PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 7, Pages 1899–1906
S 0002-9939(00)05865-2
Article electronically published on December 4, 2000

ON THE SLOPE OF BIELLIPTIC FIBRATIONS

MIGUEL A. BARJA

(Communicated by Ron Donagi)

A la memoria de Fernando

ABSTRACT. Let $\pi : S \longrightarrow B$ be a bielliptic fibration. We prove S is, up to base change, a rational double cover of an elliptic fibration and that π is isotrivial provided it is smooth. Finally, we prove that the slope of π is at least four provided the genus of the fibre is at least six.

0. INTRODUCTION

Let $\pi : S \longrightarrow B$ be a *fibration*, i.e. a surjective morphism with connected fibres, from a smooth projective surface S onto a smooth curve B. A fibration is said to be *relatively minimal* when it has no vertical (-1)-curve. Let g denote the genus of a general fibre and b the genus of B.

Let $\omega_{S/B} = \omega_S \otimes \pi^*(\omega_B^{-1})$ be the relative canonical bundle and let $\Delta(\pi) :=$ deg $\pi_*(\omega_{S/B})$. It is known that $\Delta(\pi) \geq 0$ and that $\Delta(\pi) = 0$ if and only if π is locally trivial with respect to the Zariski topology (see [3], III, theorem 18.2). Assume π is not locally trivial. Then we define the *slope* of π as

$$\lambda(\pi) := \omega_{S/B}^2 \Big/ \Delta(\pi)$$

(see [19]). There are several results on the lower slope of relatively minimal fibrations of genus $g \geq 2$. First of all we have $\lambda \geq 4 - \frac{4}{g}$ (see [8], [12], [13], [18] when the general fibre is hyperelliptic and [19] for the general case) and equality holds only in the hyperelliptic case ([9]). There are improvements in the non-hyperelliptic case for $g \leq 5$ (see [4], [7], [9], [11], [14]) but the presently-known techniques seem to have some limitations to extend these results to higher genus.

Recently Konno has being trying to find good bounds depending on some extra numerical invariants of the general fibre, such as the Clifford index. In [10], Konno finds better bounds for trigonal and plane quintic fibrations (so Clifford index 1), although they do not seem to be sharp. Also in [11] he gets general bounds depending on the Clifford index in some cases.

In this paper we deal with the case of bielliptic fibrations (i.e., when the general fibre has a 2-to-1 map onto an elliptic curve). Using the glueing results of [2] we know that if $g \geq 6$, a bielliptic fibration is a (generically) double cover of an elliptic

Received by the editors December 12, 1997 and, in revised form, October 29, 1999.
2000 *Mathematics Subject Classification.* Primary 14H10; Secondary 14J29.
Partially supported by CICYT PS93-0790 and HCM project n.ERBCHRXCT-940557.

fibration. We prove (Example 1.2) that this is not true in general if $g \leq 5$ due to the existence of several bielliptic maps in the general fibre.

Using this we get the following sharp bound for the slope of bielliptic fibrations.

Theorem 2.1. *Let $\pi : S \longrightarrow B$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let V be the relative minimal model of the elliptic fibration obtained in section 1. Then*

(a) $\lambda(\pi) \geq 4 + \dfrac{2(g-5)\chi\mathcal{O}_V}{\Delta(\pi)} \geq 4.$

(b) $\lambda(\pi) = 4$ *if and only if S is the minimal desingularization of a double cover of a smooth elliptic surface V such that:*
  - *All the fibres of the elliptic fibration $\tau : V \longrightarrow B$ are smooth and isomorphic.*
  - *The branch divisor of the double cover has only negligible singularities.*

In particular, the bound is sharp.

The author thanks, among others, Professor Juan Carlos Naranjo for his encouragement and interesting comments.

During the final revision of this paper the advisor of the author, Professor Fernando Serrano, passed away. The author would like to thank him heartfully for his support and continuous help, not only during the preparation of this work but also during the last years for his teaching and friendship.

Throughout this paper we work over the field of complex numbers $\mathbb{C}$.

1. Bielliptic fibrations

Let F be a smooth curve of genus g. The curve F is *bielliptic* if F admits a 2-to-1 map onto an elliptic smooth curve E. Such a map is always given by the quotient by an involution $\mathrm{i} \in \mathrm{Aut}(F)$, called a *bielliptic involution* on F. It is a well-known fact that such an involution is unique if $g \geq 6$ (see for example [1], page 280, exercise I-9).

