

CONJUGACIES BETWEEN RATIONAL MAPS AND EXTREMAL QUASICONFORMAL MAPS

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ABSTRACT. We show that two rational maps which are K -quasiconformally combinatorially equivalent are K -quasiconformally conjugate. We also study the relationship between the boundary dilatation of a combinatorial equivalence and the dilatation of a conjugacy.

The Teichmüller theory is a powerful tool in the study of complex analytic dynamics. In addition to the work of D. Sullivan [Su] and W. Thurston (refer [DH2]), where the classical theory of the Teichmüller spaces plays a crucial role, a beautiful theorem about extremal quasiconformal maps between open Riemann surfaces, due to R. Strebel ([S]), was employed in the work of C. McMullen [Mc2]. In this paper, we give new applications of this theorem and a theorem of H. Ohtake ([Oh]) about lifts of extremal quasiconformal maps.

Suppose that f, g are rational maps of degree bigger than one which are *quasiconformally combinatorially equivalent*, i.e. there exist quasiconformal maps ϕ and ϕ_1 from $\hat{\mathbb{C}}$ to itself such that ϕ is isotopic to $\phi_1 \text{ rel } P(f)$ and $\phi f = g \phi_1$. Here

$$P(f) = \bigcup_{n>0} \overline{f^n(C_f)}$$

is the post-critical set of f , where C_f is the set of the critical points of f . It is known that there exists a quasiconformal conjugacy ψ from f to g , which is isotopic to $\phi \text{ rel } P(f)$ ([Mc3]). In the proof, ψ was constructed by the Douady-Earle extension and hence its maximal dilatation might be bigger than the maximal dilatation of ϕ . Applying extremal quasiconformal maps, we improve the theorem to show that ψ can be chosen such that its maximal dilatation is less than or equal to the maximal dilatation of ϕ (Theorem 1) and we give an application (Theorem 2) relating extremal and boundary dilatation.

At first, let us recall Strebel's Theorem. Let $\phi : R \rightarrow R'$ be a quasiconformal map between open Riemann surfaces. The *Beltrami differential* of ϕ is

$$\mu_\phi(z) \frac{d\bar{z}}{dz} = \frac{\partial_{\bar{z}}\phi(z)}{\partial_z\phi(z)}$$

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and the *maximal dilatation* of ϕ is

$$K(\phi) = \frac{1 + \|\mu_\phi\|_\infty}{1 - \|\mu_\phi\|_\infty}.$$

The *boundary dilatation* $K_1[\phi]$ of ϕ is the infimum of the maximal dilatation of $\psi|_W$ over all quasiconformal maps ψ isotopic to ϕ rel the boundary and all neighborhoods W of ∂R .

The map ϕ is called *extremal* if $K(\phi) \leq K(\psi)$ for all quasiconformal maps ψ isotopic to ϕ rel the boundary.

Let $q(z)dz^2$ be an integrable holomorphic quadratic differential on R , i.e.

$$\int_R |q(z)| dx dy < \infty.$$

A quasiconformal map ϕ is called a *Teichmüller map* associated with q if

$$\mu_\phi(z) = k \frac{\overline{q(z)}}{|q(z)|}$$

for some constant $0 < k < 1$. A Teichmüller map is the unique extremal map.

Strebel's Theorem. *Let ϕ be an extremal quasiconformal map with $K(\phi) > 1$. If $K_1[\phi] < K(\phi)$, then ϕ is a Teichmüller map associated with an integrable holomorphic quadratic differential on R .*

Theorem 1. *Let f, g be rational maps of degree bigger than one. Suppose that ϕ, ϕ_1 are K -quasiconformal maps from $\hat{\mathbb{C}}$ to itself such that ϕ_1 is isotopic to ϕ rel $P(f)$ and $\phi f = g\phi_1$ on $\hat{\mathbb{C}}$. Then there exists a K -quasiconformal conjugacy ψ from f to g which is isotopic to ϕ rel $P(f)$.*

Proof. Clearly, $\phi(P(f)) = P(g)$ and the theorem holds when $P(f)$ consists of two points. So assume that $P(f)$ contains at least three points. Denote

$$Q(f) = \overline{\bigcup_{n>0} f^{-n}(P(f))}.$$

The Julia set $J(f)$ of f is contained in $Q(f)$.

