PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 7, Pages 1927–1937
S 0002-9939(00)05870-6
Article electronically published on November 22, 2000

THE RELATIVE PLURICANONICAL STABILITY FOR 3-FOLDS OF GENERAL TYPE

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(Communicated by Ron Donagi)

ABSTRACT. The aim of this paper is to improve a theorem of János Kollár by a different method. For a given smooth complex projective threefold X of general type, suppose the plurigenus $P_k(X) \geq 2$. Kollár proved that the $(11k+5)$-canonical map is birational. Here we show that either the $(7k+3)$-canonical map or the $(7k+5)$-canonical map is birational and that the $(13k+6)$-canonical map is stably birational onto its image. Suppose $P_k(X) \geq 3$. Then the m-canonical map is birational for $m \geq 10k+8$. In particular, ϕ_{12} is birational whenever $p_g(X) \geq 2$ and ϕ_{11} is birational whenever $p_g(X) \geq 3$.

INTRODUCTION

Let X be a smooth projective 3-fold of general type defined over $\mathbb{C}$ and denote by ϕ_m the m-canonical map of X, which is the rational map associated with the linear system $|mK_X|$. Let $P_k(X) := h^0(X, \mathcal{O}_X(kK_X))$ for any positive integer k. We usually call $P_k(X)$ *the k-th plurigenus* of X which is a birational invariant. For a given positive integer m_0, we say that ϕ_{m_0} is *stably birational* if ϕ_m is birational onto its image for all $m \geq m_0$. Since the Kodaira dimension $\mathrm{kod}(X) = 3$, ϕ_m is birational for $m \gg 0$. In this paper, we consider the following

Problem. *Suppose $P_k(X) \geq 2$. For which value $m_0(k)$, does $|m_0(k)K_X|$ define a stably birational map onto its image?*

In 1986, Kollár ([5, Corollary 4.8]) first gave an effective result and proved that the $(11k+5)$-canonical map is birational if $P_k(X) \geq 2$. However, his method cannot tell whether ϕ_m is still birational for all $m > 11k+5$. On the other hand, it seems to us that the number $11k+5$ is not the optimal one. This paper aims to present a better result as the following

Main Theorem. *Let X be a nonsingular projective threefold of general type and suppose $P_k(X) \geq 2$. Then the following hold:*
(i) either ϕ_{7k+3} or ϕ_{7k+5} is birational onto its image;
(ii) ϕ_{13k+6} is stably birational onto its image;
(iii) ϕ_{10k+8} is stably birational provided that $P_k(X) \geq 3$.
In particular, ϕ_{12} is stably birational if $p_g(X) \geq 2$ and ϕ_{11} is stably birational if $p_g(X) \geq 3$.

Received by the editors December 12, 1998 and, in revised form, November 12, 1999.
1991 *Mathematics Subject Classification.* Primary 14C20, 14E05, 14E35.
The author was supported in part by the National Natural Science Foundation of China.

Noting that the main obstacle which prevents Kollár's method from getting a better bound is the case when X admits a rational pencil of certain surfaces of general type, we shall mainly make a special study of this situation in an alternative way. First we build some birationality criteria for adjoint systems on a surface of general type. Then we reduce the problem to the surface case while finding suitable divisors on the threefold whose restrictions to the surface satisfy those criteria. The Kawamata-Viehweg vanishing theorem plays a key role throughout our argument.

Definition. Let X be a normal projective variety and D be a Weil divisor on X. Denote by $\Phi_{|D|}$ the natural rational map defined by the linear system $|D|$. $|D|$ is called *base point free* if it has neither fixed components nor base points.

If $|L|$ is a linear system on X without fixed components and $h^0(X, L) \geq 2$, we mean *a general irreducible element* S of $|L|$ as follows:

(1) If $\dim \Phi_{|L|}(X) \geq 2$, then S is a general member of $|L|$.

(2) If $\dim \Phi_{|L|}(X) = 1$, then L is linearly equivalent to a union of distinct reduced irreducible divisors of the same type. Explicitly, $L \sim_{\text{lin}} \sum S_i$. We mean by S a general S_i.

X is called *minimal* if the canonical divisor K_X is nef, i.e. $K_X \cdot C \geq 0$ for all proper curves $C \subset X$.

X is said to *be of general type* if the Kodaira dimension $\text{kod}(X) = \dim(X)$.

X is said to *have only terminal singularities* according to Reid ([7]) if the following two conditions hold:

(i) for some integer $r \geq 1$, rK_X is Cartier;

(ii) for some resolution $f : Y \longrightarrow X$, $K_Y = f^*(K_X) + \sum a_i E_i$ for $0 < a_i \in \mathbb{Q}$ for all i, where the E_i vary all the exceptional divisors on Y.

1. Preparation

Throughout our argument, the Kawamata-Viehweg vanishing theorem will be always employed as a much more effective tool. We shall use it in the following form.

Vanishing Theorem ([3] or [10]). *Let X be a nonsingular complete variety, $D \in \text{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions:*

(1) D is nef and big;

(2) the fractional part of D has supports with only normal crossings.