Let $\pi : S \longrightarrow B$ be a fibration of genus g. We say that π is bielliptic if the general fibre F of π is also. The following result clarifies the structure of such fibrations. The fibration π is said to be smooth if every fibre is smooth and it is said to be isotrivial if all the smooth fibres are mutually isomorphic.

Proposition 1.1. *Let $\pi : S \longrightarrow B$ be a bielliptic fibration of genus g. Then:*

(a) *A base change of S is a rational double cover of an elliptic surface over the base curve.*

(b) *If $g \geq 6$ the same is true without base change.*

(c) *If π is smooth, then π is isotrivial.*

Proof. (a) and (b) are consequences of general results given in [2]. We give here a sketch of the proof and refer there for details.

Given $\pi : S \longrightarrow B$ we can consider $\psi : \underline{\mathrm{Aut}}^{2,2g-2}_{S/B} \longrightarrow B$ the scheme of relative automorphisms of S over B of order 2 having $2g-2$ fixed points (which corresponds fibrewise to double covers of elliptic curves) which is a quasi-projective B-scheme. After a base change $B' \longrightarrow B$ such a map always has a section defined over a non-empty Zariski open subset of B' which corresponds to a rational automorphism Φ of the minimal desingularization S' of $S \times_B B'$ such that $\Phi_{|F_t}$ is a bielliptic involution for $t \in B'$ general. If V is a desingularization of $S'/\langle\Phi\rangle$ we have a rational double cover $S' \dashrightarrow V$ over B'.

If $g \geq 6$, then ψ is 1-to-1 and base change is not needed in order to have a section.

(c) Isotriviality can be checked after base change. Following [2], section 2, we can consider after base change

$$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & J(\pi) \arrow[r, dashed, "f"] \arrow[d] & J(\pi) \arrow[dl] \\ & B & \end{tikzcd}$$

where $J(\pi)$ is the relative Jacobian variety of S over B and f is a rational relative endomorphism of $J(\pi)$ such that $f \circ i$ produces a bielliptic map on the general fibre of π (see [2]). Let $V = \overline{(f \circ i)(S)}$. Note that V is an elliptic surface over B (possibly singular). Nevertheless classification of singular fibres of a smooth elliptic surface shows, since $J(\pi)_t$ is an abelian variety for every $t \in B$, that V is smooth and the map $\tau : V \longrightarrow B$ is also smooth. Moreover, the map $g = f \circ i : S \dashrightarrow V$ can be solved after some blow-ups but then exceptional curves must be contracted since $V \subseteq J(\pi)$. So we have that S is a double cover of a smooth elliptic fibration (perhaps after base change). In particular *every* fibre of π is bielliptic.

Consider now the double cover $g : S \longrightarrow V$. Since g has degree two the branching divisor of g must be smooth and hence it is étale over B. If we perform base changes to the irreducible components of the branching divisor, we get a fibration for which the irreducible components of the branching divisor D are sections of τ. Moreover, since $\tau : V \longrightarrow B$ is a smooth elliptic fibration it is isotrivial (see [15], thms. 6,7, chapter IV) and then, after base change, we can assume $V = B \times E$ (E: elliptic smooth curve). Let D_1 be an irreducible component of D. If D_1 is a constant section of τ, then so must be the other components and then π is clearly isotrivial. Assume D_1 is not a constant section. Then $D_1 = \{(b, \alpha(b)) \in B \times E \mid \alpha : B \longrightarrow E$ non constant map$\}$. Fix a group structure on E and consider the automorphism of V over B defined by $\beta(b, x) = (b, x + \alpha(b))$. Note that $\beta^{-1}(D_1) = B \times \{0\}$ and, hence, $\beta^{-1}(D)$ is composed of trivial horizontal sections. If we change the base

$$\begin{tikzcd} S \arrow[r, "\simeq"] \arrow[d, "\widetilde{g}"'] \arrow[dr, phantom, "\otimes"] & S \arrow[d, "g"] \\ B \times E \arrow[r, "\beta"'] & B \times E \end{tikzcd}$$

the branching divisor of $\widetilde{g}$ is just $\beta^{-1}(D)$ which is constant. Hence π is isotrivial. $\square$

A bielliptic curve of genus $g \leq 5$ can have more than one bielliptic involution; the number of such involutions are in correspondence with the elliptic components of $W^1_4(F)$, the Brill-Noether locus of linear series on F of type g^1_4. We give an example which shows that these involutions do not glue independently for a general fibration.