There exists a lift ϕ_2 of ϕ_1 , such that ϕ_2 is isotopic to ϕ_1 rel $f^{-1}(P(f))$ and $\phi_1 f = g\phi_2$. Continuing this process, we get a sequence of K -quasiconformal maps $\{\phi_n\}$, such that ϕ_{n+1} is isotopic to ϕ_n rel $f^{-n}(P(f))$ and $\phi_n f = g\phi_{n+1}$. There exist subsequences $\{\phi_{n_i}\}$, $\{\phi_{n_i+1}\}$, which uniformly converge to K -quasiconformal maps Φ and Φ_1 , respectively. Clearly, $\Phi(Q(f)) = Q(g)$, Φ_1 is isotopic to Φ rel $Q(f)$ and $\Phi f = g\Phi_1$.

Suppose that D_1 is a periodic Fatou domain of f with period $p \geq 1$. Then $D_2 = \Phi(D_1) = \Phi_1(D_1)$ is a periodic Fatou domain of g with the same type and period as D_1 . We claim that there exists a K -quasiconformal map ψ from D_1 to D_2 , such that ψ is isotopic to $\Phi|_{D_1}$ rel $Q(f)$ and $\psi f^p = g^p \psi$. (Note that a quasiconformal isotopy rel ideal boundary is equivalent to a quasiconformal isotopy rel real boundary, by [EM].)

Case 1. D_1 is a Siegel disk or a Herman ring. $D_1 - Q(f)$ is a disk U_1 or a disjoint union of annuli U_1, \dots, U_m and $f^p(U_i) = U_i$. The continuous extensions of f^p, g^p to the ideal boundary of U_i , $\Phi(U_i)$, denoted by \bar{f}^p and \bar{g}^p , are irrational rotations. Let $\bar{\Phi}$ and $\bar{\Phi}_1$ denote the continuous extensions of Φ and Φ_1 to the ideal boundary, respectively. Then $\bar{\Phi} = \bar{\Phi}_1$ and $\bar{\Phi} \bar{f}^p = \bar{g}^p \bar{\Phi}_1$. Now \bar{f}^p and \bar{g}^p are irrational rotations

with the same rotation number since they are topologically conjugate. Thus $\bar{\Phi}$ is a rotation since the orbit of every point is dense in each component of the ideal boundary.

By Strebel's Theorem, there exists a unique extremal quasiconformal map ψ_i in the isotopy class of $\Phi|_{U_i}$. Thus $\psi_i f^p = g^p \psi_i$. Let $\psi = \psi_i$ on each U_i and $\psi = \Phi$ on $Q(f)$. Then ψ is a K -quasiconformal map from D_1 to D_2 , ψ is isotopic to $\Phi|_{D_1}$ rel $Q(f)$ and $\psi f^p = g^p \psi$.

Case 2. D_1 is superattracting. $D_1 - Q(f)$ is a punctured disk U_1 or a disjoint union of annuli U_i , $i \in \mathbb{Z}$. Identify each component of the ideal boundary of U_i , $\Phi(U_i)$, with the unit circle S^1 . The continuous extensions of f^p and g^p , \bar{f}^p and \bar{g}^p , are $z \rightarrow z^d$ on S^1 for some $d > 1$. Let $\bar{\Phi}$ and $\bar{\Phi}_1$ denote the continuous extensions of Φ and Φ_1 to the ideal boundary, respectively. Then $\bar{\Phi} = \bar{\Phi}_1$ and $\bar{\Phi} f^p = \bar{g}^p \bar{\Phi}_1$. Thus $\bar{\Phi}$ is a rotation.

An argument similar to the one above shows that there is a K -quasiconformal map ψ from D_1 to D_2 such that ψ is isotopic to $\Phi|_{D_1}$ rel $Q(f)$ and $\psi f^p = g^p \psi$.

Case 3. D_1 is attracting. There exist a domain $V_i \subset D_i$, $i = 1, 2$, which contains the periodic point, and a conformal map h_i from V_i into \mathbb{C} , such that h_i maps the periodic point to the origin,

$$h_1 f^p h_1^{-1}(z) = \lambda_1(z) = \lambda_1 z \quad \text{and}$$

$$h_2 g^p h_2^{-1}(z) = \lambda_2(z) = \lambda_2 z$$

whenever they are defined. Denote

$$E_1 = h_1(Q(f) \cap V_1) \quad E_2 = h_2(Q(g) \cap V_2)$$

$$\text{and } K_i = \bigcup_{n>0} \lambda_i^{-n}(E_i).$$

Then $\lambda_i(K_i) = K_i$ and $R_i = (\mathbb{C} - K_i)/\langle \lambda_i \rangle$ is a punctured torus, where $\langle \lambda_i \rangle$ means the Abelian group generated by $z \rightarrow \lambda_i z$.