Then $H^i(X, \mathcal{O}_X(\ulcorner D \urcorner + K_X)) = 0$ for $i > 0$, where $\ulcorner D \urcorner$ is the round-up of D, i.e. the minimum integral divisor with $\ulcorner D \urcorner - D \geq 0$.

Another important principle that is tacitly used throughout the text is due to Tankeev ([9]). Explicitly, on a smooth projective variety X, if we have a base point free system $|M|$ and an effective divisor D, we want to study the birationality of the map $\Phi_{|D+M|}$. Now let S be a general irreducible element of $|M|$. Then S is a smooth divisor on X by Bertini's theorem. Suppose we have known that $\Phi_{|D+M|}$ can distinguish general irreducible elements and that $\Phi_{|D+M|}\big|_S$ is birational. Then Tankeev's principle implies the birationality of $\Phi_{|D+M|}$.

Lemma 1.1 ([8, Corollary 2]). *Let S be a nonsingular algebraic surface, L be a nef divisor on S, $L^2 \geq 10$ and let ϕ be a map defined by $|L + K_S|$. If ϕ is not birational, then S contains a base point free pencil E' with $L \cdot E' = 1$ or $L \cdot E' = 2$.*

Lemma 1.2. *Let S be a nonsingular projective surface of general type and suppose L is a divisor with $h^0(S, L) \geq 2$. Then $h^0(S, K_S + L) \geq 2$. In particular, if $\chi(\mathcal{O}_S) \geq 3$, then $h^0(S, K_S + L) \geq 4$.*

Proof. Taking a general irreducible element C in the moving part of $|L|$, then C is a nef divisor, $C \leq L$ and C is a curve of genus ≥ 2. By R-R on the surface S, we have

$$h^0(S, K_S + L) \geq h^0(S, K_S + C) \geq \frac{1}{2}(K_S \cdot C + C^2) + \chi(\mathcal{O}_S).$$

It is easy to get the result. □

Lemma 1.3. *Let S be a nonsingular projective surface of general type, L be a nef divisor, $L^2 \geq 3$ and $\dim \Phi_{|L|}(S) = 2$. Then $|K_S + 2L|$ gives a birational map.*

Proof. We have $(2L)^2 \geq 12$. If $\Phi_{|K_S+2L|}$ is not birational, then according to Lemma 1.1, there is a base point free pencil E' such that $2L \cdot E' \leq 2$, i.e. $L \cdot E' = 1$. Since $\dim \Phi_{|L|}(S) = 2$ and E' is a curve of genus ≥ 2, we see that $L \cdot E' \geq 2$, a contradiction. □

Lemma 1.4. *Let S be a nonsingular projective surface of general type, L_i be a divisor on S such that $\dim \Phi_{|L_i|}(S) \geq i$ for $i = 1,\ 2$. Then $|K_S + 2L_2 + L_1|$ gives a birational map.*

Proof. Modulo blowing-ups, we can suppose that the $|L_i|$ is base point free for $i = 1,\ 2$. This means that L_2 is nef and big and that L_1 is nef.

If the system $|L_2|$ gives a birational map, then so does $|K_S + 2L_2 + L_1|$, because $K_S + L_1$ is effective by Lemma 1.2.

Otherwise, we have $L_2^2 \geq 2$. Now we have $(2L_2 + L_1)^2 \geq 12$. If $|K_S + 2L_2 + L_1|$ does not give a birational map, then, by Lemma 1.1, there is a free pencil E' on S such that

$$(2L_2 + L_1) \cdot E' \leq 2.$$

This means $L_2 \cdot E' = 1$. Note that E' is a curve of genus ≥ 2 and $|L_2|$ gives a generically finite map. The Riemann-Roch theorem on the curve E' tells that $\deg(L_2|_{E'}) \geq 2$. We have derived a contradiction. □

Lemma 1.5. *Let X be a nonsingular projective 3-fold of general type. Suppose that L_i is a divisor on X such that $\dim \Phi_{|L_i|}(X) \geq i$ for $i = 1,\ 2,\ 3$. Then*

$$|K_X + 2L_3 + L_2 + L_1|$$

gives a birational map.

Proof. Take a birational modification $\pi : X' \longrightarrow X$ according to Hironaka such that the $|\pi^*(L_i)|$ are all base point free for $i > 0$. On X', we can study the system $|K_{X'} + 2\pi^*(L_3) + \pi^*(L_2) + \pi^*(L_1)|$. Let M_i be the moving part of $|\pi^*(L_i)|$. We have

$$|K_{X'} + 2M_3 + M_2 + M_1| \subset |K_{X'} + 2\pi^*(L_3) + \pi^*(L_2) + \pi^*(L_1)|.$$

Therefore, for simplicity, we can suppose from the beginning that the $|L_i|$ are base point free on X. So L_3 is nef and big under this assumption.

Step 1. Verifying that $K_X + 2L_3 + L_2$ is effective.

