# FRAME WAVELET SETS IN $\mathbb{R}$ 

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#### Abstract

In this paper, we try to answer an open question raised by Han and Larson, which asks about the characterization of frame wavelet sets. We completely characterize tight frame wavelet sets. We also obtain some necessary conditions and some sufficient conditions for a set $E$ to be a (general) frame wavelet set. Some results are extended to frame wavelet functions that are not defined by frame wavelet set. Several examples are presented and compared with some known results in the literature.


## §1. Introduction

A collection of elements $\left\{x_{j}: j \in \mathcal{J}\right\}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exist constants $A$ and $B, 0<A \leq B<\infty$, such that

$$
\begin{equation*}
A\|f\|^{2} \leq \sum_{j \in \mathcal{J}}\left|\left\langle f, x_{j}\right\rangle\right|^{2} \leq B\|f\|^{2} \tag{1}
\end{equation*}
$$

for all $f \in \mathcal{H}$. The supremum of all such numbers $A$ and the infimum of all such numbers $B$ are called the frame bounds of the frame and denoted by $A_{0}$ and $B_{0}$ respectively. The frame is called a tight frame when $A_{0}=B_{0}$ and is called a normalized tight frame when $A_{0}=B_{0}=1$. Any orthonormal basis in a Hilbert space is a normalized tight frame.

The concept of frame first appeared in the late 40's and early 50's (see [6], [13] and [12). The development and study of wavelet theory during the last decade also brought new ideas and attention to frames because of their close connections. For a glance of the recent development and work on frames and related topics, see [1], [3], 4], [5], 7], [8 and 10].

Let $D, T$ be dilation and translation operators respectively on $L^{2}(\mathbb{R})$. Namely $(D f)(x)=\sqrt{2} f(2 x)$ and $(T f)(x)=f(x-1)$ for any $f \in L^{2}(\mathbb{R}) . D, T$ are both unitary operators, i.e., $\|D f\|=\|T f\|=\|f\|$ for any $f \in L^{2}(\mathbb{R})$. In this paper, we are interested in frames of $L^{2}(\mathbb{R})$ of the form

$$
\begin{equation*}
\left\{\psi_{n, l}(x)\right\}=\left\{2^{\frac{n}{2}} \psi\left(2^{n} x-\ell\right): n, \ell \in \mathbb{Z}\right\}=\left\{D^{n} T^{\ell} \psi: n, \ell \in \mathbb{Z}\right\} \tag{2}
\end{equation*}
$$

where $\psi \in L^{2}(\mathbb{R})$. The function $\psi \in L^{2}(\mathbb{R})$ is called a frame wavelet for $L^{2}(\mathbb{R})$ if (22) is a frame of $L^{2}(\mathbb{R})$. Namely, there exist two positive constants $A \leq B$ such

[^0]that for any $f \in L^{2}(\mathbb{R})$,
\[

$$
\begin{equation*}
A\|f\|^{2} \leq \sum_{n, \ell \in \mathbb{Z}}\left|\left\langle f, D^{n} T^{\ell} \psi\right\rangle\right|^{2} \leq B\|f\|^{2} \tag{3}
\end{equation*}
$$

\]

When (2) is a (normalized) tight frame in $L^{2}(\mathbb{R})$, then $\psi$ is called a (normalized) tight frame wavelet of $L^{2}(\mathbb{R})$. A characterization of the normalized tight frame wavelets in $L^{2}(\mathbb{R})$ is obtained in [8].

In this paper, we will devote ourselves to the investigation of the frame wavelets defined in the following way. Let $E$ be a Lebesgue measurable set of finite measure. Define $\psi \in L^{2}(\mathbb{R})$ by $\widehat{\psi}=\frac{1}{\sqrt{2 \pi}} \chi_{E}$, where $\widehat{\psi}$ is the Fourier transform of $\psi$. If $\psi$ so defined is a frame wavelet for $L^{2}(\mathbb{R})$, then the set $E$ is called a frame wavelet set (for $L^{2}(\mathbb{R})$ ). Similarly, $E$ is called a (normalized) tight frame wavelet set if $\psi$ is a (normalized) tight frame wavelet. This study may be useful in the operator theory since from an operator theoretic point of view, frame wavelets for $L^{2}(\mathbb{R})$ are just the so-called frame vectors for the unitary system $\mathcal{U}=\left\{D^{n} T^{\ell} \mid n, \ell \in \mathbb{Z}\right\}([7])$.

A characterization of normalized tight frame wavelet sets is obtained in [7]. It can also be induced from a characterization of normalized tight frame wavelet obtained in [8]. A necessary condition for a function in $L^{2}(\mathbb{R})$ to be a frame wavelet for $L^{2}(\mathbb{R})$ is obtained in [8]. However, the question of how to characterize frame wavelets in general and frame wavelet sets in particular remains open ([7]). In this paper, we obtain a necessary condition and a sufficient condition for a set to be a frame wavelet set. Though we are still a few steps away from a characterization of frame wavelet sets, we are able to characterize tight frame wavelet sets. This result induces the known characterization of normalized tight wavelet sets. It also shows that the frame bound for the tight frame corresponding to a tight frame wavelet set is always an integer. We need to point out that it is not trivial to characterize the tight frame wavelet sets. Although a normalized tight frame wavelet can be obtained from a tight frame wavelet $\psi$ by dividing its frame bound $A_{0}=B_{0}, \psi / A_{0}$ is no longer defined by an inverse Fourier transform of a function of the form $\frac{1}{\sqrt{2 \pi}} \chi_{E}$ while $\psi$ itself is defined this way.

This paper will be organized in the following way. In section 2 , we introduce some definitions, prelimilary lemmas and the main results. In section 3, we prove several lemmas. In section 4, we prove the main theorems. In the last section, we will furnish several examples. We will also extend some of our results to frame wavelets that are not defined by frame wavelet sets. These will be compared with some known necessary conditions for frame wavelets in the literature.