Example 1.2. We are going to give an example for which the general fibre has two different bielliptic structures which are interchanged by the monodromy. Hence, the fibration is not a double cover of an elliptic fibration. Take a genus five curve F with *exactly* two bielliptic involutions $\sigma_i : F \longrightarrow E_i$ such that $E_1 \not\cong E_2$, with

E_i having no exceptional automorphisms (a count of constants shows that such an F can be chosen). Then $\sigma_1 \times \sigma_2 : F \longrightarrow E_1 \times E_2$ embeds F as a smooth curve, $F \in |\ell_1^*(2p_1) \otimes \ell_2^*(2p_2)|$, where $\ell_i : E_1 \times E_2 \longrightarrow E_i$ are the projections and $(p_1, p_2) \in E_1 \times E_2$. Since Aut $(E_1 \times E_2)$ acts transitively on $E_1 \times E_2$ we have that for every $(q_1, q_2) \in E_1 \times E_2$ there exists $\widetilde{F} \in |\ell_1^*(2q_1) \otimes \ell_2^*(2q_2)|$, $\widetilde{F} \cong F$.

Let B be any smooth curve having an involution ι and let $g : B \longrightarrow \bar{B} = B_{/\langle \iota \rangle}$. Consider a morphism $\kappa : B \longrightarrow \mathbb{P}^1$ with no factorization through $\bar{B}$. Take a fixed $\bar{t} \in \bar{B}$ such that $g^{-1}(\bar{t}) = \{t_1, t_2\}$ with $\kappa(t_1) \neq \kappa(t_2)$. After an automorphism of $\mathbb{P}^1$ we can suppose that $\kappa(t_i)$ is the modular invariant of E_i in $\mathbb{C} \subseteq \mathbb{P}^1$.

Then, by [3], p. 160, there exists an elliptic fibration $\tau : V \longrightarrow B$ with a section, such that $\tau^{-1}(t_i) \cong E_i$. Let B' be the image in V of the section of τ. Consider the following Cartesian diagram:

$$\begin{tikzcd} Z := V \times_B V \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] \arrow[dr, "\xi"] & V \arrow[d, "\tau"] \\ V \arrow[r, "\iota \circ \tau"'] & B \end{tikzcd}$$

Then, for $t \in B$ we have $Z_t = \xi^{-1}(t) = E_{\iota(t)} \times E_t$, where $E_m = \tau^{-1}(m)$. The natural involution on $V \times_{\mathbb{C}} V$ induces commutative diagrams

$$\begin{tikzcd} Z \arrow[r, "\bar{\iota}"] \arrow[d, "\xi"] & Z \arrow[d, "\xi"] \\ B \arrow[r, "\iota"] & B \end{tikzcd}$$

and

$$\begin{tikzcd} Z \arrow[r, "\bar{g}"] \arrow[d, "\xi"] & \bar{Z} := Z_{/\langle \iota \rangle} \arrow[d, "\bar{\xi}"] \\ B \arrow[r, "g"] & \bar{B} \end{tikzcd}$$

Note that $\bar{Z}$ is a threefold fibred over $\bar{B}$ and the fibre over a general $g(t) \in \bar{B}$ is $E_{\iota(t)} \times E_t$. We can assume $\bar{Z}$ is already smooth.

Let $B'' = \bar{g}(\xi_2^{-1}(B'))$ and $\mathcal{L} = \mathcal{O}_{\bar{Z}}(2B'')$. We have that $\mathcal{L}_{|\bar{Z}_{\bar{t}}} \cong \ell_1^*(2q_1) \otimes \ell_2^*(2q_2)$ for some $(q_1, q_2) \in E_1 \times E_2$. Note that if $\mathfrak{a} \in$ Pic $\bar{B}$ is ample enough we have an epimorphism

$$H^0(\bar{Z}, \mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})) \longrightarrow H^0(E_1 \times E_2, \mathcal{L}_{|\bar{Z}_{\bar{t}}}) \,.$$