We have $\dim \Phi_{|L_2|}(X) \geq 2$. So a general member $S \in |L_2|$ is a nonsingular projective surface of general type. Using the vanishing theorem to the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + 2L_3) \longrightarrow \mathcal{O}_X(K_X + 2L_3 + S) \longrightarrow \mathcal{O}_S(K_S + 2L_3|_S) \longrightarrow 0,$$

we get the surjective map

$$H^0(X, K_X + 2L_3 + S) \longrightarrow H^0(S, K_S + 2L_3|_S) \longrightarrow 0.$$

From Lemma 1.2, we know $K_S + 2L_3|_S$ is effective, so is $K_X + 2L_3 + L_2$.

Step 2. Reduction to surface case.

Taking a 1-dimensional sub-system of $|L_1|$, then this system defines a rational map onto $\mathbb{P}^1$. Taking further blowing-up if necessary, we can also suppose that this system defines a morphism $f : X \longrightarrow \mathbb{P}^1$. Taking the Stein factorization of f, one obtains a derived fibration $g : X \longrightarrow C$. A general fibre of f can be written as a disjoint union $\sum F_i$. Let F be a general fibre of g; then it is a nonsingular projective surface of general type and we have $F \leq L_1$. Now considering the system $|K_X + 2L_3 + L_2 + \sum F_i|$, it can distinguish general fibres of g because $K_X + 2L_3 + L_2$ is effective and $2L_3 + L_2$ is nef and big. Using the vanishing theorem again, we have

$$|K_X + 2L_3 + L_2 + \sum F_i| \;\big|_F = |K_F + 2L_3' + L_2'|,$$

where $L_3' := L_3|_F$ and $L_2' := L_2|_F$. Lemma 1.4 tells that the right system gives a birational map, so does $|K_X + 2L_3 + L_2 + L_1|$. The proof is completed. □

Lemma 1.6. *Let X be a nonsingular variety of dimension n, $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ be a $\mathbb{Q}$-divisor on X. Then we have the following:*

(i) if S is a smooth irreducible divisor on X and S is not a fractional component of D, then $\ulcorner D \urcorner|_S \geq \ulcorner D|_S \urcorner$;

(ii) if $\pi : X' \longrightarrow X$ is a birational morphism, then $\pi^*(\ulcorner D \urcorner) \geq \ulcorner \pi^*(D) \urcorner$.

Proof. These are trivial. □

Lemma 1.7. *Let X be a nonsingular projective threefold of general type. Let D be a divisor on X with $h^0(X, D) \geq 2$ and suppose $|D|$ has no fixed components. Denote by F a general irreducible element of $|D|$. If L is another divisor such that* $\dim \Phi_{|L|}(F) \geq 1$*, then $mK_X + L + D$ is effective and* $\dim \Phi_{|mK_X + L + D|}(F) \geq 1$ *for all $m \geq 2$.*

Proof. According to the 3-dimensional MMP ([4] and [6]), X has a minimal model X_0 which is normal projective with only $\mathbb{Q}$-factorial terminal singularities. Let $\alpha : X \dashrightarrow X_0$ be the contraction which is a rational map. Take a common resolution X' with $\pi' : X' \longrightarrow X$ and $\pi : X' \longrightarrow X_0$ such that $\pi = \alpha \circ \pi'$ and that

(1) both $|{\pi'}^*(L)|$ and $|{\pi'}^*(D)|$ have no base points (they may have fixed components);

(2) $\pi^*(K_{X_0})$ has supports with only normal crossings.

This is possible because of Hironaka's big theorem. Since ${\pi'}^*(mK_X + L + D) \leq mK_{X'} + {\pi'}^*(L) + {\pi'}^*(D)$ and

$$\begin{aligned} {\pi'}_*\mathcal{O}_{X'}(mK_{X'} + {\pi'}^*(L) + {\pi'}^*(D)) &= \mathcal{O}_X(mK_X + L + D) \\ &= {\pi'}_*{\pi'}^*\mathcal{O}_X(mK_X + L + D), \end{aligned}$$

then $h^0\big(X', \pi'^*(mK_X + L + D)\big) = h^0\big(X', mK_{X'} + \pi'^*(L) + \pi'^*(D)\big)$, so

$$\Phi_{|\pi'^*(mK_X+L+D)|} \text{ and } \Phi_{|mK_{X'}+\pi'^*(L)+\pi'^*(D)|}$$

have the same behavior. Let S be a general irreducible element of the moving part of $|\pi'^*(D)|$; then $\dim \Phi_{|\pi'^*(L)|}(S) \geq 1$ by assumption. Therefore it is sufficient to show

$$\dim \Phi_{|mK_{X'}+\pi'^*(L)+\pi'^*(D)|}(S) \geq 1$$

for $m \geq 2$. Let H be the moving part of $|\pi'^*(L)|$; then H is nef since $|H|$ is base point free. We have

$$|K_{X'} + \ulcorner(m-1)\pi^* K_{X_0}\urcorner + H + S| \subset |mK_{X'} + \pi'^*(L) + \pi'^*(D)|.$$

