PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 7, Pages 1979–1986
S 0002-9939(00)05948-7
Article electronically published on November 22, 2000

DOMINATION BY POSITIVE DISJOINTLY STRICTLY SINGULAR OPERATORS

JULIO FLORES AND FRANCISCO L. HERNÁNDEZ

(Communicated by Joseph A. Ball)

ABSTRACT. We prove that each positive operator from a Banach lattice E to a Banach lattice F with a disjointly strictly singular majorant is itself disjointly strictly singular provided the norm on F is order continuous. We prove as well that if $S : E \to E$ is dominated by a disjointly strictly singular operator, then S^2 is disjointly strictly singular.

1. INTRODUCTION

The classical problem of domination for positive compact operators on Banach lattices was solved by Dodds and Fremlin ([5]) for a pair of positive operators $0 \leq S \leq T$ defined on a Banach lattice E with order continuous dual norm and taking values in a Banach lattice F with order continuous norm: we can guarantee that S is compact if T is so. A full answer to this problem was given by Aliprantis and Burkinshaw in [2], namely if $E = F$ and either the norm on E or the norm on E' is order continuous, then the compactness of T is inherited by the operator S^2. They also show that for an arbitrary Banach lattice, T compact always implies S^3 compact. More recently Wickstead has given in [14] not only sufficient but necessary conditions for the problem of domination for positive compact operators to have a solution.

The problem of domination for weakly compact operators was first considered by Abramovich in [1], giving a positive solution for a Banach lattice E and a KB-space F. Later on, a general result was obtained by Wickstead in [13] where it was shown that the problem has a positive answer if and only if either the norm on E' or F is order continuous. Again Aliprantis and Burkinshaw settled the question by considering the case $E = F$ and showing that T weakly compact implies S^2 weakly compact ([3]).

The aim of this paper is to study the problem of domination for positive disjointly strictly singular operators. We recall that an operator T between a Banach lattice E and a Banach space Y is said to be *disjointly strictly singular* (DSS) if there is no disjoint sequence of non-null vectors $(x_n)_n$ in E such that the restriction of T to the subspace $[x_n]$ spanned by the vectors $(x_n)_n$ is an isomorphism. DSS operators were introduced by Rodríguez-Salinas and the second author in [9]. This class of operators, a generalization of the class of strictly singular (or Kato) operators, is a

Received by the editors October 14, 1999.
2000 *Mathematics Subject Classification.* Primary 47B65, 46B42.
This work was partially supported by DGES PB97-0240.

useful tool to compare the lattice structure of Banach function lattices ([7]). Recall that an operator T between two Banach spaces X and Y is said to be *strictly singular* if the restriction of T to any infinite dimensional closed subspace is not an isomorphism. Every strictly singular operator is DSS but the converse is not true (f.i. take the natural inclusion $i : L^p[0,1] \to L^q[0,1]$ with $p > q \geq 1$). However if E is a Banach lattice with a Schauder basis of mutually disjoint vectors or a $C(K)$-space, then every DSS operator from E to Y is strictly singular. The set of all DSS operators between E and Y is a vector space which is stable under the composition by the left but not by the right ([8, Prop. 1]).

The main results of the paper are presented now.

Theorem 1.1. *Let E and F be Banach lattices and $0 \leq S \leq T : E \to F$ two positive operators. If the norm on F is order continuous and T is DSS, then S is also DSS.*

Moreover, if F is σ-Dedekind complete and every positive operator from E to F dominated by a DSS operator is DSS, then either the norm on F or the norm on E' is order continuous.

In the case $E = F$ we obtain the following:

Theorem 1.2. *Let $0 \leq S \leq T$ be two operators on a Banach lattice E. If T is DSS, then S^2 is DSS.*

For any unexplained terms from Banach lattices and regular operators theory we refer to [4], [12] or [15].

2. Proofs

Let us start by recalling a couple of well-known facts.