## §2. Definitions and main Results

Throughout this paper, we use $\widehat{\psi}$ to denote the Fourier transformation of $\psi$. The Fourier transformation is normalized so that it is a unitary operator. Since (3) is equivalent to

$$
\begin{equation*}
A\|f\|^{2} \leq \sum_{n, \ell \in \mathbb{Z}}\left|\left\langle f, \widehat{D}^{n} \widehat{T}^{\ell} \widehat{\psi}\right\rangle\right|^{2} \leq B\|f\|^{2}, \forall f \in L^{2}(\mathbb{R}) \tag{4}
\end{equation*}
$$

we may work with (4) instead of (3).
Let $E$ be a measurable set. $x, y \in E$ are $\stackrel{\delta}{\sim}$ equivalent if $x=2^{n} y$ for some integer $n$. The $\delta$-index of a point $x$ in $E$ is the number of elements in its $\stackrel{\delta}{\sim}$ equivalent
class and is denoted by $\delta_{E}(x)$. Let $E(\delta, k)=\left\{x \in E: \delta_{E}(x)=k\right\}$. Then $E$ is the disjoint union of the sets $E(\delta, k)$. Let

$$
\delta(E)=\bigcup_{n \in \mathbb{Z}} 2^{-n}\left(E \cap\left(\left[-2^{n+1} \pi,-2^{n} \pi\right) \cup\left[2^{n} \pi, 2^{n+1} \pi\right)\right)\right)
$$

The above is a disjoint union if and only if $E=E(\delta, 1)$, as one can easily check. The proof of the following lemma can be found in [2].

Lemma 1. If $E$ is a Lebesgue measurable set, then each $E(\delta, k)(k \geq 1)$ is also Lebesgue measurable. Furthermore, each $E(\delta, k)$ is a disjoint union of $k$ measurable sets $\left\{E^{j}(\delta, k)\right\}, 1 \leq j \leq k$, such that $E^{j}(\delta, k) \stackrel{\delta}{\sim} E^{j^{\prime}}(\delta, k)$ for any $1 \leq j, j^{\prime} \leq k$.

Similarly, $x, y \in E$ are $\stackrel{\tau}{\sim}$ equivalent if $x=y+2 n \pi$ for some integer $n$. The $\tau$-index of a point $x$ in $E$ is the number of elements in its $\stackrel{\tau}{\sim}$ equivalent class and is denoted by $\tau_{E}(x)$. Let $E(\tau, k)=\left\{x \in E: \tau_{E}(x)=k\right\}$. Then $E$ is the disjoint union of the sets $E(\tau, k)$. Define $\tau(E)=\bigcup_{n \in \mathbb{Z}}(E \cap[2 n \pi, 2(n+1) \pi)-2 n \pi)$. Again, this is a disjoint union if and only if $E=E(\tau, 1)$. The proof of the following lemma can also be found in [2].

Lemma 2. If $E$ is a Lebesgue measurable set, then each $E(\tau, k)(k \geq 1)$ is also Lebesgue measurable. Furthermore, each $E(\tau, k)$ is a disjoint union of $k$ measurable sets $\left\{E^{(j)}(\tau, k)\right\}, 1 \leq j \leq k$, such that $E^{(j)}(\tau, k) \stackrel{\tau}{\sim} E^{j^{\prime}}(\tau, k)$ for any $1 \leq j, j^{\prime} \leq k$.

Remark 1. If $E$ is of finite measure, then $E(\tau, \infty)$ is of zero measure.
Remark 2. The decompositions of $E(\delta, k)$ (resp. $E(\tau, k)$ ) into $E^{j}(\delta, k)$ (resp. $\left.E^{(j)}(\tau, k)\right)$ are not unique in general. However, one of them is guaranteed by the procedure of construction in [2]. To avoid confusion, in this text we always assume that these sets are (uniquely) defined in that way. But our results do not depend on the decomposition as long as all sets involved are measurable.

Now let $E$ be a Lebesgue measurable set with finite measure. For any $f \in L^{2}(\mathbb{R})$, let $H_{E} f$ be the following formal summation:

$$
\begin{equation*}
\left(H_{E} f\right)(s)=\sum_{n, \ell \in \mathbb{Z}}\left\langle f, \widehat{D}^{n} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}\right\rangle \widehat{D}^{n} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}(s) \tag{5}
\end{equation*}
$$

Notice that if $H_{E} f$ converges to a function in $L^{2}(\mathbb{R})$ under the $L^{2}(\mathbb{R})$ norm, then equation (4) (with $\widehat{\psi}=\frac{1}{\sqrt{2 \pi}} \chi_{E}$ ) is equivalent to

$$
\begin{equation*}
A\|f\|^{2} \leq\left\langle H_{E} f, f\right\rangle \leq B\|f\|^{2} \tag{6}
\end{equation*}
$$

We outline the main results obtained in this paper below.
Theorem 1. Let $E$ be a Lebesgue measurable set with finite measure. Then the following statements are equivalent:
(i) $H_{E}$ defines a bounded linear operator in $L^{2}(\mathbb{R})$, that is, $H_{E} f$ converges in $L^{2}(\mathbb{R})$ for any $f \in L^{2}(\mathbb{R})$ and $\left\|H_{E} f\right\| \leq b\|f\|$ for some constant $b>0$.
(ii) There exists a constant $B>0$ such that $\sum_{n, \ell \in \mathbb{Z}}\left|\left\langle f, \widehat{D}^{n} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}\right\rangle\right|^{2} \leq B\|f\|^{2}$ for all $f \in L^{2}(\mathbb{R})$.
(iii) There exists a constant $M>0$ such that $\mu(E(\delta, m))=0$ and $\mu(E(\tau, m))=0$ for any $m>M$.

Theorem 2. Let $E$ be a Lebesgue measurable set with finite measure. Then $E$ is a frame wavelet set if (i) $\bigcup_{n \in \mathbb{Z}} 2^{n} E(\tau, 1)=\mathbb{R}$ and (ii) there exists $M>0$ such that $\mu(E(\delta, m))=0$ and $\mu(E(\tau, m))=0$ for any $m>M$.

Furthermore, in this case, the lower frame bound is at least 1, and the upper frame bound is at most $M^{\frac{5}{2}}$.

Theorem 3. Let $E$ be a Lebesgue measurable set with finite measure. If $E$ is a frame wavelet set, then (i) $\bigcup_{n \in \mathbb{Z}} 2^{n} E=\mathbb{R}$ and (ii) there exists $M>0$ such that $\mu(E(\delta, m))=0$ and $\mu(E(\tau, m))=0$ for any $m>M$.