Since by hypothesis there exists $F \in |\mathcal{L}_{|\bar{Z}_{\bar{t}}}|$ we get $\bar{S} \in |\mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})|$, a surface fibred over $\bar{B}$, smooth at a general fibre and such that $\bar{S}_{\bar{t}} = F$. Again, we can suppose $\bar{S}$ is smooth. Let $\bar{\pi} : \bar{S} \longrightarrow \bar{B}$ and $F_{\bar{m}} = \bar{\pi}^{-1}(\bar{m})$. For $\bar{m} \in \bar{B}$ general we have that $F_{\bar{m}}$ is a smooth curve of genus 5 having *at least* two bielliptic involutions given by the inclusion $F_{\bar{m}} \subseteq E_{\iota(m)} \times E_m$ (if $g(m) = \bar{m}$) as a $(2,2)$-divisor. We claim that in general $\bar{m} \in \bar{B}$, $F_{\bar{m}}$ has exactly two bielliptic involutions. Since this is the case for $F = F_{\bar{t}}$ we only have to prove that having *at most* two of them is an open condition. Consider $W_4^1(\bar{\pi}) \longrightarrow \bar{B}$, the relative Brill-Noether locus of

$\bar{\pi}$ (at least over an open set of B, see [16]), after a base change if necessary. The number of bielliptic involutions of $F_{\bar{m}}$ is given by the number of elliptic components of $W^1_4(F_{\bar{m}}) \cong W^1_4(\bar{\pi})_{\bar{m}}$. Then, having at most two of such components is obviously an open condition.

We claim that $\bar{S}$ is not a (birational) double cover of any elliptic fibration $\bar{\tau} : \bar{V} \longrightarrow \bar{B}$. Indeed, assume we have a double cover $\bar{f} : \bar{S} \longrightarrow \bar{V}$ (we can suppose $\bar{f}$ everywhere defined after some blow-ups). Consider the base change diagram

$$\begin{tikzcd}
Z \arrow[r] & \bar{Z} \\
S \arrow[r] \arrow[u, hook] \arrow[d, "\widetilde{f}"'] & \bar{S} \arrow[u, hook] \arrow[d, "\bar{f}"] \\
\widetilde{V} \arrow[r] \arrow[d, "\widetilde{\tau}"'] & \bar{V} \arrow[d, "\bar{\tau}"] \\
B \arrow[r] & \bar{B}
\end{tikzcd}$$

For S we have three double covers of elliptic fibrations over B:

$$\begin{aligned}
&\widetilde{f} : S \longrightarrow \widetilde{V}, \\
&f_i : S \longrightarrow V, \qquad\quad f_i = \xi_{i|S}, \quad i = 1, 2.
\end{aligned}$$

Set $U = \{m \in B \,|\, E_m \not\cong E_{\iota(m)};\, E_m, E_{\iota(m)}$ and $\widetilde{E}_m$ are smooth and F_m has exactly two bielliptic involutions$\}$ (where $\widetilde{E}_m = \widetilde{\tau}^{-1}(m)$). We have that U is a non-empty open set of B. Since $f_{1|F_m}$, $f_{2|F_m}$, $\widetilde{f}_{|F_m}$ are double covers of $E_{\iota(m)}$, E_m and $\widetilde{E}_m$ respectively, we have that for every $m \in U$, $\widetilde{E}_m \cong E_{\iota(m)}$ or $\widetilde{E}_m \cong E_m$.

If $g_1 = g \circ \iota_{|U} : U \longrightarrow \mathbb{P}^1$, $g_2 = g_{|U} : U \longrightarrow \mathbb{P}^1$ and $\widetilde{g} : U \longrightarrow \mathbb{P}^1$ are the modular morphisms induced by $\iota \circ \tau$, τ and $\widetilde{\tau}$ over U respectively, we have that $\widetilde{g} = g_1$ or $\widetilde{g} = g_2$. Assume $\widetilde{g} = g_2$.

As we have $t_1, t_2 \in U$ and $\iota(t_1) = t_2$ we get

$$E_{t_1} = \tau^{-1}(t_1) = \widetilde{\tau}^{-1}(t_1) \cong \widetilde{\tau}^{-1}(t_2) = \tau^{-1}(t_2) = E_{t_2}$$

since $\widetilde{\tau}$ is induced by $\bar{\tau} : \bar{V} \longrightarrow \bar{B}$ and then $\widetilde{\tau}^{-1}(m) \cong \widetilde{\tau}^{-1}(\iota(m))$ for all $m \in B$. But this is imposible since by hypothesis $E_{t_1} = E_1 \not\cong E_2 = E_{t_2}$. □

2. Double covers and the slope of bielliptic fibrations

We recall some basic facts about double covers (see [6], [3]).

By a double cover we mean a finite, degree two map between surfaces, $f_0 : S_0 \longrightarrow V_0$. This map is determined by a divisor Z_0 on V_0 (the branch divisor) and a line bundle $\mathcal{L}_0$ such that $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(Z_0)$. If V_0 is smooth, S_0 is normal (respectively smooth) if and only if Z_0 is reduced (respectively smooth).