Lemma 2.1. *Let E be an L-space. Then every weakly-null sequence of positive vectors is convergent to zero.*

Lemma 2.2. *Let (Ω, Σ, μ) be a finite measure space and $(f_n)_n$ be a weakly convergent sequence in $L^1(\mu)$. If $(f_n)_n$ converges to zero in μ-measure, then $(f_n)_n$ converges to zero in norm.*

Proof. We can assume w.l.o.g. that $\mu(\Omega) = 1$. The sequence $(f_n)_n$ is uniformly absolutely continuous since it is weakly convergent (cf. [6, Cor. IV.8.11]). Hence for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|\chi_B f_n\|_1 < \varepsilon/2$ for every integer n and every $B \in \Sigma$ with $\mu(B) < \delta$. Consider $B_n = \{t \in \Omega : |f_n(t)| > \varepsilon/2\}$. By assumption there exists an integer n_0 such that $\mu(B_n) < \delta$ for $n \geq n_0$. Thus, for $n \geq n_0$ we have

$$\|f_n\|_1 = \int_{B_n} |f_n| + \int_{B_n^c} |f_n| \leq \|\chi_{B_n} f_n\|_1 + \varepsilon/2 < \varepsilon.$$

□

The following result will be used in the sequel (cf. [12, Cor. 3.4.14 and Thm. 3.4.17]).

Proposition 2.3. *Let E and F be two Banach lattices such that the norm on F is order continuous. If $T : E \to F$ is a positive operator, then T preserves an isomorphic copy of l^1 if and only if T preserves a lattice isomorphic copy of l^1.*

The proposition given next is essential in the proof of Theorem 1.1. Notice that in the definition of a DSS operator the disjoint vectors are not in general positive.

Proposition 2.4. *Let T be a positive operator defined on a Banach lattice E and with values in a Banach lattice F with order continuous norm. Then T is DSS if and only if there is no disjoint sequence of non-null positive vectors $(y_n)_n$ in E such that the restriction of T to the span $[y_n]$ is an isomorphism.*

Proof. We just need to prove the non-trivial implication. Suppose that T is not DSS; then there exists a (normalized) sequence in E of pairwise disjoint elements $(x_n)_n$ such that the restriction of T to the span $[x_n]$ is an isomorphism, that is,

$$\left\| T\left(\sum_{n=1}^{\infty} a_n x_n \right) \right\| \geq \alpha \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \quad \text{for some } \alpha > 0$$

(note that $(x_n)_n$ is an unconditional basic sequence, being disjoint). Since the span of the sequence $(Tx_n)_n$ is a separable subspace of the Banach lattice F, we can find a closed order ideal J of F with a weak unit which contains $[Tx_n]$ (cf. [11, Prop. 1.a.9]); furthermore J is complemented in F by a positive projection P (cf. [11, Prop. 1.a.11]). Consider the operator $PT : E \to J$. Clearly the restriction of PT to the span $[x_n]$ is not DSS. On the other hand the assumption on T shows that the operator PT is not invertible on the span of any disjoint sequence of *positive* vectors. Therefore there is no loss of generality in assuming that F itself has a weak unit. In such a case we can represent F as an (in general not closed) order ideal of $L^1(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ), which is continuously included in $L^1(\Omega, \Sigma, \mu)$ (cf. [11, Thm. 1.b.14]).

$(*)$ Let us consider the sublattice $[\,|x_n|\,]$; we claim that $[\,|x_n|\,]$ contains no lattice isomorphic copy of c_0.

To prove this we will assume the contrary and get a contradiction by showing that T is invertible on a sublattice of $[|x_n|]$. Suppose then that there exists a normalized sequence $(z_k)_k \subset [\,|x_n|\,]$ of mutually disjoint positive vectors equivalent to the unit basis $\{e_k\}_k$ of c_0. Let us write $z_k = \sum_{j=1}^{\infty} a_j^k |x_j|$ with $a_j^k \geq 0$ for all j. If $\|\,T\,(z_{k_l})\| \underset{l}{\to} 0$ for some subsequence $(k_l)_l$, then $\left\| T\left(\sum_{j=1}^{\infty} a_j^{k_l} x_j \right) \right\| \underset{l}{\to} 0$ and hence $\|z_{k_l}\| = \left\| \sum_{j=1}^{\infty} a_j^{k_l} x_j \right\| \underset{l}{\to} 0$ since the restriction of T to the span $[x_n]$ is an isomorphism; however this is a contradiction with $\|z_k\| = 1$ for all k. Thus we may assume that $\inf_k \|Tz_k\| \geq \delta > 0$ for some $\delta > 0$. The sequence $(z_k)_k$ is $\sigma([z_k], [z_k]')$-null being equivalent to the unit basis of c_0. Hence $(Tz_k)_k$ is weakly null in F being T bounded. Note that $(Tz_k)_k$ is also convergent to zero in the weak topology of $L^1(\mu)$ since F is continuously included in $L^1(\mu)$. In fact $(T\,z_k)_k$ converges to zero in $L^1(\mu)$ since $Tz_k \geq 0$ for all k (cf. Lemma 2.1).