Theorem 4. Let $E$ be a Lebesgue measurable set with finite measure. Then $E$ is a tight frame set if and only if $E=E(\tau, 1)=E(\delta, k)$ for some $k \geq 1$ and $\bigcup_{n \in \mathbb{Z}} 2^{n} E=\mathbb{R}$.
Corollary 1. If $E$ is a tight frame wavelet set, then the frame bound is an integer.
Corollary 2. $E$ is a normalized tight frame wavelet set if and only if $E=E(\tau, 1)=$ $E(\delta, 1)$ and $\bigcup_{n \in \mathbb{Z}} 2^{n} E=\mathbb{R}$.
Corollary 3. If $E=E(\tau, 1), \bigcup_{n \in \mathbb{Z}} 2^{n} E(\tau, 1)=\mathbb{R}$ and there exist $1 \leq k_{1} \leq k_{2}$ such that $\mu(E(\delta, m))=0$ for $m<k_{1}$ and $m>k_{2}, \mu\left(E\left(\delta, k_{1}\right)\right) \mu\left(E\left(\delta, k_{2}\right)\right) \neq 0$, then $E$ is a frame bound with lower bound $k_{1}$ and upper bound $k_{2}$.

## §3. LEMMAS

Let $f$ be in $L^{2}(\mathbb{R})$ and let $E$ be a Lebesgue measurable set in $\mathbb{R}$. First we define $f_{m j}^{k}$ to be the $2^{k+1} \pi$ periodical extension of $f \cdot \chi_{2^{k} E^{(j)}(\tau, m)}$ over $\mathbb{R}$. In particular we define $f_{m j}^{0}$ to be the $2 \pi$ periodical extension of $f \cdot \chi_{E^{(j)}(\tau, m)}$ over $\mathbb{R}$. Also for $k \in \mathbb{Z}$, we define

$$
\begin{equation*}
H_{E}^{k} f=\sum_{\ell \in \mathbb{Z}}\left\langle f, \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}\right\rangle \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E} \tag{7}
\end{equation*}
$$

When we speak of the convergence of the sum which defines $H_{E}^{k} f$, we always mean the convergence under the $L^{2}(\mathbb{R})$ norm unless otherwise stated. We now give the following elementary lemma without proof.

Lemma 3. Let $f$ be a $2 \pi$ periodical function that is square integrable over $[0,2 \pi]$. Then for any measurable sets $E, G$ such that $E=E(\tau, 1), G=G(\tau, 1)$ and $\tau(E)=$ $\tau(G)$, we have $\left\|f \cdot \chi_{E}\right\|=\left\|f \cdot \chi_{G}\right\|=\left\|f \cdot \chi_{\tau(E)}\right\|$ and $\left\langle f, \chi_{E}\right\rangle=\left\langle f, \chi_{G}\right\rangle=\left\langle f, \chi_{\tau(E)}\right\rangle$.

Lemma 4. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$ with finite measure. Then (i) $f=H_{E}^{0} f$ for all $f \in L^{2}(\mathbb{R})$ with $\operatorname{supp}(f) \subset E$ if and only if (ii) $E=E(\tau, 1)$.

Proof. (ii) $\Rightarrow$ (i). Let $F=[0,2 \pi) \backslash \tau(E)$ and $G=F \cup E$. Then $G=G(\tau, 1)$ and $\tau(G)=[0,2 \pi)$ since $E=E(\tau, 1)$. If $f \in L^{2}(\mathbb{R})$ and $\operatorname{supp}(f) \subset E$, then $f \in$ $L^{2}(\mathbb{R}) \cdot \chi_{G}$. However, $\left\{\frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{G}: \ell \in \mathbb{Z}\right\}$ is an orthonormal basis for $L^{2}(\mathbb{R}) \cdot \chi_{G}$, so we have $f(s)=\sum_{\ell \in \mathbb{Z}}\left\langle f, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{G}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{G}(s)=H_{G}^{0} f$. Using $f=f \cdot \chi_{E}$, $\chi_{E} \cdot \chi_{G}=\chi_{E}$ and $\left\langle f, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{G}\right\rangle=\left\langle f, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E}\right\rangle$, we get $f=\chi_{E} f=$ $\chi_{E} \cdot \sum_{\ell \in \mathbb{Z}}\left\langle f, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{G}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{G}(s)=H_{E}^{0} f$.
$(\mathrm{i}) \Rightarrow(\mathrm{ii})$. Assume that $E$ is a set which satisfies (i) but not (ii). Then $\mu(E(\tau, k))>$ 0 for some $k>1$ where $\mu$ is the Lebesgue measure. Let $g(s)=\chi_{E^{(1)}(\tau, k)}(s)-$ $\chi_{E^{(2)}(\tau, k)}(s)$. By Lemma33, $\int_{E^{(1)}(\tau, k)} e^{i \ell s} d s=\int_{E^{(2)}(\tau, k)} e^{i \ell s} d s \forall \ell \in \mathbb{Z}$. It then follows
that $\left\langle g, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E}\right\rangle=0$ for all $\ell \in \mathbb{Z}$. Since $\operatorname{supp}(g) \subset E$ and $g \in L^{2}(\mathbb{R})$, (i) holds for $g$. That leads to $g=0$, a contradiction.

Lemma 5. Let $E, F$ be Lebesgue measurable sets of finite measure such that $\tau(E(\tau, k)) \cap \tau(F)=\emptyset$ for some natural number $k$. Then for any $f \in L^{2}(\mathbb{R})$, we have

$$
\begin{equation*}
\sum_{\ell \in \mathbb{Z}}\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E(\tau, k)}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \chi_{F(\tau, m)}=0 \tag{8}
\end{equation*}
$$

for any $m \geq 1$ under the $L^{2}(\mathbb{R})$ norm. Consequently,

$$
\sum_{\ell \in \mathbb{Z}}\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E(\tau, k)}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s}
$$

converges to 0 for almost all $s \in \mathbb{R}$ that is not $\stackrel{\tau}{\sim}$ to any point in $E(\tau, k)$.
Proof. For any $1 \leq j \leq k$ and $1 \leq n \leq m$, consider $G=E^{(j)}(\tau, k) \cup F^{(n)}(\tau, m)$. We have $G=G(\tau, 1)$. By Lemma 4, $f \chi_{G}=H_{G}^{0} f$. Multiplying both sides of this by $\chi_{F^{(n)}(\tau, m)}$, we get

$$
\begin{equation*}
f \chi_{F^{(n)}(\tau, m)}=\sum_{\ell \in \mathbb{Z}}\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{G}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{F^{(n)}(\tau, m)} \tag{9}
\end{equation*}
$$

On the other hand, by Lemma, 4, we also have $f \chi_{F^{(n)}(\tau, m)}=H_{F^{(n)}(\tau, m)}^{0} f$. Subtracting this from (91), we get $\sum_{\ell \in \mathbb{Z}}\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E^{(j)}(\tau, k)}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{F^{(n)}(\tau, m)}=0$. This then leads to $\sum_{\ell \in \mathbb{Z}}\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E(\tau, k)}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{F(\tau, m)}=0$. The last statement in the lemma is obvious since the convergence under $L^{2}(\mathbb{R})$ norm implies almost everywhere convergence.

