PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 7, Pages 2011–2016
S 0002-9939(01)05945-7
Article electronically published on January 17, 2001

ELEMENTS WITH GENERALIZED BOUNDED CONJUGATION ORBITS

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(Communicated by Joseph A. Ball)

ABSTRACT. For a pair of linear bounded operators T and A on a complex Banach space X, if T commutes with A, then the orbits $\{A^nTA^{-n}\}$ of T under A are uniformly bounded. The study of the converse implication was started in the 1970s by J. A. Deddens. In this paper, we present a new approach to this type of question using two localization theorems; one is an operator version of a theorem of tauberian type given by Katznelson-Tzafriri and the second one is on power-bounded operators by Gelfand-Hille. This improves former results of Deddens-Stampfli-Williams.

Let $\mathcal{B}(X)$, $\mathcal{B}(H)$ and $\mathcal{A}$ denote respectively the algebra of all bounded linear operators on a complex Banach space X, the algebra of all bounded linear operators on the complex separable infinite dimensional Hilbert space H, and a complex Banach algebra. The symbols $Sp(S)$ and $r(S)$ denote respectively the spectrum and the spectral radius of the operator $S \in \mathcal{B}(X)$, and as usual S is called quasi-nilpotent if $Sp(S) = \{0\}$. Given an invertible operator A, the study of operators T whose conjugation orbit $\{A^nTA^{-n}\}$ is bounded, that is, for $T \in \mathcal{B}(X)$ and A invertible,

$$\sup_{n\geq 0} \|A^nTA^{-n}\| < \infty, \tag{1}$$

was initiated by J. A. Deddens in the 1970s when he gave a characterization of nest algebras in terms of the algebra B_A, where B_A is the set of operators T in $\mathcal{B}(X)$ satisfying (1). It is clear that B_A contains the commutant $\{A\}'$ of A, but in general this inclusion may be proper. J. A. Deddens and T. K. Wong showed that, if $A \in \mathcal{B}(H)$ is of the form $A = \alpha I + N$ where $0 \neq \alpha \in \mathbb{C}$ and N is nilpotent, then $B_A = \{A\}'$. Furthermore, if H is finite dimensional, then the converse holds ; see [3] and [4]. In [4], Deddens raised the question : does the converse result still hold in infinite dimensional Hilbert spaces? A negative answer to Deddens' question was provided by P. Roth [10]. The algebra B_A was further studied by J. P. Williams [13] and J. Stampfli [12].

Recently, a quantitative version of these results was given in [6]. It provides us with a bound on $\|e^STe^{-S} - T\|$ in terms of the spectral radius $r(\Delta_S)$ of the commutator $\Delta_S(T) = ST - TS$,

Received by the editors November 1, 1999.
2000 *Mathematics Subject Classification.* Primary 47B10, 47B15.
Key words and phrases. Bounded conjugation orbit, spectrum, spectral radius.
Research of the first author partially supported by grants from Kuwait University.

On the other hand, in 1941 I. Gelfand proved that if a is a doubly power-bounded element of $\mathcal{A}$, that is, $\sup_n \|a^n\| < \infty$, and the spectrum of a is reduced to the singleton $\{1\}$, then $a = 1_{\mathcal{A}}$. In 1944, N. Dunford and E. Hille tried independently to find out whether all the assumptions are needed in Gelfand's theorem. In 1950, G. E. Shilov [11] solved the conjecture and gave an example showing that the boundedness of only the positive power of a is not sufficient.

It is very surprising to see these two research areas living in a parallel manner with no interconnections. Our aim in this paper is to bring these two areas together, show that they are closely related, and use a new approach to improve former results of J.A. Deddens, J.P. Williams, and J. Stampfli (cf. Corollary 4 below).

Let a and b be in $\mathcal{A}$ and k be a positive integer. Put

$$B^k_{a,b} = \{x \in \mathcal{A} : \|a^n x b^n\| = O(n^k),\ \text{as } n \to \infty\},$$

and

$$R_{a,b} = \{x \in \mathcal{A} : \lim_{n\to\infty} \|a^n x b^n\| = 0\}.$$

For the case $b = a^{-1}$ and $k = 0$, we recover Deddens' classes $B_a = B^0_{a,a^{-1}}$ and $R_a = R_{a,a^{-1}}$.