Let us consider the *Kadec-Pełczynski* set $M(\varepsilon) = \{y \in F : \mu(\sigma(y, \varepsilon)) \geq \varepsilon\}$, where $\varepsilon > 0$ and $\sigma(y, \varepsilon) = \{t \in \Omega : |y(t)| \geq \varepsilon \|y\|\}$. If $(Tz_k)_k \subset M(\varepsilon)$ for some $\varepsilon > 0$, then $\|\,T\,z_k\|_1 \geq \varepsilon^2 \|Tz_k\|$ for all k; hence $\|Tz_k\| \underset{k}{\to} 0$, which is a contradiction with $\inf_k \|Tz_k\| \geq \delta > 0$. Thus we may assume that $(Tz_k)_k \not\subseteq M(\varepsilon)$ for every $\varepsilon > 0$; then, by Kadec-Pełczynsky's disjointification process (cf. [11, Prop. 1.c.8]) we may choose a subsequence $(Tz_{k_j})_j$ equivalent to a disjoint sequence in F; it follows that $(Tz_{k_j})_j$ is an unconditional basic sequence with unconditional constant, say $\beta > 0$.

For every integer j we have

$$\Big\|\sum_{j=1}^{\infty} a_j\, T\, z_{k_j}\Big\| \geq \beta^{-1}\Big\|\sum_{j=1}^{\infty}|a_j|\, T\, z_{k_j}\Big\| \geq \beta^{-1}|a_j|\,\|\, T\, z_{k_j}\| \geq \beta^{-1}|a_j|\delta$$

(note that in the previous inequalities we use that $Tz_{k_j} \geq 0$ for all j). Thus

$$\Big\| T\Big(\sum_{j=1}^{\infty} a_j z_{k_j}\Big)\Big\| \geq \beta^{-1}\delta\Big(\bigvee_{j=1}^{\infty}|a_j|\Big) \geq K\Big\|\sum_{j=1}^{\infty} a_j z_{k_j}\Big\|,$$

where K is a positive constant. Hence the operator T preserves a lattice copy of c_0, which is a contradiction.

Now if we apply Rosenthal's dichotomy theorem (cf. [10, Thm. 2.e.5]) to the sequence $(|x_n|)_n$, we obtain a subsequence $(|x_{n_j}|)_j$ satisfying either (1) $(|x_{n_j}|)_j$ is equivalent to the unit basis of l^1 or (2) $(|x_{n_j}|)_j$ is a weakly Cauchy sequence. Suppose first that (1) holds. Then

$$\sum_j a_j x_{n_j} < \infty \Leftrightarrow \sum_j a_j |x_{n_j}| < \infty \Leftrightarrow \sum_j |a_j| < \infty;$$

hence T preserves an isomorphic copy of l^1 or, equivalently by Proposition 2.3, T preserves a lattice copy of l^1. Contradiction.

Finally let us show that case (2) also leads to contradiction. Indeed, once the statement $(*)$ has been proved we may assume that the Banach lattice $[\,|x_n|\,]$ is weakly sequentially complete or equivalently a KB-space (cf. [4, Thm. 14.12]); hence the subsequence $(|x_{n_j}|)_j$ must be weakly convergent. Thus the separable lattice $[\,|x_n|\,]$ has an order continuous norm and a weak unit, and hence it can be considered as a continuously-included order ideal in $L^1(\Omega', \Sigma', \mu')$ for some probability space (Ω', Σ', μ') (cf. [11, Thm. 1.b.14]). It follows that $(|x_{n_j}|)_j$ is convergent in the weak topology of $L^1(\mu')$ to a function f. In fact $f = 0$ since the sequence $(|x_{n_j}|)_j$ converges to zero in μ'-measure being pairwise disjoint (cf. Lemma 2.2). Since T is bounded, the sequence $(T|x_{n_j}|)_j$ converges to zero in the weak topologies of F and $L^1(\mu)$; hence $\|\, T|x_{n_j}|\,\|_1 \underset{j}{\to} 0$ by Lemma 2.1.