Lemma 6. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$ with finite positive measure. Then $H_{E(\tau, m)}^{0} f=\sum_{j=1}^{m} f_{m j}^{0} \cdot \chi_{E(\tau, m)}$ for any $f \in L^{2}(\mathbb{R})$.

Proof. First, we have

$$
\sum_{j=1}^{m} f_{m j}^{0} \cdot \chi_{E(\tau, m)}=\sum_{j=1}^{m} f_{m j}^{0}\left(\sum_{i=1}^{m} \chi_{E^{(i)}(\tau, m)}\right)=\sum_{i, j=1}^{m} f_{m j}^{0} \cdot \chi_{E^{(i)}(\tau, m)}
$$

So by Lemma 4 .

$$
f_{m j}^{0} \cdot \chi_{E^{(i)}(\tau, m)}=\sum_{\ell \in \mathbb{Z}}\left\langle f_{m j}^{0}, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E^{(i)}(\tau, m)}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \chi_{E^{(i)}(\tau, m)}
$$

Since

$$
\begin{aligned}
\left\langle f_{m j}^{0}, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E^{(i)}(\tau, m)}\right\rangle & =\left\langle f_{m j}^{0}, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E^{(j)}(\tau, m)}\right\rangle \\
& =\left\langle f, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E^{(j)}(\tau, m)}\right\rangle
\end{aligned}
$$

by Lemma 3, we get

$$
\begin{aligned}
\sum_{j=1}^{m} f_{m j}^{0} \cdot \chi_{E(\tau, m)} & =\sum_{i, j=1}^{m} \sum_{\ell \in \mathbb{Z}}\left\langle f, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E^{(j)}(\tau, m)}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \chi_{E^{(i)}(\tau, m)} \\
& =\sum_{\ell \in \mathbb{Z}} \sum_{i, j=1}^{m}\left\langle f, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E^{(j)}(\tau, m)}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \chi_{E^{(i)}(\tau, m)} \\
& =\sum_{\ell \in \mathbb{Z}}\left\langle f, \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E(\tau, m)}\right\rangle \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \chi_{E(\tau, m)}=H_{E(\tau, m)}^{0} f .
\end{aligned}
$$

Lemma 7. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$ with finite positive measure. The following statements are equivalent:
(i) There exists a constant $a>0$ such that $\sum_{\ell \in \mathbb{Z}}\left|\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E}\right\rangle\right|^{2} \leq a\|f\|^{2}$ for all $f \in L^{2}(\mathbb{R})$.
(ii) There exists $M>0$ such that $\mu(E(\tau, m))=0$ for all $m \geq M$.

Proof. (ii) $\Rightarrow$ (i). By Lemmas 5 and $6 H_{E}^{0} f=\sum_{m=1}^{M} \sum_{j=1}^{m} f_{m j}^{0} \cdot \chi_{E(\tau, m)}$ where the convergence is under the $L^{2}(\mathbb{R})$ norm. By Lemma 3

$$
\int_{\mathbb{R}}\left|f_{m j}^{0} \chi_{E(\tau, m)}\right|^{2} d s=\int_{E(\tau, m)}\left|f_{m j}^{0}\right|^{2} d s=m \int_{E^{(j)}(\tau, m)}|f|^{2} d s
$$

So

$$
\int_{E(\tau, m)}\left|\sum_{j=1}^{m} f_{m j}^{0}\right|^{2} d s \leq m^{2} \int_{E(\tau, m)}|f|^{2} d s
$$

Therefore,

$$
\begin{align*}
\left\|H_{E}^{0} f\right\|^{2} & =\int_{E}\left|\sum_{m=1}^{M} \sum_{j=1}^{m} f_{m j}^{0}\right|^{2} d s \leq M \sum_{m=1}^{M} m^{2} \int_{E(\tau, m)}|f|^{2} d s \\
& \leq M^{3} \int_{E}|f|^{2} d s \leq M^{3} \int_{\mathbb{R}}|f|^{2} d s=M^{3}\|f\|^{2} \tag{10}
\end{align*}
$$

It follows that $\sum_{\ell \in \mathbb{Z}}\left|\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E}\right\rangle\right|^{2}=\left\langle H_{E}^{0} f, f\right\rangle \leq\left\|H_{E}^{0} f\right\| \cdot\|f\| \leq M^{\frac{3}{2}}\|f\|^{2}$.
(i) $\Rightarrow$ (ii). Assume this is not true; then (i) holds for some $E$ that does not satisfy (ii). Thus, $\mu\left(E\left(\tau, m_{0}\right)\right)>0$ for some $m_{0}>a$. Define $f=\chi_{E\left(\tau, m_{0}\right)} \in L^{2}(\mathbb{R})$. We have $\|f\|^{2}=\mu\left(E\left(\tau, m_{0}\right)\right)$. By the assumption, we have $\sum_{\ell \in \mathbb{Z}} \left\lvert\,\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s}\right.$. \right. $\left.\chi_{E}\right\rangle\left.\right|^{2} \leq a\|f\|^{2}=a \mu\left(E\left(\tau, m_{0}\right)\right)$. On the other hand, the left hand side in the above inequality is $\left\langle f, H_{E}^{0} f\right\rangle=\left\langle f, H_{E\left(\tau, m_{0}\right)}^{0} f\right\rangle$ since $\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s} \cdot \chi_{E}\right\rangle=\left\langle f(s), \frac{1}{\sqrt{2 \pi}} e^{-i \ell s}\right.$. $\left.\chi_{E\left(\tau, m_{0}\right)}\right\rangle$. By Lemma 6

$$
H_{E}^{0} f=\sum_{j=1}^{m_{0}} f_{m_{0} j}^{0} \chi_{E\left(\tau, m_{0}\right)}=\sum_{j=1}^{m_{0}} \chi_{E\left(\tau, m_{0}\right)}=m_{0} \chi_{E\left(\tau, m_{0}\right)}=m_{0} f
$$

Therefore, $\left\langle f, H_{E}^{0} f\right\rangle=m_{0} \mu\left(E\left(\tau, m_{0}\right)\right)=m_{0}\|f\|^{2}$. This contradicts the assumption that $a<m_{0}$.