The Kawamata-Viehweg vanishing theorem gives

$$\begin{aligned}&|K_{X'} + \ulcorner(m-1)\pi^* K_{X_0}\urcorner + H + S|\,\big|_S \\ &= |K_S + \ulcorner(m-1)\pi^* K_{X_0}\urcorner|_S + M| \supset |K_S + \ulcorner B\urcorner + M|,\end{aligned}$$

where $B := (m-1)\pi^* K_{X_0}|_S$ is nef and big on S and $M := H|_S$. From the assumption, we have $h^0(S, M) \geq 2$. Choosing a 1-dimensional sub-system $|C|$ in $|M|$, modulo blowing-ups, we can suppose $|C|$ to be base point free. Also from the vanishing theorem, we have

$$|K_S + \ulcorner B\urcorner + C|\,\big|_C = |K_C + D|,$$

where $D := \ulcorner B\urcorner|_C$ is a divisor on the curve C with positive degree since $D \geq \ulcorner B|_C\urcorner$ by Lemma 1.6(i). Because $g(C) \geq 2$, we have $h^0(K_C + D) \geq 2$. This means $|K_C + D|$ gives a generically finite map and

$$\dim \Phi_{|K_S+\ulcorner B\urcorner+C|}(C) = 1;$$

thus $K_{X'} + \ulcorner(m-1)\pi^* K_{X_0}\urcorner + \pi'^*(L) + \pi'^*(D)$ is effective and the image of S through the map defined by this divisor is at least 1. The proof is completed. □

2. Proof of the main theorem

2.1 Basic formula. Let X be a nonsingular projective threefold, $f : X \longrightarrow C$ be a fibration onto a nonsingular curve C. From the spectral sequence:

$$E_2^{p,q} := H^p(C, R^q f_* \omega_X) \Rightarrow E^n := H^n(X, \omega_X),$$

we get by direct calculation that

$$h^2(X, \mathcal{O}_X) = h^1(C, f_*\omega_X) + h^0(C, R^1 f_* \omega_X),$$

$$q(X) := h^1(X, \mathcal{O}_X) = b + h^1(C, R^1 f_* \omega_X),$$

where b denotes the genus of C.

2.2 Review of Kollár's technique. Let X be a smooth projective 3-fold of general type and suppose $P_k(X) \geq 2$. Choose a 1-dimensional sub-system of $|kK_X|$ and replace X by a birational model X' where this pencil defines a morphism $g : X' \longrightarrow \mathbb{P}^1$. (For simplicity, we can suppose $X' = X$.) Let S be a general irreducible element of this pencil. Then a general fibre of g is a disjoint union of some surfaces with the same type as S and S is a smooth projective surface of general type. Let $t = k(2p+1)+p$. Then $H^0(\omega_X^t) = H^0(\mathbb{P}^1, g_*\omega_X^t)$ and we have an injection $\mathcal{O}(1) \hookrightarrow g_*\omega_X^k$, and hence an injection $\mathcal{O}(2p+1) \hookrightarrow g_*\omega_X^{k(2p+1)}$. This gives an injection

$$\mathcal{O}(2p+1) \otimes g_*\omega_X^p \hookrightarrow g_*\omega_X^t,$$

where $\mathcal{O}(2p+1) \otimes g_*\omega_X^p = \mathcal{O}(1) \otimes g_*\omega_{X/\mathbb{P}^1}^p$. Now it is well-known that $g_*\omega_{X/\mathbb{P}^1}^p$ is a sum of line bundles of non-negative degree on $\mathbb{P}^1$. If $p \geq 5$, the local sections of $g_*\omega_X^p$ give a birational map for S and all these extend to global sections of $\mathcal{O}(2p+1) \otimes g_*\omega_X^p$. Moreover its sections separate the fibres from each other, and hence ϕ_t is a birational map for X.

From the above method, according to [1] and [11], we have

(1) ϕ_{5k+2} is generically finite for X if S is not a surface with $p_g(S) = q(S) = 0$ and $K_{S_0}^2 = 1$, where S_0 is the minimal model of S. Otherwise, we have at least $\dim \phi_{5k+2}(X) \geq 2$;

(2) ϕ_{7k+3} is birational for X if S is not a surface with

$$(K_{S_0}^2, p_g(S)) = (1,2) \text{ or } (2,3).$$

2.3 Proof of the main theorem. According to the 3-dimensional MMP, we can suppose X to be a minimal model with at worst $\mathbb{Q}$-factorial terminal singularities. This means that K_X is a nef and big $\mathbb{Q}$-divisor. We begin from a minimal model in order to make use of the Kawamata-Viehweg vanishing theorem.