Assume that a, b are invertible and $\sup_{n\geq 0} \|b^{-n}a^{-n}\| < \infty$. Then it is easy to see that $B_{a,b} = B^0_{a,b}$ is a sub-algebra of $\mathcal{A}$, which is not necessarily closed. In that case $R_{a,b}$ becomes a two-sided ideal which is contained in the Jacobson radical $Rad(B_{a,b})$ of $B_{a,b}$.

For each x in $B_{a,b}$, let $C(x) = \sup_{n\geq 0}(\|b^{-n}a^{-n}\|\|a^n x b^n\|)$. Then $C(.)$ defines a new norm for which $B_{a,b}$ is a Banach algebra, and $R_{a,b}$ is a closed two-sided ideal. Furthermore, $C(x)$ and $\|x\|$ are equivalent norms if and only if x is in $B_{a,b}$.

For a, b in $\mathcal{A}$, let $D_{a,b}(x) = axb$ for each $x \in \mathcal{A}$. We denote by $\{a, b\}'$ the set of fixed points of $D_{a,b}$. Caution : $\{a, b\}'$ is not the commutant of the set $\{a, b\}$. For the case $b = a^{-1}$, the class $\{a, b\}'$ reduces to the commutant $\{a\}'$. The Cesáro means of T are defined by $M_n(T) = \frac{I+T+...+T^{n-1}}{n}$. We shall be using the identity

$$(T - I)M_n(T) = \frac{T^n - I}{n}. \tag{2}$$

The condition x in $B_{a,b}$ is equivalent to $D_{a,b}$ being locally power-bounded in $\mathcal{B}(\mathcal{A})$, that is, $\sup_{n\geq 0} \|D^n_{a,b}(x)\| < \infty$.

Theorem 1. *Assume that $Sp(a) = Sp(b) = \{1\}$. Then $B_{a,b} \cap B^k_{a^{-1},b^{-1}} = \{a, b\}'$, for all k.*

Proof. By [8, Theorem 10], we obtain $Sp(D_{a,b}) = Sp(a)Sp(b) = \{1\}$. Assume that x is in $B_{a,b} \cap B^k_{a^{-1},b^{-1}}$. Then $\|D^n_{a,b}(x)\| = O(1),$ as $n \to \infty$ and $\|D^n_{a,b}(x)\| = O(n^k),$ as $n \to -\infty$. By the local Gelfand-Hille theorem [1, Theorem 3.4], we obtain $(D_{a,b} - I)^{k+1}x = 0$.

Suppose that $(D_{a,b} - I)^p x = 0$, for some $p \geq 2$, and let $y = (D_{a,b} - I)^{p-1}x$. Then $D_{a,b}(y) = y$. This implies that $M_n(D_{a,b})y = y$. On the other hand, from (2) applied to the operator $T = D_{a,b}$, we obtain

$$\begin{aligned} M_n(D_{a,b})y &= (D_{a,b} - I)^{p-2} M_n(D_{a,b})(D_{a,b} - I)x \\ &= (D_{a,b} - I)^{p-2}\,(\frac{D^n_{a,b} - I}{n})x \to 0,\ \text{as } n \to \infty. \end{aligned}$$

Hence, $(D_{a,b} - I)^{p-1}x = y = 0$. By induction, we obtain that $(D_{a,b} - I)x = 0$, which completes the proof.

Theorem 2. *Fix x in $\mathcal{A}$, and let a, b be in $\mathcal{A}$. Assume that*
(i) $Sp(a) = Sp(b) = \{1\}$;
(ii) $\|\frac{1}{n}\sum_{k=0}^{n-1} a^k x b^k\| = O(1),$ as $|n| \to \infty$.
Then $x \in \{a, b\}'$.

Proof. By [8, Theorem 10], we have $Sp(D_{a,b}) = \{1\}$. By (2) and the local Gelfand-Hille theorem [1, Theorem 3.4], condition (ii) implies $(D_{a,b} - I)^2 x = 0$. By induction, we obtain $D^n_{a,b}(x) = nD_{a,b}(x) - (n-1)x$. Hence

$$M_n(D_{a,b})x = \frac{1}{n}\sum_{k=0}^{n-1} D^k_{a,b}(x) = \frac{n-1}{2}(D_{a,b}(x) - x) + x. \tag{3}$$

By (ii) and (3), we can conclude that $D_{a,b}(x) - x = 0$.