## §4. Proofs of the theorems

Proof of Theorem 1] (i) $\Rightarrow$ (ii). This is obvious from $\|\langle f, g\rangle\| \leq\|f\|\|g\|$.
(iii) $\Rightarrow$ (i). Substitute $s / 2^{k}=t$ in (7); we get (using Lemma 7)

$$
H_{E}^{k} f=\sum_{m=1}^{M} \sum_{j=1}^{m} f_{m j}^{k} \cdot \chi_{2^{k} E(\tau, m)}
$$

by definition of $f_{m j}^{k}$. Similar to the proof of Lemma 7, we have

$$
\begin{equation*}
\int_{\mathbb{R}}\left|H_{E}^{k} f\right|^{2} d s \leq \int_{2^{k} E}\left(\sum_{m=1}^{M} \sum_{j=1}^{m}\left|f_{m j}^{k}\right|\right)^{2} d s \leq M^{3} \int_{2^{k} E}|f|^{2}=M^{3}\left\|f \cdot \chi_{2^{k} E}\right\|^{2} \tag{11}
\end{equation*}
$$

Notice that $\sum_{k \in \mathbb{Z}}\left|H_{E}^{k} f\right|$ converges pointwise since for each $s \in \mathbb{R}$, there are at most $M$ nonzero terms in there. We now proceed to prove that $\sum_{k \in \mathbb{Z}}\left|H_{E}^{k} f\right| \in L^{2}(\mathbb{R})$. Note that the given condition implies $\sum_{j \in \mathbb{Z}} \chi_{2^{j} E} \leq M$. Since the support of $\left|H_{E}^{k} f\right|$ is in $2^{k} E$, for any $L_{1}, L_{2}>0$, we have

$$
\begin{align*}
& \int_{\mathbb{R}}\left(\sum_{-L_{1} \leq k \leq L_{2}}\left|H_{E}^{k} f\right|\right)^{2} d s \leq \sum_{-L_{1} \leq p, q \leq L_{2}} \int_{2^{p} E \cap 2^{q} E}\left|H_{E}^{p} f\right| \cdot\left|H_{E}^{q} f\right| d s  \tag{12}\\
\leq & \frac{1}{2} \sum_{-L_{1} \leq p, q \leq L_{2}}\left(\int_{2^{p} E \cap 2^{q} E}\left|H_{E}^{p} f\right|^{2} d s+\int_{2^{p} E \cap 2^{q} E}\left|H_{E}^{q} f\right|^{2} d s\right) \\
= & \sum_{-L_{1} \leq p \leq L_{2}} \int_{2^{p} E}\left|H_{E}^{p} f\right|^{2} \sum_{-L_{1} \leq q \leq L_{2}} \chi_{2^{q} E} d s \leq M \sum_{-L_{1} \leq p \leq L_{2}} \int_{2^{p} E}\left|H_{E}^{p} f\right|^{2} d s \\
= & M \sum_{-L_{1} \leq p \leq L_{2}} \int_{\mathbb{R}}\left|H_{E}^{p} f\right|^{2} d s \leq M^{4} \sum_{-L_{1} \leq p \leq L_{2}} \int_{2^{p} E}|f|^{2} d s \\
= & M^{4} \int_{\mathbb{R}}|f|^{2} \sum_{-L_{1} \leq p \leq L_{2}} \chi_{2^{p} E} d s \leq M^{5} \int_{\mathbb{R}}|f|^{2} d s=M^{5}\|f\|^{2} .
\end{align*}
$$

Therefore, $\int_{\mathbb{R}}\left(\sum_{k \in \mathbb{Z}}\left|H_{E}^{k} f\right|\right)^{2} d s \leq M^{5}\|f\|^{2}$ by Fatou's lemma. This also leads to

$$
\begin{equation*}
\lim _{K_{1}, K_{2} \rightarrow \infty} \int_{\mathbb{R}}\left(\sum_{k \leq-K_{1}, k \geq K_{2}}\left|H_{E}^{k} f\right|\right)^{2} d s=0 \tag{13}
\end{equation*}
$$

That is, $\sum_{k \in \mathbb{Z}}\left|H_{E}^{k} f\right|$ (hence $\sum_{k \in \mathbb{Z}} H_{E}^{k} f$ as well) converges in $L^{2}(\mathbb{R})$.
Let $a_{k \ell}=\left\langle f, \frac{1}{\sqrt{2 \pi}} \widehat{D}^{k} \widehat{T}^{\ell} \chi_{E}\right\rangle$ for convenience. By LemmaZ7, we have $\sum_{\ell \in \mathbb{Z}}\left|a_{k \ell}\right|^{2}=$ $\left\langle H_{E}^{k} f, f\right\rangle \leq M^{\frac{3}{2}}\left\|f \cdot \chi_{2^{k} E}\right\|^{2}$. It follows that

$$
\begin{equation*}
\sum_{k, \ell \in \mathbb{Z}}\left|a_{k \ell}\right|^{2} \leq M^{\frac{3}{2}} \sum_{k \in \mathbb{Z}}\left\|f \cdot \chi_{2^{k} E}\right\|^{2} \leq M^{\frac{5}{2}}\|f\|^{2} \tag{14}
\end{equation*}
$$

We now show that

$$
\begin{equation*}
H_{E} f \longrightarrow \sum_{k \in \mathbb{Z}} H_{E}^{k} f=\sum_{k \in \mathbb{Z}} \sum_{m=1}^{M} \sum_{j=1}^{m} f_{m j}^{k} \cdot \chi_{2^{k} E(\tau, m)} \tag{15}
\end{equation*}
$$

in $L^{2}(\mathbb{R})$ by showing that

$$
\begin{equation*}
\lim _{K_{1}, K_{2}, L_{1}, L_{2} \rightarrow \infty} \int_{\mathbb{R}}\left|\sum_{-K_{1} \leq k \leq K_{2},-L_{1} \leq \ell \leq L_{2}} a_{k \ell} \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}-\sum_{k \in \mathbb{Z}} H_{E}^{k} f\right|^{2}=0 \tag{16}
\end{equation*}
$$