Theorem 2.3.1. *Let X be a nonsingular projective 3-fold of general type and suppose $P_k(X) \geq 2$. Then either ϕ_{7k+3} or ϕ_{7k+5} is birational.*

Proof. Suppose X is a minimal model with at worst $\mathbb{Q}$-factorial terminal singularities. Choose a 1-dimensional sub-system Λ of $|kK_X|$ and take a birational modification $\pi : X' \longrightarrow X$ such that

(i) X' is nonsingular;

(ii) $\pi^*\Lambda$ gives a morphism;

(iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

This is possible because of Hironaka's big theorem. Set $g_1 := \Phi_\Lambda \circ \pi$ and let $X' \xrightarrow{f_1} W_1 \xrightarrow{s_1} \mathbb{P}^1$ be the Stein factorization of g_1. Denote $b := g(W_1)$, the geometric genus of the curve W_1.

If $b > 0$, then the moving part of Λ is base point free. Let $\sum S_i$ be the moving part of Λ; then $\sum S_i \leq kK_X$ and a general S_i is a smooth projective surface of general type, since the singularities on X are isolated. Using Kawamata's vanishing theorem ([4]) to $\mathbb{Q}$-Cartier Weil divisors on minimal threefold X, we see that $|(a+1)K_X + \sum S_i|$ can distinguish general S_i for $a > 0$ and

$$H^0(X, (a+1)K_X + \sum S_i) \longrightarrow \bigoplus H^0(S_i, (a+1)K_{S_i})$$

is surjective. Therefore it is obvious that ϕ_m is effective whenever $m \geq k+2$, generically finite whenever $m \geq 2k+2$, birational whenever $m \geq 2k+4$.

So, from now on, we can suppose that $b = 0$. We have a fibration $f_1 : X' \longrightarrow \mathbb{P}^1$. Let F be a general fibre of f_1. By virtue of 2.2(2), we can suppose that F is a surface with invariants $(K_{F_0}^2, p_g(F)) = (1,2)$ or $(2,3)$, where F_0 is the minimal model of F. F is the moving part of $\pi^*\Lambda$ and $F \leq_{\mathbb{Q}} \pi^*(kK_X)$. We automatically have $q(F) = 0$. First we study the system $|K_{X'} + \ulcorner k\pi^*(K_X)\urcorner + F|$. For a general fibre F, the vanishing theorem gives that

$$|K_{X'} + \ulcorner k\pi^*(K_X)\urcorner + F| \,\big|_F = |K_F + \ulcorner k\pi^*(K_X)\urcorner|_F \,\big|,$$

where $\ulcorner k\pi^*(K_X)\urcorner|_F$ is effective. This means that $(2k+1)K_{X'}$ is effective and $\dim \phi_{2k+1}(F) \geq 1$. By Lemma 1.7, we see that $mK_{X'}$ is effective and $\dim \phi_m(F) \geq 1$ for $m \geq 3k+3$.

Actually, we have $\dim \phi_{3k+2}(F) = 2$. In fact, we have

$$|K_{X'} + \ulcorner (2k+1)\pi^*(K_X)\urcorner + F| \,\big|_F \supset |K_F + M_{2k+1}|_F \,\big|,$$

where M_{2k+1} is the moving part of $|\ulcorner(2k+1)\pi^*K_X\urcorner|$. It is easy to check that $\big| \, K_F + M_{2k+1}|_F \, \big|$ gives a generically finite map because $q(F) = 0$ and $p_g(F) > 0$. Thus

$$\dim \Phi_{|K_{X'} + \ulcorner(2k+1)\pi^*(K_X)\urcorner + F|}(F) \geq 2.$$

We have $|K_{X'} + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner + F| \subset |(7k+5)K_{X'}|$. $K_{X'} + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner$ is effective by the above argument. So $|K_{X'} + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner + F|$ can distinguish general fibre F. On the other hand, the Kawamata-Viehweg vanishing theorem gives

$$\begin{aligned} |K_{X'} + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner + F| \,\big|_F &= \big| \, K_F + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner|_F \, \big| \\ &\supset |K_F + 2L_{3k+2}|, \end{aligned}$$

where $L_{3k+2} := M_{3k+2}|_F$. It is sufficient to show that $|K_F + 2L_{3k+2}|$ gives a birational map for F. We have already known that $|L_{3k+2}|$ gives a generically finite map for F. Excluding the fixed components of $|L_{3k+2}|$, we can suppose that $|L_{3k+2}|$ are moving on the surface F. So L_{3k+2} is nef. If $|L_{3k+2}|$ gives a birational map, then so does $|K_F + 2L_{3k+2}|$. Otherwise,

$$L_{3k+2}^2 \geq 2(h^0(F, L_{3k+2}) - 2).$$

Consider the following three natural maps:

$$\begin{aligned} &H^0(X', M_{3k+2}) \xrightarrow{\alpha} H^0(F, L_{3k+2}), \\ &H^0(X', K_{X'} + \ulcorner(2k+1)\pi^*(K_X)\urcorner + F) \\ &\qquad\qquad\qquad \xrightarrow{\beta} H^0(F, K_F + \ulcorner(2k+1)\pi^*(K_X)\urcorner|_F) \longrightarrow 0, \\ &H^0\big(X', (3k+2)K_{X'}\big) \xrightarrow{\gamma} H^0\big(F, (3k+2)K_F\big) \end{aligned}$$