We apply again the Kadec-Pełczynski method: if $T(|x_{n_j}|)_j \subset M(\varepsilon)$ for some $\varepsilon > 0$, then $\|\, T\,|x_{n_j}|\,\|_1 \underset{j}{\to} 0$ implies $\|\, T\,|x_{n_j}|\,\| \underset{j}{\to} 0$ and $\|x_{n_j}\| \underset{j}{\to} 0$ follows; this is a contradiction with the initial choice of $(x_n)_n$. Thus we may assume $(T\,|x_{n_j}|)_j \not\subseteq M(\varepsilon)$ for all $\varepsilon > 0$; in this case we may choose a subsequence, still denoted by $(T|x_{n_j}|)_j$, which is equivalent to a disjoint sequence in F. It follows that $(T\,|x_{n_j}|)_j$ is an unconditional basic sequence with unconditional constant, say $K > 0$. And

$$\begin{aligned}\alpha\Big\|\sum_{j=1}^{\infty} a_j|x_{n_j}|\Big\| &= \alpha\Big\|\sum_{j=1}^{\infty} a_j x_{n_j}\Big\| \leq \Big\|\sum_{j=1}^{\infty} a_j\, T\, x_{n_j}\Big\| \\ &\leq \Big\|\sum_{j=1}^{\infty}|a_j|\, T\,|x_{n_j}|\Big\| \leq K\,\Big\|\sum_{j=1}^{\infty} a_j\, T\,|x_{n_j}|\Big\| = K\,\Big\| T\,\Big(\sum_{j=1}^{\infty} a_j|x_{n_j}|\Big)\Big\|,\end{aligned}$$

that is, T is invertible on the span $[\,|x_{n_j}|\,]$. This contradiction concludes the proof. □

Remark 2.5. If the dual norm on E' and the norm on F are simultaneously order continuous, then the previous proof becomes shorter by noticing that the sequence $(|x_n|)_n$ is weakly null (cf. [12, Thm. 2.4.14]).

We recall that an operator T between a Banach lattice E and a Banach space Y is said to be *order-weakly compact* if $T[-x,x]$ is relatively weakly compact for every $x \in E_+$. It is known that $T : E \to Y$ is order-weakly compact if and only if T does not preserve a sublattice isomorphic to c_0 whose unit ball is order bounded in E (cf. [12, Cor. 3.4.5]). Consequently every DSS operator is order-weakly compact. The following characterization will be used in the sequel (cf. [12, Prop. 3.4.9]).

Proposition 2.6. *Let E be a Banach lattice, F be a Banach space and $T : E \to F$ a bounded operator. Then T is order-weakly compact if and only if T transforms every order bounded and weakly null sequence of positive vectors in E in a sequence convergent to zero.*

We pass now to prove the main result of the paper.

Proof of Theorem 1.1. We prove first that if the norm on F as well as the norm on E' are order continuous, then T DSS implies S DSS.

Assume that this is not the case. By Proposition 2.4 there exists a disjoint (normalized) sequence $(x_n)_n$ of positive vectors in E and $\alpha > 0$ such that $\|Sx\| \geq \alpha\|x\|$ for all $x \in [x_n]$. As in the proof of Proposition 2.4 the Banach lattice F can be considered to have a weak unit. Hence there exist a probability space (Ω, Σ, μ), an order ideal I of $L^1(\Omega, \Sigma, \mu)$, a lattice norm $\| \|_I$ on I and an order isometry ψ between F and $(I, \| \|_I)$, such that the canonical inclusion from I in $L^1(\mu)$ is continuous with $\|f\|_1 \leq \|f\|_I$ (cf. [11, Thm. 1.b.14]). Note that $\psi T : E \to I$ is DSS and that $0 \leq \psi S \leq \psi T$. Note too that if ψS were DSS, then S would also be DSS. This observation allows us to reduce the proof to the case that F is an order ideal in $L^1(\Omega, \Sigma, \mu)$.