In light of (13), this is equivalent to

$$
\lim _{K_{1}, K_{2}, L_{1}, L_{2} \rightarrow \infty} \int_{\mathbb{R}}\left|\sum_{-K_{1} \leq k \leq K_{2}}\left(\sum_{-L_{1} \leq \ell \leq L_{2}} a_{k \ell} \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}-H_{E}^{k} f\right)\right|^{2} d s=0
$$

Similar to the approach used in obtaining (12), we have

$$
\begin{align*}
& \int_{\mathbb{R}}\left|\sum_{-K_{1} \leq k \leq K_{2}}\left(\sum_{-L_{1} \leq \ell \leq L_{2}} a_{k \ell} \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}-H_{E}^{k} f\right)\right|^{2} d s \\
\leq & M \sum_{-K_{1} \leq k \leq K_{2}} \int_{\mathbb{R}}\left|\sum_{-L_{1} \leq \ell \leq L_{2}} a_{k \ell} \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}-H_{E}^{k} f\right|^{2} d s . \tag{17}
\end{align*}
$$

Since each $H_{E}^{k} f$ converges in $L^{2}(\mathbb{R})$,

$$
\begin{align*}
& \int_{\mathbb{R}}\left|\sum_{-L_{1} \leq \ell \leq L_{2}} a_{k \ell} \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}-H_{E}^{k} f\right|^{2} d s \\
= & \lim _{L_{3}, L_{4} \rightarrow \infty} \int_{\mathbb{R}}\left|\sum_{(3,4)} a_{k \ell} \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}\right|^{2} d s \tag{18}
\end{align*}
$$

where $\sum_{(3,4)}$ is the summation over $\left\{-L_{3} \leq \ell \leq-L_{1}, L_{2} \leq \ell \leq L_{4}\right\}$. Let $u=\frac{s}{2^{k}}$ in the above and let $g_{\ell}(u)=\frac{1}{\sqrt{2 \pi}} e^{-i \ell u}$ for short; we get

$$
\begin{aligned}
& \int_{\mathbb{R}}\left|\sum_{(3,4)} a_{k \ell} \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E}\right|^{2} d s=\int_{\mathbb{R}}\left|\sum_{(3,4)} a_{k \ell} g_{\ell}(u) \chi_{E}\right|^{2} d u \\
= & \sum_{m=1}^{M} \int_{\mathbb{R}}\left|\sum_{(3,4)} a_{k \ell} g_{\ell}(u) \chi_{E(\tau, m)}\right|^{2} d u=\sum_{m=1}^{M} m \int_{\mathbb{R}}\left|\sum_{(3,4)} a_{k \ell} g_{\ell}(u) \chi_{\tau(E(\tau, m))}\right|^{2} d u \\
\leq & M \int_{\mathbb{R}}\left|\sum_{(3,4)} a_{k \ell} g_{\ell}(u) \chi_{\tau(E)}\right|^{2} d u=M \int_{0}^{2 \pi}\left|\sum_{(3,4)} a_{k \ell} g_{\ell}(u)\right|^{2} \chi_{\tau(E)} d u \\
\leq & M \int_{0}^{2 \pi}\left|\sum_{(3,4)} a_{k \ell} g_{\ell}(u)\right|^{2} d u=M \sum_{(3,4)}\left|a_{k \ell}\right|^{2} \leq M\left(\sum_{\ell \leq-L_{1}, \ell \geq L_{2}}\left|a_{k \ell}\right|^{2}\right) .
\end{aligned}
$$

Therefore, (17) is bounded by $M^{2} \sum_{k \in \mathbb{Z}} \sum_{\ell \leq-L_{1}, \ell \geq L_{2}}\left|a_{k \ell}\right|^{2}$. However, this goes to 0 as $L_{1}, L_{2} \rightarrow \infty$ because of (14).
(ii) $\Rightarrow$ (iii). If there exists $m_{0}>B$ such that $\mu\left(E\left(\tau, m_{0}\right)\right)>0$, then we will derive a contradiction the same way as we did in the proof of Lemma 7 since $\sum_{\ell \in \mathbb{Z}}\left|a_{0 \ell}\right|^{2} \leq$ $\sum_{k, \ell \in \mathbb{Z}}\left|a_{k \ell}\right|^{2} \leq B\|f\|^{2}$. So $\mu(E(\tau, m))=0$ for all $m>B$. Now, if $\mu\left(E\left(\delta, m_{0}\right)\right)>0$ for some $m_{0}>B$ (this includes the case $m_{0}=\infty$ ), then there exists a subset $F$ in $E$ such that $2^{k_{j}} F \subset E$ for some $q>B$ integers $k_{q-1}>k_{q-2}>\cdots>k_{0}=0$. To see this, first let $E_{n}=E\left(\delta, m_{0}\right) \cap\left(\left(-\infty,-2^{n}\right) \cup\left(2^{n}, \infty\right)\right)$. Notice that $\mu\left(E_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$, so when $n$ is large enough, $\mu\left(E_{n}\right) \ll \mu\left(E\left(\delta, m_{0}\right)\right)$. It follows that $\mu\left(\bigcup_{k \leq 0} 2^{k} E_{n}\right) \ll \mu\left(E\left(\delta, m_{0}\right)\right)$ when $n$ is large enough, say $n \geq n_{0}$ for some $n_{0}>0$. Let $D=E\left(\delta, m_{0}\right) \backslash E_{n_{0}}$. Then a point in $D$ is not 2-dilation equivalent to any point
of $E_{n_{0}}$ by its definition, that is, all its equivalent points are in $\left[-2^{n_{0}}, 2^{n_{0}}\right]$. Consider $D_{k}=D \cap\left(\left[-2^{n_{0}-k},-2^{n_{0}-k-1}\right) \cup\left[2^{n_{0}-k-1}, 2^{n_{0}-k}\right)\right), k \geq 0$. Since $\mu(D)>0$ and $\left[-2^{n_{0}}, 2^{n_{0}}\right)=\bigcup_{k \geq 0}\left(\left[-2^{n_{0}-k},-2^{n_{0}-k-1}\right) \cup\left[2^{n_{0}-k-1}, 2^{n_{0}-k}\right)\right)$, there exists a $k_{0} \geq 0$ such that $\mu\left(D_{k_{0}}\right)>0$ but $\mu\left(D_{k}\right)=0$ if $k<k_{0}$. Define $D_{k_{0}}=F_{1}$. Now we can consider the sets $2^{-k} F_{1}, k>0$. Since $\mu\left(F_{1}\right)>0, \mu\left(2^{k} F_{1} \cap E\right)=0$ for any $k>0$ and $F_{1}=F_{1}\left(\delta, m_{0}\right)$ by its choice, there exists $k_{1}>0$ so that $\mu\left(E \cap 2^{-k_{1}} F_{1}\right)>0$ and $\mu\left(E \cap 2^{-k} F_{1}\right)=0$ if $0<k<k_{1}$. Let $F_{2}=E \cap 2^{-k_{1}} F_{1}$. This process can now be repeated at least $[B]$ times and the last set obtained is the $F$ we need.