Corollary 3. *If $Sp(a) = Sp(b) = \{1\}$, then $B_{a,b} \cap B_{b,a} = \{a, b\}'$.*

For the particular case $b = a^{-1}$, we obtain, as a consequence of Corollary 3, the following result of J.P. Williams [13, Theorem 2].

Corollary 4. *If $Sp(a) = \{1\}$, then $B_{a,a^{-1}} \cap B_{a^{-1},a} = \{a\}'$.*

Remark 1. Since $\{a, b\}' \cap R_{a,b} = \{0\}$, a natural question that can be raised is whether $B_{a,b} = \{a, b\}' + R_{a,b}$. In general, this is not true. To see this, take $\mathcal{A} = \mathcal{B}(H)$, a as an unitary operator with $a \neq \alpha I$, and $b = a^{-1}$. Then $B_{a,b} = \mathcal{B}(H)$, $R_{a,b} = \{0\}$, and $\{a\}' \neq \mathcal{B}(H)$.

For a and b elements of a Banach algebra $\mathcal{A}$, let $\pi_R : B_{a,b} \to B_{a,b}/R_{a,b}$ be the canonical surjective mapping. We have the following interesting commutativity theorem.

Theorem 5. *Let a, b be in $\mathcal{A}$, such that $Sp(a) = Sp(b) = \{1\}$. Then $\pi_R(B_{a,b}) = \{\pi_R(a), \pi_R(b)\}'$.*

Proof. Let $x \in B_{a,b}$. Using the local version of the Katznelson-Tzafriri theorem (see [2, Theorem 1] or [5, Corollary 3.8]), we obtain $\|D^n_{a,b}(x) - D^{n+1}_{a,b}(x)\| \to 0,$ as $n \to \infty$. Hence

$$\|a^n x b^n - a^{n+1} x b^{n+1}\| = \|a^n(x - axb)b^n\| \to 0, \text{ as } n \to \infty.$$

This implies that $x - axb \in R_{a,b}$. Since $R_{a,b}$ is an ideal, the conclusion follows.

Following the same steps as in the proof of Theorem 5, we obtain

Corollary 6. *Let a be in $\mathcal{A}$. Then*
(i) $Sp(a) = \{1\} \Longrightarrow \pi_R(B_a) = \{\pi_R(a)\}'$;
(ii) $Sp(a) = \{1\}$ and $R_a = \{0\} \Longrightarrow B_a = \{a\}'$.

Remark 2. **(i)** In the proofs of Theorem 5 and Corollary 6, it is sufficient to use the following weaker version of Katznelson-Tzafriri given by J. Esterle.

Let a be an element of norm 1. If $Sp(a) = \{1\}$, then $\|a^n - a^{n+1}\| \to 0$, as $n \to \infty$. In fact, since $D_{a,b}(B_{a,b}) \subset B_{a,b}$, we obtain a contraction which we will denote also by $D_{a,b}$ without losing any generality. Since $Sp(D_{a,b}) = \{1\}$, by applying Esterle's theorem, we obtain for any x in $B_{a,b}$

$$\|D^n_{a,b}(x) - D^{n+1}_{a,b}(x)\| \to 0, \text{ as } n \to \infty.$$

The rest of the the proof is as in the proof of Theorem 5 and in Corollary 6 when $b = a^{-1}$.

(ii) From the result obtained in Corollary 6 (ii), a natural and interesting question would be whether the reverse implication of (ii) holds, that is, $B_A = \{A\}'$ implies $Sp(A) = \{1\}$. Unfortunately, the answer is negative (see [10, Example 2.12, p. 506] for an elegant counterexample).

Let us introduce the class $B_{a,b}^{cv} = \{x \in \mathcal{A} : (a^n x b^n)_{n\geq 0} \text{ converges}\}$. It is easy to see that $\{a,b\}' + R_{a,b} \subset B_{a,b}^{cv} \subset B_{a,b}$.