where β is surjective by the Kawamata-Viehweg vanishing theorem. We see that

$$\dim_{\mathbb{C}}\big(\mathrm{im}(\alpha)\big) = \dim_{\mathbb{C}}\big(\mathrm{im}(\gamma)\big) \geq \dim_{\mathbb{C}}\big(\mathrm{im}(\beta)\big) = h^0(F, K_F + D_{2k+1})$$

where $D_{2k+1} := \ulcorner(2k+1)\pi^*(K_X)\urcorner|_F$ and $h^0(F, D_{2k+1}) \geq 2$. So

$$h^0(F, K_F + D_{2k+1}) \geq 4,$$

according to Lemma 1.2, because we have $\chi(\mathcal{O}_F) \geq 3$ in this case. Thus

$$L_{3k+2}^2 \geq 2\big(h^0(F, L_{3k+2}) - 2\big) \geq 2\Big(\dim_{\mathbb{C}}\big(\mathrm{im}(\alpha)\big) - 2\Big) \geq 4$$

and then $|K_F+2L_{3k+2}|$ gives a birational map by Lemma 1.3. So ϕ_{7k+5} is birational.

Finally, for all $m \geq 10k+7$, set $t := m-7k-5 \geq 3k+2$; then $\dim \phi_t(F) \geq 1$. In particular, $tK_{X'}$ is effective. So ϕ_m is birational for all $m \geq 10k+7$ in this case. □

Corollary 2.3.1. *Let X be an irregular nonsingular 3-fold of general type and suppose $P_k(X) \geq 2$. Then ϕ_{7k+3} is birational. Therefore at least ϕ_{143} is birational according to Kollár and Fletcher.*

Proof. In the proof of the last theorem, if $b>0$, then ϕ_m is birational for $m \geq 2k+4$. If $b=0$, we can use the formula of $q(X)$ to the fibration $f_1 : X' \longrightarrow \mathbb{P}^1$. When $q(X)>0$, then we must have $q(F)>0$. Then $\Phi_{|3K_F|}$ is birational for the fibre F, and so is $\Phi_{|(7k+3)K_X|}$ by 2.2(2). Moreover, we have $P_{20}(X) \geq 2$ for any irregular 3-fold of general type according to Kollár ([5]) and Fletcher ([2]). Thus ϕ_{143} is birational. □

Theorem 2.3.2. *Let X be a nonsingular projective threefold of general type and suppose $P_k(X) \geq 2$. Then ϕ_m is birational for $m \geq 13k+6$.*

Proof. Suppose X is a minimal model with at worst $\mathbb{Q}$-factorial terminal singularities. Make a birational modification $\pi : X' \longrightarrow X$ such that:

(i) X' is nonsingular;
(ii) $|kK_{X'}|$ gives a morphism;
(iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Set $g := \Phi_{|kK_X|} \circ \pi$ and $W' := \overline{\Phi_{|kK_X|}(X)}$. Let $X' \xrightarrow{f} W \xrightarrow{s} W'$ be the Stein factorization of g.

We would like to formulate our proof through two steps as follows.

Case 1. $\dim \phi_k(X) \geq 2$.

Set $kK_{X'} \sim_{\text{lin}} M_k + Z_k$, where M_k is the moving part and Z_k is the fixed part. Then a general member $S \in |M_k|$ is an irreducible nonsingular projective surface of general type. Write $K_{X'} = \pi^*(K_X) + \sum a_i E_i$, where the E_i are exceptional divisors for π, $0 < a_i \in \mathbb{Q}$ for each i. Obviously, $\ulcorner \pi^*(K_X) \urcorner \leq K_{X'}$. Because $h^0(X', \ulcorner \pi^*(kK_X) \urcorner) = h^0(X', kK_{X'})$, we can see that M_k is actually also the moving part of $|\ulcorner \pi^*(kK_X) \urcorner|$. Thus we have

$$\pi^*(kK_X) \geq_{\mathbb{Q}} M_k + \sum b_i E_i,$$

where $0 \leq b_i \in \mathbb{Q}$ for each i.

We claim that $mK_{X'}$ is always effective for $m \geq 2k+1$. In fact, for any $t \in \mathbb{Z}^+$, we consider the system

$$|K_{X'} + \ulcorner \pi^*((t+k)K_X) \urcorner + S|.$$

It is a sub-system of $|(2k+t+1)K_{X'}|$. By the Kawamata-Viehweg vanishing theorem, we have a surjective map

$$H^0(X', K_{X'} + \ulcorner \pi^*((t+k)K_X) \urcorner + S) \longrightarrow H^0(S, K_S + \ulcorner \pi^*((t+k)K_X) \urcorner|_S) \longrightarrow 0.$$