Now define $f=\chi_{F}$. By Lemma $7, H_{E}^{k} f$ converges for each $k$. In particular, for $k=0,-k_{1}, \ldots,-k_{q-1}$, we have $H_{E}^{k} f=\sum_{m=1}^{M} \sum_{j=1}^{m} f_{m j}^{k} \chi_{2^{k} E} \geq f$. It then follows that $\sum_{k, \ell \in \mathbb{Z}}\left|a_{k \ell}\right|^{2} \geq \sum_{j=-(q-1)}^{0} \sum_{\ell \in \mathbb{Z}}\left|a_{k_{j} \ell}\right|^{2}=\sum_{j=-(q-1)}^{0}\left\langle H_{E}^{-k_{j}} f, f\right\rangle \geq q\|f\|^{2}>$ $B\|f\|^{2}$. This contradicts the assumption.

Note. (15) implies the following decomposition of $H_{E} f$ :

$$
\begin{equation*}
H_{E} f=H_{E(\tau, 1)} f+H_{E(\tau, 2)} f+\ldots+H_{E(\tau, m)} f \tag{19}
\end{equation*}
$$

Proof of Theorem (2) By (14) and Theorem [1] we have $\left|\left\langle H_{E} f, f\right\rangle\right| \leq M^{\frac{5}{2}}\|f\|^{2}$. On the other hand,

$$
\sum_{j=1}^{m} f_{m j}^{k} \cdot \chi_{2^{k} E(\tau, m)}=\sum_{\ell \in \mathbb{Z}}\left\langle f, \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E(\tau, m)}\right\rangle \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E(\tau, m)}
$$

hence $\left\langle\sum_{j=1}^{m} f_{m j}^{k} \cdot \chi_{2^{k} E(\tau, m)}, f\right\rangle=\sum_{\ell \in \mathbb{Z}}\left|\left\langle f, \widehat{D}^{k} \widehat{T}^{\ell} \frac{1}{\sqrt{2 \pi}} \chi_{E(\tau, m)}\right\rangle\right|^{2} \geq 0$. It follows that $\left\langle H_{E} f, f\right\rangle=\sum_{k \in \mathbb{Z}} \sum_{m=1}^{M} \sum_{j=1}^{m}\left\langle f_{m j}^{k} \cdot \chi_{2^{k} E(\tau, m)}, f\right\rangle \geq \sum_{k \in \mathbb{Z}}\left\langle f \cdot \chi_{2^{k} E(\tau, 1)}, f\right\rangle=$ $\int_{\mathbb{R}}|f|^{2}\left(\sum_{k \in \mathbb{Z}} \chi_{2^{k} E(\tau, 1)}\right) d s \geq\|f\|^{2}$ since $\sum_{k \in \mathbb{Z}} \chi_{2^{k} E(\tau, 1)} \geq 1$ by the given condition.

Proof of Theorem [3, This is obvious from Theorem 1
Proof of Theorem 4. If $E=E(\delta, k)=E(\tau, 1)$ for some $k \geq 1$ and $\bigcup_{n \in \mathbb{Z}} 2^{n} E=\mathbb{R}$ (modulo a null set), then $\sum_{n \in \mathbb{Z}} \chi_{2^{n} E}=k$ for almost all $s \in \mathbb{R}$. So for any $f \in L^{2}(\mathbb{R})$, we have $\left\langle H_{E} f, f\right\rangle=\left\langle\sum_{n \in \mathbb{Z}} f \chi_{2^{n} E}, f\right\rangle=\int_{\mathbb{R}}|f|^{2} \sum_{n \in \mathbb{Z}} \chi_{2^{n} E} d s=k\|f\|^{2}$.

Now assume that $E$ is a tight frame wavelet set but $\mu\left(E\left(\tau, m_{0}\right)\right)>0$ for some $m_{0}>1$. Let $g=\chi_{E^{(1)}\left(\tau, m_{0}\right)}, h=\chi_{E^{(2)}\left(\tau, m_{0}\right)}$ and $f_{1}=g+h, f_{2}=g-h$. Since $\left\|f_{1}\right\|=\left\|f_{2}\right\|$, we must have $\left\langle H_{E}\left(f_{1}\right), f_{1}\right\rangle=\left\langle H_{E}\left(f_{2}\right), f_{2}\right\rangle$. However, on the other hand, we have

$$
\left\langle H_{E}\left(f_{1}\right), f_{1}\right\rangle=\left\langle H_{E} g+H_{E} h, g+h\right\rangle=\left\langle H_{E} g, g\right\rangle+\left(\left\langle H_{E} g, h\right\rangle+\left\langle H_{E} h, g\right\rangle\right)+\left\langle H_{E} h, h\right\rangle .
$$

Similarly, $\left\langle H_{E}\left(f_{2}\right), f_{2}\right\rangle=\left\langle H_{E} g, g\right\rangle-\left(\left\langle H_{E} g, h\right\rangle+\left\langle H_{E} h, g\right\rangle\right)+\left\langle H_{E} h, h\right\rangle$. Since $H_{E} g=$ $\sum_{k \in \mathbb{Z}} \sum_{m=1}^{M} \sum_{j=1}^{m} g_{m j}^{k} \chi_{2^{k} E(\tau, m)}$ contains the term $\chi_{E\left(\tau, m_{0}\right)}$ (with $k=0, m=m_{0}$ and $j=1$ ) and the other terms in it are all nonnegative, we see that $\left\langle H_{E} g, h\right\rangle>0$. $\left\langle H_{E} h, g\right\rangle>0$ similarly. Thus $\left\langle H_{E}\left(f_{1}\right), f_{1}\right\rangle \neq\left\langle H_{E}\left(f_{2}\right), f_{2}\right\rangle$, a contradiction. Finally, assume that $\mu\left(E\left(\delta, k_{1}\right)\right) \neq 0$ and $\mu\left(E\left(\delta, k_{2}\right)\right) \neq 0$ for some $k_{1} \neq k_{2}$. Let $f_{1}=\chi_{E\left(\delta, k_{1}\right)}$ and $f_{2}=\chi_{E\left(\delta, k_{2}\right)}$. We leave it to our reader to check that $H_{E}\left(f_{1}\right)=k_{1} f_{1}$ and $H_{E}\left(f_{2}\right)=k_{2} f_{2}$. This leads to $\left\langle H_{E}\left(f_{1}\right), f_{1}\right\rangle=k_{1}\left\|f_{1}\right\|^{2}$ and $\left\langle H_{E}\left(f_{2}\right), f_{2}\right\rangle=k_{2}\left\|f_{2}\right\|^{2}$. Again, we get a contradiction.