Now $h_2 \Phi h_1^{-1}(\lambda_1 z) = \lambda_2 h_2 \Phi_1 h_1^{-1}(z)$ is K -quasiconformal on a neighborhood of the origin. There exists a subsequence of $\{\lambda_2^{-n} h_2 \Phi h_1^{-1} \lambda_1^n\}_{n=1}^\infty$ which uniformly converges to a K -quasiconformal map Ψ of \mathbb{C} . Let $\Psi_1(z) = \lambda_2^{-1} \Psi(\lambda_1 z)$; then Ψ_1 is isotopic to Ψ rel K_1 .

Ψ defines an isotopy class of homeomorphisms from R_1 to R_2 . Let $\tilde{\Psi}_0$ be the unique extremal quasiconformal map in this isotopy class. Then $\tilde{\Psi}_0$ is K -quasiconformal since $\mathbb{C} - K_i \rightarrow R_i$ is a normal covering with Abelian transformation group (refer to [Oh]). Lifting $\tilde{\Psi}_0$, we get a K -quasiconformal map Ψ_0 of \mathbb{C} such that Ψ_0 is isotopic to Ψ rel K_1 and $\Psi_0(\lambda_1 z) = \lambda_2 \Psi_0(z)$.

Let $\psi_0 = h_2^{-1} \Psi_0 h_1$ and pullback ψ_0 along f^{np} and g^{np} to the Fatou domain D_1 . We get a K -quasiconformal map ψ from D_1 to D_2 such that ψ is isotopic to $\Phi|_{D_1}$ rel $Q(f)$ and $\psi f^p = g^p \psi$.

Case 4. D_1 is parabolic associated to a parabolic periodic point z_1 . The quasiconformality of Φ implies that there exists an attracting petal $V_1 \subset D_1$ such that $\Phi(V_1)$ is contained in an attracting petal D_2 . By an attracting petal $V_1 \subset D_1$, we mean that $f^p(\bar{V}) \subset V \cup \{z_1\}$ and for any $z \in D_1$, $f^n(z) \in V$ for some $n > 0$. A similar argument as in Case 3 shows the existence of a K -quasiconformal conjugacy ψ from (f^p, D_1) to (g^p, D_2) in the isotopy class of $\Phi|_{D_1}$ rel $Q(f)$.

Now pullback ψ along f^n and g^n . We get a K -quasiconformal conjugacy ψ from $(f, F(f))$ to $(g, F(g))$ in the isotopy class of $\Phi|_{F(f)} \text{ rel } Q(f)$. Let $\psi = \Phi$ on $J(f)$. Then ψ is a K -quasiconformal map of $\hat{\mathbb{C}}$ by Rickman's Theorem (refer to [Ri] or [DH1], p. 303, Lemma 2), ψ is isotopic to $\phi \text{ rel } P(f)$ and $\psi f = g\psi$ on $\hat{\mathbb{C}}$. \square

We now give an application. Let $\phi : (\hat{\mathbb{C}}, P(f)) \rightarrow (\hat{\mathbb{C}}, P(g))$ be a quasiconformal combinatorial equivalence between rational maps f, g . Denote by $K_0[\phi]$ the infimum of the maximal dilatations over all quasiconformal conjugacies between f and g in the isotopy class of ϕ . Denote by $K_1[\phi]$ the infimum of the maximal dilatations of $\psi|_W$ over the quasiconformal maps ψ in the isotopy class of ϕ and all open sets $W \supset P(f)$. Recall that $K_1[\phi]$ is the boundary dilatation of ϕ on $\hat{\mathbb{C}} - P(f)$ if $P(f) \neq \hat{\mathbb{C}}$. Obviously, $K_1[\phi] \leq K_0[\phi]$. An interesting problem is to characterize when equality holds.

Examples. 1. $K_1[\phi] = K_0[\phi] = 0$ whenever f and g are postcritically finite and not covered by integral torus endomorphisms, since Thurston rigidity for such maps implies that they are in fact globally conformally conjugate. So equality holds in this case.

2. Integral torus endomorphisms have a one-parameter family of deformations, so in this case $K_1[\phi] = 0$ but $K_0[\phi] > 0$.