Consider a double cover as above with S_0 normal and V_0 smooth. Then there exists a *canonical resolution of singularities* for S_0 which consists on a finite sequence of maps

$$\begin{tikzcd}
S_k \arrow[r, "\sigma_k"] \arrow[d, "f_k"'] & S_{k-1} \arrow[r] \arrow[d, "f_{k-1}"'] & \cdots \arrow[r] & S_1 \arrow[r, "\sigma_1"] \arrow[d, "f_1"] & S_0 \arrow[d, "f_0"] \\
V_k \arrow[r, "\alpha_k"'] & V_{k-1} \arrow[r] & \cdots \arrow[r] & V_1 \arrow[r, "\alpha_1"'] & V_0
\end{tikzcd}$$

satisfying:

(i) α_j is the blow-up of V_{j-1} at a singular point p_{j-1} of Z_{j-1} (the branching divisor of f_{j-1}).

(ii) f_j is the double cover of V_j defined by $\mathcal{L}_j^{\otimes 2} \cong \mathcal{O}(Z_j)$, with $Z_j = \alpha_j^*(Z_{j-1}) - 2m_{j-1}E_j$, $\mathcal{L}_j = \alpha_j^*(\mathcal{L}_{j-1}) \otimes \mathcal{O}_{V_j}(-m_{j-1}E_j)$, where E_j is the exceptional divisor of α_j and p_{j-1} is a singular point of Z_{j-1} of multiplicity $2m_{j-1}$ or $2m_{j-1}+1$.

(iii) σ_j is a birational morphism induced by the cartesian product of α_j and f_{j-1}.

(iv) Z_k is smooth and, hence, S_k is a smooth surface.

Now we can use this as follows. Recall from section 1 that we have obtained $f : \widetilde{S} \longrightarrow V$ a generically 2-to-1 morphism (we can suppose that f is everywhere defined up to performing blow-ups) from a blow-up of S onto an elliptic fibration V over B which we can suppose relatively minimal after some blow-downs. Suppose that π is relatively minimal.

Now consider

$$\begin{tikzcd}
\widetilde{S} \arrow[rrrd, "u"] \arrow[dd, "\sigma"] & & & \\
& \bar{S} = S_k \arrow[r] \arrow[ld] \arrow[dd, "f_k"'] & \cdots \arrow[r] & S_0 \arrow[dd, "f_0"'] \\
S \arrow[dd, "\pi"] & & \cdots & \\
& \bar{V} = V_k \arrow[r] & \cdots \arrow[r] & V_0 = V \arrow[llld] \\
B & & &
\end{tikzcd}$$

where:

- $f = f_0 \circ u$ is the Stein factorization of f, with u birational, f_0 finite (so it is a double cover) and S_0 normal.
- $f_k : S_k \longrightarrow V_k$ is the canonical resolution of singularities of $f_0 : S_0 \longrightarrow V_0$.
- $\bar{\sigma} : S_k \longrightarrow S$ is the birational morphism defined by the relative minimality of π.

Theorem 2.1. *Let $\pi : S \longrightarrow B$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let V be the relative minimal model of the elliptic fibration obtained in section 1. Then:*

(a) $\omega^2_{S/B} - 4\Delta(\pi) \geq 2(g-5)\chi\mathcal{O}_V$. *In particular, if* π *is not locally trivial*

$$\lambda(\pi) \geq 4 + \frac{2(g-5)\chi\mathcal{O}_V}{\Delta(\pi)} \geq 4.$$

(b) $\lambda(\pi) = 4$ *if and only if* S *is the minimal desingularization of a double cover* $S_0 \longrightarrow V$ *of a smooth elliptic surface such that:*

- *All the fibres of the elliptic fibration* $\tau : V \longrightarrow B$ *are smooth and isomorphic.*
- *The branch divisor of the double cover has only negligible singularities (i.e., all the multiplicities* m_j *in the above process are* 2 *or* 3 *(see* [13], [17]*)).*

In particular, the bound is sharp.