We claim that $(Tx_n)_n \not\subseteq M(\varepsilon)$ for all $\varepsilon > 0$ where $M(\varepsilon)$ denotes a Kadec-Pełczynski set as above. Indeed, the norm-bounded disjoint sequence $(x_n)_n$ is weakly null since the norm on E' is order continuous (cf. [12, Thm. 2.4.14]). Hence $(Tx_n)_n$ is a weakly null sequence in $L^1(\mu)$; in fact $\|Tx_n\|_1 \to 0$ by Lemma 2.1. If $(Tx_n)_n \subseteq M(\varepsilon)$ for some $\varepsilon > 0$, then $\|Tx_n\|_1 \geq \varepsilon^2\|Tx_n\|$; hence $(Tx_n)_n$ converges to zero in F. The inequalities $0 \leq Sx_n \leq Tx_n$ for all n show that $(Sx_n)_n$ converges to zero in F, and hence $\|x_n\| \underset{n}{\to} 0$ since $\|Sx_n\| \geq \alpha\|x_n\|$ for all n; however this is a contradiction with the choice of $(x_n)_n$. Now, by Kadec-Pełczynski's disjointification process (cf. [11, Prop. 1.c.8]), we may choose a subsequence, still denoted by $(Tx_n)_n$, equivalent to a pairwise disjoint sequence in F; hence $(Tx_n)_n$ is an unconditional basic sequence with unconditional constant, say $K > 0$. We have

$$\left\|T\left(\sum_{n=1}^{\infty} a_n x_n\right)\right\| = \left\|\sum_{n=1}^{\infty} a_n T x_n\right\| \geq K^{-1}\left\|\sum_{n=1}^{\infty} |a_n| T x_n\right\| \geq K^{-1}\left\|\sum_{n=1}^{\infty} |a_n| S x_n\right\|$$
$$\geq \alpha K^{-1}\left\|\sum_{n=1}^{\infty} |a_n| x_n\right\| = \alpha K^{-1}\left\|\sum_{n=1}^{\infty} a_n x_n\right\|,$$

for all $x \in [x_n]$. However this is impossible since T is a DSS operator.

We can now prove the first part of Theorem 1.1 in the general case. Assume the opposite, that is, there is a disjoint sequence $(x_n)_n$ of positive vectors in E such that the restriction of S to the sublattice $[x_n]$ is an isomorphism while T is DSS. By the lines above we can assume that the dual norm on the sublattice $[x_n]'$ is not order continuous. Then $[x_n]$ contains a lattice copy of l^1 (cf. [12, Thm. 2.4.14]) which is preserved by S. It follows that the adjoint operator S' is not order-weakly compact (cf. [12, Cor. 3.4.14]) and so is T' by Proposition 2.6 (note that $0 \leq S' \leq T'$); hence T preserves a copy of l^1 (cf. [12, Cor. 3.4.14]) or equivalently T preserves a lattice copy of l^1 (cf. Proposition 2.3). Contradiction.

To prove the second part of Theorem 1.1 assume that the norms on E' and F are simultaneously not order continuous. Since the dual norm on E' is not order continuous, we can find in E a lattice copy of l^1 complemented by a positive projection (cf. [12, Thm. 2.4.14 and Prop. 2.3.11]); let H_1 be the sublattice of E lattice isomorphic to l^1, ϕ_1 the lattice isomorphism between H_1 and l^1, and P_1 the positive projection from E onto H_1. On the other hand, since the norm on F is not order continuous and F is σ-Dedekind complete, there exist a sublattice H_2 in F and a lattice isomorphism ϕ_2 between H_2 and l^∞ (cf. [12, Cor. 2.4.3]). Consider the operators defined from l^1 into l^∞ by

$$T(a = (a_n)) = \left(\sum_{n=1}^{\infty} a_n\right)(1, 1, \cdots) \quad \text{and} \quad S(a = (a_n)) = \left(\left(\sum_{n=1}^{\infty} x_{k,n} a_n\right)_{k=1}^{\infty}\right),$$

where $S \equiv (x_{k,n})$ is the infinite matrix with $\{0, 1, -1\}$-entries defined as follows:

$$S \equiv (x_{k,n}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & -1 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Note that for a given $a \in l^1$ there exists an integer k with $2 + 2^2 + \cdots + 2^{n-1} < k \leq 2 + 2^2 + \cdots + 2^n$ satisfying

$$\sum_{m=1}^{n} |a_m| = \sum_{j=1}^{n} x_{k,j} a_j.$$

Clearly S is a linear isometry from l^1 into l^∞. Indeed,

$$\|S(a)\|_\infty = \sup_k \left(\left| \sum_{n=1}^{\infty} x_{k,n} a_n \right| \right) \leq \sum_{n=1}^{\infty} |a_n| = \|a\|_1.$$