3. If f has a Herman ring or a Siegel disc containing a point of $P(f)$ with infinite forward orbit, then one can deform f by fattening an annulus compactly contained in the interior of the Siegel disc or Herman ring. Thus again one can produce g and ϕ for which $K_1[\phi] = 0$ while $K_0[\phi] > 0$.

The last two examples for which equality need not hold have the property that the rational maps contain Thurston obstruction; see [Mc1]. The following theorem shows that in the absence of Thurston obstruction we have equality.

Theorem 2. *Suppose that f is a rational map which has no Thurston obstruction. If f and g are combinatorially equivalent via a quasiconformal map ϕ , then $K_0[\phi] = K_1[\phi]$.*

Proof. Suppose that $K_0[\phi] = K_1[\phi] + 2\epsilon$ for some $\epsilon > 0$ otherwise. Let ϕ_0 be a quasiconformal map in the isotopy class of ϕ such that ϕ_0 is extremal on each component of $\hat{\mathbb{C}} - P(f)$. Then $K(\phi_0) = K_0[\phi]$ by Theorem 1. Let K_U be the maximal dilatation of the restriction of ϕ_0 to the component U of $\hat{\mathbb{C}} - P(f)$. By Strebel's Theorem, either $K_U \leq K_1[\phi] + \epsilon$, or $\phi_0|_U$ is a Teichmüller map and hence is the unique extremal map. By the definition of $K_1[\phi]$, there are at most finitely many components such that $K_U > K_1[\phi] + \epsilon$. Let V be a component such that $K_V \geq K_U$ for all the components U . Then $K_V = K(\phi_0) = K_0[\phi]$.

There exists a quasiconformal map ψ_0 in the isotopy class of ϕ such that $K(\psi_0|_W) < K_1[\phi] + \epsilon = K_0[\phi] - \epsilon$, where W is a neighborhood of $P(f)$. Let ψ_k be the isotopic lift of ψ_{k-1} with $\psi_{k-1}f = g\psi_k$. Then $K(\psi_k|_{f^{-k}(W)}) = K(\psi_0|_W)$. Let Ψ be a limit of the sequence $\{\psi_k\}$. Then $K(\Psi|_D) \leq K(\psi_0|_W) \leq K_0[\phi] - \epsilon$ for each attracting, superattracting or parabolic domain D of f since any compact set in D is contained in $f^{-k}(W)$ for sufficiently large integer k . Moreover, Ψ is isotopic to $\phi \text{ rel } Q(f)$. Thus V is not contained in attracting, superattracting and parabolic domains.

Since there is no Thurston obstruction for f , f has no Herman ring and a Siegel disk of f (if it exists) contains no $P(f)$ point except the center. By the argument

in Case 1 of the proof of Theorem 1, $\phi_0|_U$ is conformal when U is contained in a Siegel disc. So $V \cap J(f) \neq \emptyset$.

Let ϕ_k ($k \geq 1$) be the isotopic lift of ϕ_{k-1} with $\phi_{k-1}f = g\phi_k$. Then $K(\phi_k) = K(\phi_0)$ and hence $\phi_k|_V = \phi_0|_V$ since $\phi_0|_V$ is a Teichmüller map. For each component U , since $V \cap J(f) \neq \emptyset$, there exists a domain $W \subset V$ such that $f^k(W) \subset U$ for some $k > 0$. Thus $K_U \geq K(\phi_k|_W) = K_V$ since $\phi_k|_V$ is a Teichmüller map. Thus $\phi_0|_U$ is also a Teichmüller map with constant maximal dilatation and $\phi_k = \phi_0$ for all integers $k > 0$, i.e. ϕ_0 is actually a conjugacy.

If the Fatou set $F(f)$ of f is nonempty, then the restriction of ϕ_0 to each component of $F(f) - Q(f)$ is a Teichmüller map and hence is the unique extremal map. We know f has no Herman ring. If f has a Siegel disk D , then $\phi_0|_D$ is conformal. This is a contradiction. If f has an attracting, superattracting or parabolic domain D , then $K(\Psi|_D) \leq K(\phi_0) - \epsilon$. This is also a contradiction. Hence $J(f) = \hat{\mathbb{C}}$.

By Lemma 3.9 in [Mc1], f is critically finite, the signature of the orbifold of f is $(2, 2, 2, 2)$ and f is covered by an integral torus endomorphism, i.e. there is a Thurston obstruction for f . It is a contradiction. \square

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