Proof. (a) First of all we have

$$\begin{aligned}\omega^2_{S/B} - 4\Delta(\pi) &= (K^2_S - 4\chi\mathcal{O}_S) - 4(b-1)(g-1)\\ &\geq (K^2_{\bar{S}} - 4\chi\mathcal{O}_{\bar{S}}) - 4(b-1)(g-1)\,.\end{aligned} \tag{1}$$

For smooth double covers $f_k : \bar{S} \longrightarrow \bar{V}$ we have (see [3], p. 183):

$$\begin{aligned}\chi\mathcal{O}_{\bar{S}} &= 2\chi\mathcal{O}_{\bar{V}} + \frac{1}{2}\mathcal{L}_k K_{\bar{V}} + \frac{1}{2}\mathcal{L}_k\mathcal{L}_k,\\ K^2_{\bar{S}} &= 2K^2_{\bar{V}} + 4\mathcal{L}_k K_{\bar{V}} + 2\mathcal{L}_k\mathcal{L}_k,\end{aligned}$$

so we have

$$K^2_{\bar{S}} - 4\chi\mathcal{O}_{\bar{S}} = 2[K^2_{V_k} - 4\chi\mathcal{O}_{V_k}] + 2\mathcal{L}_k K_{V_k}\,. \tag{2}$$

Moreover, in each blow-up $\alpha_j : V_j \longrightarrow V_{j-1}$ we get

$$\chi\mathcal{O}_{V_j} = \chi\mathcal{O}_{V_{j-1}}; \quad K_{V_j} = \alpha^*_j K_{V_{j-1}} + E_j; \quad \mathcal{L}_j = \alpha^*_j\mathcal{L}_{j-1} - m_{j-1}E_j\,.$$

Then

$$\begin{aligned}2[K^2_{V_j} - 4\chi\mathcal{O}_{V_j}] + 2\mathcal{L}_j K_{V_j} &= 2[K^2_{V_{j-1}} - 4\chi\mathcal{O}_{V_{j-1}}]\\ +2\mathcal{L}_{j-1}K_{V_{j-1}} + 2(m_{j-1}-1) &\geq 2[K^2_{V_{j-1}} - 4\chi\mathcal{O}_{V_{j-1}}] + 2\mathcal{L}_{j-1}K_{V_{j-1}}\,.\end{aligned} \tag{3}$$

Finally as $\tau : V \longrightarrow B$ is an elliptic minimal fibration, numerically we have $K_V \equiv \left[2(b-1) + \chi\mathcal{O}_V + \sum_i \frac{(n_i-1)}{n_i}\right] E$ ([3], p. 162) where E denotes a smooth fibre of τ and $\{n_i\}$ are the multiplicities of singular fibres of τ. In particular $K^2_V \equiv 0$.

As $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(Z_0)$ and Z_0 is the branch divisor of f_0 we get $\mathcal{L}_0 E = (g-1)$ by Hurwitz formula. So

$$\begin{aligned}&2[K^2_{V_0} - 4\chi\mathcal{O}_{V_0}] + 2\mathcal{L}_0 K_{V_0} = -8\chi\mathcal{O}_{V_0}\\ &+2\mathcal{L}_0 E\left[2(b-1) + \chi\mathcal{O}_{V_0} + \sum_i \frac{(n_i-1)}{n_i}\right] \geq 4(b-1)(g-1) + 2(g-5)\chi\mathcal{O}_V\,.\end{aligned} \tag{4}$$

Then (a) follows from (1), (2), (3) and (4) and from the fact that $\chi\mathcal{O}_V \geq 0$ for elliptic fibrations.

(b) Looking at the proof of (a) we see that $\lambda = 4$ iff $\chi\mathcal{O}_V = 0$ and equality holds in (1), (2), (3) and (4). So we have $\lambda = 4$ iff S is the minimal desingularization of a double cover of an elliptic, relatively minimal, fibration $\tau : V \longrightarrow B$ such that:

- τ has no multiple fibres ($\forall i \quad n_i = 1$).
- $\chi\mathcal{O}_V = 0$.

- The branch divisor Z_0 of the double cover has only *negligible singularities* (see [13], [17]), i.e. all the multiplicities of the singularities of the branch divisors in the process of canonical resolution are 2 or 3.

But the first two conditions are equivalent to the fact that τ is smooth and isotrivial (see [15], thms. 6, 7, Ch. IV). This allows us to construct examples with $\lambda(\pi) = 4$ which are essentially the same as in [19], example 4.3. So the bound is sharp. $\square$

Remark 2.2. Although we cannot use double covers for the case of bielliptic fibrations of genus 5 we already know that $\lambda \geq 4$ also holds for such fibrations (see [9] thm. 5.1, [11]).

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Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

E-mail address: barja@ma1.upc.es