) \quad \text{for all } x \in M(L^\infty) \text{ and } f \in B. \]

Since \(1 \in B\), we have
\[ (2.10) \quad \psi \in B. \]

First, we prove that
\[ (2.11) \quad B \circ \varphi \subset B. \]

To prove this, suppose not. Let \(A\) be the Douglas algebra generated by \(B\) and \(B \circ \varphi\). Then \(B \subset A\). By (2.9) and (2.10), we have \(\psi A \subset B\). Hence by [15, Theorem 1],
\[ (2.12) \quad M(B) \setminus M(A) \subset Z_B(\psi). \]

By Lemma 2, \(M(A) \cap \overline{(M(B) \setminus M(A))}\) is an uncountable set. Let \(y \in M(A) \cap \overline{(M(B) \setminus M(A))}\) and \(f \in B\). Since \(\psi, f \circ \varphi \in A\) and \(y \in M(A)\), we have \((Tf)(y) = (\psi f \circ \varphi)(y) = \psi(y)(f \circ \varphi)(y)\). Since \(y \in M(B) \setminus M(A)\), by (2.12) \(\psi(y) = 0\). Hence \((Tf)(y) = 0\), so that
\[ M(A) \cap (M(B) \setminus M(A)) \subset Z_A(Tf) \subset Z_B(Tf) \quad \text{for every } f \in B. \]

By [1, Proposition 3.1], \(\bigcap \{Z_B(Tf); f \in B\}\) is a finite set. Hence by the above, \(M(A) \cap (M(B) \setminus M(A))\) is a finite set. This is a contradiction. Thus we obtain (2.11).

Next, we prove that
\[ (2.13) \quad \overline{\psi} \notin B. \]

To prove this, suppose not. Then \(\overline{\psi} \in B\) and \(\psi\) is invertible in \(B\). Put \(T_0f = \overline{\psi} Tf\) for \(f \in B\). Then \(T_0\) is a codimension 1 linear isometry on \(B\). By (2.9), \(T_0B = B \circ \varphi\)
and there exists $\lambda \in B$ such that $B = B \circ \varphi + C\lambda$ and $\lambda \notin B \circ \varphi$, where $C$ is the set of complex numbers. Then we have

$$B \circ \varphi^{-1} = B + C\lambda \circ \varphi^{-1} \quad \text{and} \quad \lambda \circ \varphi^{-1} \notin B.$$ 

Since $\varphi$ is a homeomorphism of $M(L^\infty)$, $B \circ \varphi^{-1}$ is a closed subalgebra of $L^\infty$ and $H^\infty \subset B$. Then by the above, $B \circ \varphi^{-1}$ is a Douglas algebra which contains $B$ properly. And $B$ is a linear subspace of $B \circ \varphi^{-1}$ of codimension 1. By Chang and Marshall’s theorem, there exists an interpolating Blaschke product $b$ such that $b \in B \circ \varphi^{-1}$ and $b \notin B$. Since $\overline{b} \in B \circ \varphi^{-1} \setminus B$ and $\{\overline{b}^n; n = 1, 2, \ldots\}$ is linearly independent, $B$ is not a linear subspace of $B \circ \varphi^{-1}$ of codimension 1. This is a contradiction. Thus we get (2.13).

By (2.13), there exists $x_0 \in M(B)$ such that $\psi(x_0) = 0$. By (2.9), (2.10), and (2.11), $Tf = 0$ on $Z_B(\psi)$ for every $f \in B$. Then we have

$$(2.14) \quad Z_B(\psi) = \{x_0\}.$$ 

For, if $y \in Z_B(\psi)$ and $y \neq x_0$, there exist $f, g \in B$ such that $f(y) = 1, f(x_0) = 0$, and $g(y) = 0, g(x_0) = 1$. Then $f, g \notin TB$ and $f, g$ are linear independent. Hence $T$ is not a codimension 1 linear isometry.

Also we have

$$(2.15) \quad TB = \{f \in B; f(x_0) = 0\}$$ 

since $TB \subset \{f \in B; f(x_0) = 0\}$ and $TB$ is a linear subspace of $B$ of codimension 1.

Then we have

$$\psi B \subset \{f \in B; f(x_0) = 0\} \quad \text{by} \quad (2.14)$$

$$= \psi B \circ \varphi \quad \text{by} \quad (2.9) \text{and} \quad (2.15)$$

$$\subset \psi B \quad \text{by} \quad (2.11).$$

Thus we get

$$(2.16) \quad B = B \circ \varphi,$$

$$(2.17) \quad \psi B = \{f \in B; f(x_0) = 0\}.$$ 

We prove that $B \neq B_0$. By (2.13), it is sufficient to prove $\overline{\psi} \in B_0$. To show this, let $\{f_n\}_n \subset B$ such that $f_n \rightharpoonup 0$ weakly in $B$. Then $f_n(x_0) \rightharpoonup 0$. By (2.17), $f_n - f_n(x_0) \in \psi B$. Then there exists $g_n \in B$ such that $f_n - f_n(x_0) = \psi g_n$. Hence we have $\|f_n - \overline{\psi} - B\| \leq \|f_n - g_n\| \leq |f_n(x_0)| \to 0$ as $n \to \infty$. Therefore $\overline{\psi} \in B_0$.

Now we complete the proof. Condition (i) follows from (2.16). Since $x_0 \in M(B)$, $B_{|\text{supp } \mu_{x_0}} = (H^\infty)_{|\text{supp } \mu_{x_0}}$. By (2.10), there exists $h \in H^\infty$ such that $h_{|\text{supp } \mu_{x_0}} = \psi_{|\text{supp } \mu_{x_0}}$. If ord $(h, x_0) \geq 2$, there is a factorization $h = h_1 h_2$ such that $h_1(x_0) = h_2(x_0) = 0$ and $h_1, h_2 \in H^\infty$ (see [11] Section 5). Since $h_i \in B$, by (2.17) we have $h_i = \psi g_i$ for some $g_i \in B$ for $i = 1, 2$. Then $h = \psi^2 g_1 g_2$. Hence $\psi_{|\text{supp } \mu_{x_0}} = (\psi^2 g_1 g_2)_{|\text{supp } \mu_{x_0}}$. Since $|\psi| = 1$ on $M(L^\infty)$, $1 = (\psi^2 g_1 g_2)_{|\text{supp } \mu_{x_0}}$. Since $(\psi^2 g_1 g_2)(x_0) = 0$, we have a contradiction. Hence ord $(h, x_0) = 1$ and $x_0$ is a non-trivial point. Therefore by (2.14) and [12] Corollary 4.5, there are an interpolating Blaschke product $b$ and an invertible unimodular function $\nu$ in $B$ such that $\psi = ub$ and $Z_B(b) = \{x_0\}$. Then by [8] Theorem 3.2, $\{y \in M(B); |b(y)| < 1\} = P(x_0)$. This completes the proof.
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