On the other hand, for a given $\varepsilon > 0$ there exists an integer n such that $\sum_{m=1}^{n} |a_m| \geq (\|a\|_1 - \varepsilon)$; hence there is an integer k satisfying $\sum\limits_{j=1}^{n} x_{k,j} a_j = \sum\limits_{m=1}^{n} |a_m| \geq (\|a\|_1 - \varepsilon)$. The relations $\|Sa\|_\infty \geq \|a\|_1 - \varepsilon$ and $\|Sa\|_\infty = \|a\|_1$ follow.

Consider now the operators S^+ and S^- defined by the sequences $(x'_{k,n})_{n,k}$ and $(x''_{k,n})_{n,k}$ where

$$x'_{k,n} = \begin{cases} x_{k,n} & \text{if } x_{k,n} > 0, \\ 0 & \text{if } x_{k,n} \leq 0, \end{cases} \qquad x''_{k,n} = \begin{cases} -x_{k,n} & \text{if } x_{k,n} < 0, \\ 0 & \text{if } x_{k,n} \geq 0. \end{cases}$$

The operator $\widetilde{S}(a) = (\sum_{j=1}^{\infty} |x_{k,j}| a_j)_{k=1}^{\infty}$ from l^1 into l^∞ clearly factorizes through the space c of all convergent sequences (which is isomorphic to c_0); thus $\widetilde{S}$ is strictly singular. It follows from the equalities $\widetilde{S} = S^+ + S^-$ and $S = 2S^+ - \widetilde{S}$ that S^+ is not strictly singular. Hence S^+ is neither DSS as an operator from l^1 into l^∞ (in fact the inequality $\|S^+a\| \geq 1/2\|a\|_1$ holds for all $a \in l^1$). Finally it is clear that $S^+ \leq T$ and that T is DSS being a rank-one operator.

Consider the operators $S' = \phi_2 S^+ \phi_1 P_1$ and $T' = \phi_2 T \phi_1 P_1$ defined on E and with values in F. Clearly $0 \leq S' \leq T'$; moreover $\phi_2 S^+ \phi_1$ is not strictly singular since ϕ_1 and ϕ_2 are isomorphisms and S^+ is not strictly singular. The inequalities

$$m\|h\|_E \leq \|S^+\phi_1(h)\|_{l^\infty} \leq \|\phi_2 S^+ \phi_1(h)\|_F \leq M\|h\|_H$$

show that S' is invertible on H, or equivalently that S' is not DSS. □

Remark 2.7. The proof actually shows that if the norms on E' and F are simultaneously not order continuous and F is σ-Dedekind complete, then the problem of domination for strictly singular operators has in general a negative answer. This problem requires its own study which will be carried out elsewhere.

We consider next the problem of domination in the case $E = F$. To this end we recall the following *factorization* result due to Aliprantis and Burkinshaw (cf. [4, Thm. 18.7] or [12, Thm. 3.4.6]): Let E and Y be a Banach lattice and a Banach space respectively and $T : E \to Y$ an order-weakly compact operator. Then there exist a Banach lattice F with order continuous norm, a lattice homomorphism Q from E into F and a bounded operator S from F into Y such that $T = SQ$. Moreover, if Y is a Banach lattice and $0 \leq T_1 \leq T$, then $0 \leq S_1 \leq S$.