The corollaries of the theorems now follow trivially and the proofs are omitted.

## §5. Examples and generalizations

Example 1. Let $E=\left[-\pi,-\frac{\pi}{2^{n}}\right) \cup\left[\frac{\pi}{2^{n}}, \pi\right)$ where $n \geq 1$ is an integer. Then $E=$ $E(\tau, 1)$ and $E=E(\delta, n)$; hence $E$ is a tight frame wavelet set of frame bound $n$.

Example 2. Let $E=\left[-\pi,-\frac{\pi}{2}\right) \cup\left[\frac{\pi}{4}, \pi\right)$. By Corollary 3, $E$ is a frame wavelet set with frame bounds $a=1$ and $b=2$.

Example 3. Let $E=\left[-\frac{3 \pi}{2},-\frac{\pi}{2}\right) \cup\left[\frac{\pi}{4}, \pi\right)$. Then $E$ is a frame wavelet set with lower bound 1 since $E(\tau, 1)=\left[-\pi,-\frac{\pi}{4}\right) \cup\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ satisfies condition (i) of Theorem 2 and $E(\delta, m)=E(\tau, m)=\emptyset$ for $m>2$.

Example 4. Let $E=[-3 \pi,-\pi) \cup[\pi, 2 \pi)$. Then $E$ does not satisfy the conditions in Theorem 2 since $E_{1}=E(\tau, 1)=[-2 \pi,-\pi)$ so $\bigcup_{n \in \mathbb{Z}} 2^{n} E_{1} \neq \mathbb{R}$. However, one can prove that $E$ is indeed a frame wavelet set with a positive lower bound and $4 \sqrt{2}$ an upper bound. The upper bound is easy to see since $E(\delta, m)=E(\tau, m)=\emptyset$ for all $m>2$. To see that a positive lower frame bound exists, let $f \in L^{2}(\mathbb{R})$ and define $f_{r}=f \cdot \chi_{[0, \infty)}$ and $f_{l}=f \cdot \chi_{(-\infty, 0)}$. Apparently $\|f\|^{2}=\left\|f_{l}\right\|^{2}+\left\|f_{r}\right\|^{2}$. By (19) and the proof of Theorem 2] we have

$$
\left\langle H_{E} f, f\right\rangle=\left\langle H_{E_{1}} f, f\right\rangle+\left\langle H_{E_{2}} f, f\right\rangle \geq \sum_{k \in \mathbb{Z}}\left\|f \cdot \chi_{2^{k} E_{1}}\right\|^{2}=\left\|f_{l}\right\|^{2}
$$

where $E_{2}=E(\tau, 2)=[-3 \pi,-2 \pi) \cup[\pi, 2 \pi)$. If $\left\|f_{l}\right\|^{2} \geq \alpha\|f\|^{2}$ for some small positive constant $\alpha$ (to be determined later), there is nothing to prove. So we only need to consider the case $\left\|f_{l}\right\|^{2}<\alpha\|f\|^{2}$, that is, $\left\|f_{r}\right\|^{2}>(1-\alpha)\|f\|^{2}$. Notice that $\left\langle H_{E_{2}} f_{r}, f_{r}\right\rangle=\left\|f_{r}\right\|^{2}$; we get

$$
\begin{aligned}
\left\langle H_{E} f, f\right\rangle & \geq\left\langle H_{E_{2}} f, f\right\rangle \\
& =\left\langle H_{E_{2}} f_{l}, f_{l}\right\rangle+\left\langle H_{E_{2}} f_{r}, f_{r}\right\rangle+\left\langle H_{E_{2}} f_{l}, f_{r}\right\rangle+\left\langle H_{E_{2}} f_{r}, f_{l}\right\rangle \\
& \geq\left\|f_{r}\right\|^{2}-\left(\left\|H_{E_{2}} f_{r}\right\| \cdot\left\|f_{l}\right\|+\left\|f_{r}\right\| \cdot\left\|H_{E_{2}} f_{l}\right\|\right. \\
& \geq\left\|f_{r}\right\|^{2}-8 \sqrt{2}\left\|f_{r}\right\| \cdot\left\|f_{l}\right\| \\
& \geq((1-\alpha)-8 \sqrt{2 \alpha(1-\alpha)})\|f\|^{2} .
\end{aligned}
$$

Apparently, if $\alpha$ is small enough, the above is greater than $\alpha\|f\|^{2}$. In fact, $\alpha$ can be chosen to be 0.005 .

This example shows that the condition given in Theorem 2 is not a necessary condition of a frame wavelet set.

Example 5. Let $E=\left[-\pi,-\frac{\pi}{2}\right) \cup[\pi, 2 \pi)$. Then $E$ is not a frame wavelet set. We leave this to our reader to verify as an exercise. Compare this example with the above example.

Hint: Let $f=\chi_{\left[-\pi,-\frac{\pi}{2}\right)}-\chi_{\left[\pi, \frac{3 \pi}{2}\right)}$ and calculate $H_{E} f$.
Not many results are known for general frame wavelet functions. One sufficient condition for a general frame function is given in [8, chapter 8 (Theorem 3.2)]. It is easy to see that the condition is not a sufficient condition as one can check that Examples 3 and 4 above do not satisfy the conditions given there.

We conclude this paper with the following Theorem, which is a generalization of Theorem [2, The proof is left to our reader. This points to a direction where the results obtained in this paper may be applied.

Theorem 5. Let $\psi \in L^{2}(\mathbb{R})$ such that the support $E$ of $\widehat{\psi}$ is of finite measure. Then $\psi$ is a general frame function if (i) E satisfies the conditions in Theorem $\mathbf{Q}_{\text {, }}$ (ii) $|\widehat{\psi}|$ is bounded above and (iii) $|\widehat{\psi}|$ is bounded below by a positive constant on $E(\tau, 1)$.

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