Noting that $\ulcorner \pi^*((t+k)K_X) \urcorner \geq \ulcorner \pi^*(tK_X) \urcorner + M_k$, also by Lemma 1.6(i), it is sufficient to show that $K_S + \ulcorner \pi^*(tK_X)|_S \urcorner + M_k|_S$ is effective. When $t=0$, then $h^0(S, K_S + M_k|_S) \geq 2$ by Lemma 1.2, because $h^0(S, M_k|_S) \geq 2$. When $t>0$, choose a 1-dimensional sub-system $|C|$ in the moving part of $\big| \ M_k|_S \ \big|$. Modulo blowing-ups, we can suppose $|C|$ to be free from base points and then C is nef and

$C \leq M_k|_S$. We have $g(C) \geq 2$. Because $\pi^*(tK_X)|_S$ is a nef and big $\mathbb{Q}$-divisor on S, by the Kawamata-Viehweg vanishing theorem, we also get a surjective map

$$H^0(S, K_S + \ulcorner \pi^*(tK_X)|_S \urcorner + C) \longrightarrow H^0(C, K_C + D) \longrightarrow 0,$$

where $D := \ulcorner \pi^*(tK_X)|_S \urcorner \big|_C$ is a divisor on C with positive degree. Thus we have $h^0(C, K_C + D) \geq 2$. This leads to the effectiveness of $(2k + t + 1)K_{X'}$. Moreover, actually we have proved that $\dim \phi_m(S) \geq 1$ for $m \geq 2k + 1$.

Now we prove that ϕ_{3k+1} is generically finite. Considering the system

$$|K_{X'} + \ulcorner 2k\pi^*(K_X)\urcorner + M_k|,$$

as we have shown above that $(2k+1)K_{X'}$ is effective, so $|K_{X'} + \ulcorner 2k\pi^*(K_X)\urcorner + M_k|$ can distinguish general S. By the Kawamata-Viehweg vanishing theorem, we have

$$|K_{X'} + \ulcorner 2k\pi^*(K_X)\urcorner + S| \big|_S = |K_S + \ulcorner 2k\pi^*(K_X)\urcorner|_S \big| .$$

We have

$$\big| K_S + \ulcorner 2k\pi^*(K_X)\urcorner|_S \big| \supset |K_S + \ulcorner k\pi^*(K_X)|_S \urcorner + M_k|_S \big| .$$

Noting that $h^0(S, M_k|_S) \geq 2$, $K_S + \ulcorner k\pi^*(K_X)|_S\urcorner \geq K_S + M_k|_S$, which is also effective by Lemma 1.2, and $k\pi^*(K_X)|_S$ is a nef and big $\mathbb{Q}$-divisor on S, it is easy to verify that $\big| K_S + \ulcorner k\pi^*(K_X)|_S\urcorner + M_k|_S \big|$ gives a generically finite map. In fact, choose a 1-dimensional sub-system $|C|$ in the moving part of $\big| M_k|_S \big|$. For the same reason, we can suppose $|C|$ to be free from base points. $\big| K_S + \ulcorner k\pi^*(K_X)|_S\urcorner + C \big|$ can distinguish general C, and we have

$$|K_S + \ulcorner k\pi^*(K_X)|_S\urcorner + C| \big|_C = |K_C + D|,$$

where D is a divisor on C with positive degree. Because $g(C) \geq 2$, it follows that $h^0(K_C + D) \geq 2$ and $|K_C + D|$ gives a generically finite map.

Finally, we want to show that ϕ_m is birational for $m \geq 9k+4$. Let $t := m-7k-3$; then $t \geq 2k+1$. Denote by M_{3k+1} the moving part of $|(3k+1)K_{X'}|$ and by M_t the moving part of $|tK_{X'}|$. We have

$$|K_{X'} + \ulcorner (t + 6k + 2)\pi^*(K_X)\urcorner + M_k| \subset |mK_{X'}|.$$

Because $t + 6k + 3 > 2k + 1$, $K_{X'} + \ulcorner (t + 6k + 2)\pi^*(K_X)\urcorner$ is effective; thus the left system in the above can distinguish general S. Furthermore, the vanishing theorem gives

$$|K_{X'} + \ulcorner (t + 6k + 2)\pi^*(K_X)\urcorner + M_k| \big|_S = |K_S + L|,$$

where $L := \ulcorner (t + 6k + 2)\pi^*(K_X)\urcorner \big|_S \geq 2M_{3k+1}|_S + M_t|_S$. By Lemma 1.4, $|K_S + L|$ gives a birational map, and so does $|mK_{X'}|$.

Case 2. $\dim \phi_k(X) = 1$.

In this case, W is a nonsingular curve of genus b. Let F be a general fibre of f; then F is an irreducible smooth projective surface of general type. We have $M_k \sim_{\text{lin}} \sum F_i$, where the F_i are fibres of f for each i.

By a parallel argument as in the proof of Theorem 2.3.1, we see that ϕ_m is birational for $m \geq 2k + 4$ if $b > 0$. And if $b = 0$ while F is a surface with the invariants $\big(K^2_{F_0}, p_g(F)\big) = (1, 2)$ or $(2, 3)$, then ϕ_m is birational for $m \geq 10k + 7$.