Proposition 2.8. *Let E_i, $i = 1, 2, 3$, be Banach lattices and $0 \leq S_i \leq T_i$ operators defined on E_i and taking values in E_{i+1} for $i = 1, 2$. If T_1 is DSS and T_2 is order-weakly compact, then S_2S_1 is DSS.*

Proof. Given $0 \leq S_2 \leq T_2 : E_2 \to E_3$, we may find, by the above result, a Banach lattice G with order continuous norm, a lattice homomorphism Q from E_2 into G and two positive operators $\widetilde{S_2} \leq \widetilde{T_2}$ from G into E_3 such that $T_2 = \widetilde{T_2}Q$ and $S_2 = \widetilde{S_2}Q$. Consider the operators $S = QS_1$ and $T = QT_1$ from E_1 into G. Note that T is DSS since T_1 is so and we are composing by the left; hence S is DSS by Theorem 1.1 and so is $\widetilde{S_2}S$. Finally the equality $S_2S_1 = \widetilde{S_2}S$ concludes that S_2S_1 is DSS. □

Now Theorem 1.2 is a direct consequence of Proposition 2.8.

Remark 2.9. Theorem 1.2 is the best possible. Indeed, consider $E = l^1 \oplus l^\infty$ and the operators $0 \leq \widetilde{S} \leq \widetilde{T}$ on E defined via the matrices

$$\widetilde{S} = \begin{pmatrix} 0 & 0 \\ S^+ & 0 \end{pmatrix}, \qquad \widetilde{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$$

where S^+ and T are the operators defined in the proof of Theorem 1.1. Clearly $\widetilde{T}$ is DSS and $\widetilde{S}$ is not.

Recall that an *orthomorphism* on a Banach lattice E is a band preserving operator which is also order bounded. An easy consequence of Theorem 1.2, which is obtained by reasoning as in [4, Thm. 16.21], is

Corollary 2.10. *Let S and T be two positive operators on a Dedekind complete Banach lattice E such that $0 \leq S \leq T$ holds. If T is DSS and S is an orthomorphism, then S is DSS.*

The authors wish to thank Y. Abramovich, J. Mendoza and Y. Raynaud for some helpful discussions. They also appreciate the suggestions and remarks made by the referee.

References

1. Y. A. Abramovich, *Weakly compact sets in topological K-spaces*, Teor. Funkcii Funkional. Anal. i Prilozen **15** (1972), 27–35.
2. C.D. Aliprantis and O. Burkinshaw, *Positive compact operators on Banach lattices*, Math. Z. **174** (1980), 289–298. MR **81m:**47053
3. ———, *On weakly compact operators on Banach lattices*, Proc. Amer. Math. Soc. **83** (1981), 573–578. MR **82j:**47057
4. ———, *Positive operators*, Academic Press, 1985. MR **87h:**47086
5. P.G. Dodds and D.H. Fremlin, *Compact operators in Banach lattices*, Israel J. of Math. **34** (1979), 287–320. MR **81g:**47037
6. N. Dunford and J.T. Schwartz, *Linear Operators. Part I, General Theory*, Pure and applied Mathematics, vol. VII, Interscience, New York, 1958. MR **22:**8302
7. A. García Del Amo, F. L. Hernández, and C. Ruiz, *Disjointly strictly singular operators and interpolation*, Proc. Royal Soc. of Edinburgh **126A** (1996), 1011–1026. MR **97i:**47075
8. F. L. Hernández, *Disjointly Strictly-Singular Operators in Banach Lattices* , Proc. 18 Winter School on Abstract Analysis (Srni). Acta Universitatis Carolinae-Mathematica et Physica **31** (1990), 35–40. MR **92i:**47038
9. F. L. Hernández and B. Rodríguez-Salinas, *On l^p-complemented copies in Orlicz spaces II*, Israel J. of Math. **68** (1989), 27–55. MR **91c:**46041
10. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, 1977. MR **58:**17766
11. ———, *Classical Banach Spaces II*, Springer-Verlag, 1979. MR **81c:**46001
12. P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, 1991. MR **93f:**46025
13. A.W. Wickstead, *Extremal structure of cones of operators*, Quart. J. Math. Oxford Ser. (2) **32** (1981), 239–253. MR **82i:**47069
14. ———, *Converses for the Dodds-Fremlin and Kalton-Saab theorems*, Math. Proc. Camb. Phil. Soc. **120** (1996), 175–179. MR **96m:**47067
15. A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983. MR **86b:**46001

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain
E-mail address: jflores@sunam1.mat.ucm.es

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain
E-mail address: pacoh@eucmax.sim.ucm.es