Otherwise, we use Kollár's method. From 2.2, we know that ϕ_{7k+3} is birational and $\dim \phi_{5k+2}(X) \geq 2$. Thus, by Lemma 1.7, $mK_{X'}$ is effective for $m \geq 6k + 4$. Since we have $|K_{X'} + \ulcorner (5k + 2)\pi^*(K_X)\urcorner + F| \big|_F = |K_F + D|$ where $D := \ulcorner (5k + 2)\pi^*(K_X)\urcorner \big|_F$ is effective and $h^0(F, D) \geq 2$, we see that $K_F + D$ is effective and thus $(6k+3)K_{X'}$ is effective. So ϕ_m is birational for $m \geq 13k+6$, which means that ϕ_{13k+6} is stably birational. □

Theorem 2.3.3. *Let X be a nonsingular projective threefold of general type and suppose $P_k(X) \geq 3$. Then ϕ_m is birational for all $m \geq 10k+8$.*

Proof. When $\dim \phi_k(X) \geq 2$, we know from Case 1 of Theorem 2.3.2 that ϕ_m is birational for $m \geq 9k+4$. When $|kK_X|$ is composed of a pencil, from the proof of Theorem 2.3.1, we see that ϕ_k will derive a fibration $f : X' \longrightarrow W$ onto a nonsingular curve. If $b := g(W) > 0$, then ϕ_m is birational for $m \geq 2k+4$.

The remaining case is the one when $b = 0$. We have an injection $\mathcal{O}(2) \hookrightarrow f_*\omega^k_{X'}$. So, for each $p > 0$, we have

$$\mathcal{O}(1) \otimes f_*\omega^p_{X'/\mathbb{P}^1} = \mathcal{O}(2p+1) \otimes f_*\omega^p_{X'} \hookrightarrow f_*\omega^{k(p+1)+p}_{X'}.$$

Thus Kollár's method tells that ϕ_{6k+5} is birational, ϕ_{4k+3} is generically finite and that $\dim \phi_{3k+2}(X) \geq 2$. Now using our method, we can see that $mK_{X'}$ is effective for $m \geq 4k+4$ by Lemma 1.7. Since $(4k+3)K_{X'}$ is also effective, ϕ_m is birational for $m \geq 10k+8$. □

Corollary 2.3.2. *Let X be a nonsingular projective threefold of general type and suppose $p_g(X) \geq 3$. Then ϕ_m is birational for $m \geq 11$.*

Proof. Keep the same notation as in the proof of Theorem 2.3.2. When $\dim \phi_1(X) \geq 2$, we set $L_3 := 4K_{X'}$, $L_2 = L_1 := K_{X'}$. Then $|L_3|$ gives a generically finite map by virtue of Case 1, Theorem 2.3.2. Using Lemma 1.5, we see that $|K_{X'} + 2L_3 + L_2 + L_1|$ gives a birational map. Thus ϕ_{11} is birational.

When $\dim \phi_1(X) = 1$, we see from the proof of Theorem 2.3.3 that ϕ_{11} is also birational. □

Theorem 2.3.1, Theorem 2.3.2, Theorem 2.3.3 and Corollary 2.3.2 imply the main theorem.

3. Open problems

3.1. Let X be a nonsingular projective variety of general type of dimension n. We define

$k_0(X) := min\{k|\ P_k(X) \geq 2\}$;

$k_s(X) := min\{k|\ \phi_m$ is birational for $m \geq k\}$, which is called the *canonical stability of X*;

$\mu_s(X) := \frac{k_s(X)}{k_0(X)}$, which is called the *relative canonical stability* of X. Obviously, $\mu_s(X)$ is a birational invariant.

$\mu_s(n) := \sup\{\mu_s(X)|\ X$ is a n-fold of general type$\}$, which is called the *n-th relative canonical stability.*

It is well-known that $\mu_s(1) = 3$ and $\mu_s(2) = 5$ ([1]). From the main theorem, we have $\mu_s(3) \leq 16$. What is the exact value of $\mu_s(3)$? It is also interesting to study $\mu_s(n)$ for $n \geq 4$, even if we don't know whether we should have $\mu_s(n) < +\infty$.

3.2. We would like to ask a very natural question which never happens in the surface case.

Question. *Do there exist a smooth projective threefold X of general type and two positive integers $k_1 < k_2$ such that ϕ_{k_1} is birational while ϕ_{k_2} is not birational?*

Of course, it may happen for some threefold that $P_{k_1} > P_{k_2}$ even if $k_1 < k_2$. But we have not found any counterexample yet to the above question.

Acknowledgment

I would like to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy for support during my visit. I also wish to thank Prof. L. Ein and Prof. M. S. Narasimhan for encouragement. Thanks are also due to Prof. J. Kollár and Prof. M. Reid, who privately gave me much more useful information on the problem considered in this paper. During the preparation of this note, I visited both the Università di Roma II and the Università degli Studi di Padova by the invitation of Prof. Ciro Ciliberto and Prof. Ezio Stagnaro whom I take this chance to thank for their hospitality. Finally, the referee's skillful comments greatly helped me to organize this paper into the present form.

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