Theorem 7. *If $Sp(a) = Sp(b) = \{1\}$, then $\{a,b\}' \oplus R_{a,b} = B_{a,b}^{cv}$.*

Proof. It is sufficient to prove the inclusion $B_{a,b}^{cv} \subset \{a,b\}' + R_{a,b}$. Let x be in $B_{a,b}^{cv}$. Using the local version of the Katznelson-Tzafriri theorem (see [2, Theorem 1] or [5, Corollary 3.8]), we obtain that $x - a^n x b^n \in R_{a,b}$ for every $n \geq 0$. On the other hand, since x is in $B_{a,b}^{cv}$, the sequence $y_n = a^n x b^n$ converges to y and y is in $\{a,b\}'$. Put $z = x - y$. We claim that $z \in R_{a,b}$. In fact,

$$\|a^n z b^n\| = \|a^n (x-y) b^n\| = \|a^n x b^n - y\| \to 0.$$

Hence, $x = y + z \in \{a,b\}' + R_a$.

Corollary 8. *Let $a \in \mathcal{A}$ and suppose that $Sp(a) = \{1\}$. Then the following are equivalent.*

(i) $R_a = \{0\}$,
(ii) $B_a = \{a\}'$,
(iii) $B_a^{cv} = \{a\}'$.

Example. Using the example given by Roth [10], we will show that there exists an operator A on a Banach space X for which $Sp(A) = \{1\}$, and $B_A \neq \{A\}' \oplus R_A$. In fact, take V as the Volterra integral operator defined by $(Vf)(t) = \int_0^t f(s)ds$, for f in $L_2[0,1]$. It is well known that $Sp(V) = \{0\}$ (cf. [7, Problem 146]). Thus, the operator $(I+V)^{-1}$ has its spectrum reduced to $\{1\}$. Moreover, it is easy to see that $\|(I+V)^{-n}\| = 1$ for $n \geq 0$ (cf. [7, Problem 150]). Define A and T on $L_2[0,1] \oplus L_2[0,1]$ by $A = \begin{pmatrix} I+V & 0 \\ 0 & I \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$. Then for all integer n, we obtain $A^n T A^{-n} = \begin{pmatrix} 0 & 0 \\ (I+V)^{-n} & 0 \end{pmatrix}$. Thus T is in B_A, but $T \notin \{A\}' + R_A$, because for the opposite case, by Theorem 7, T will be in B_V^{cv}. This implies that $(I+V)^{-n}$ converges to a projection P. Since $Sp((I+V)^{-1}) = \{1\}$, we obtain $Sp(P) = \{1\}$. Hence, $P = I$. But P is a projection on $Ker(I-(I+V)^{-1})$. Therefore, $(I+V)^{-1} = I$, which is a contradiction.

Remark 3. **(i)** In general, statements (i) and (ii) of Corollary 8 are not equivalent. To see this take the unitary operator A, with $A \neq \lambda I$. Then $B_A = \mathcal{B}(H)$, and $R_A = Rad(B_A) = Rad(\mathcal{B}(H)) = \{0\}$. But $\{A\}' \neq \mathcal{B}(H)$.

(ii) Theorem 1 cannot be extended to the case where the spectrum is countable instead of $Sp(a) = \{1\}$. To see this, let $(e_n)_{n\geq 1}$ be a Hilbert basis in H, and consider the operator A defined by $Ae_1 = -e_1$, and $Ae_n = e_n$, for every $n \geq 2$. Then $A = A^* = A^{-1}$, and $Sp(A) = \{-1, 1\}$. Since A is unitary, we have $B_A = B_{A^{-1}} = \mathcal{B}(H)$, and therefore $\{A\}' \neq B_A \cap B_{A^{-1}}$.

Finally, we present, in the situation of Banach spaces, a version of the result of J. Stampfli [12, Theorem 1] and J.P. Williams [13, Theorem 1]. For the proof we need the following lemma which is probably well known, but we could not find any reference for it.

Lemma 9. *Let $A \in \mathcal{B}(X)$ be an invertible. Then the following statements are equivalent.*

(1) There exists an $M > 0$, such that $\|A^n\| \leq M$, for every n in $\mathbf{Z}$.
(2) There exists an $M > 0$, such that, for every $n \in \mathbb{N}$ and for every $x \in X$,

$$\frac{1}{M}\|x\| \leq \|A^n x\| \leq M\|x\|.$$

(3) There exists an equivalent norm on X for which A is an isometry.
(4) A is similar to an invertible isometry.

Proof. Implications (1) $\Rightarrow$ (2) and (4) $\Longrightarrow$ (1) are trivial. For (2) $\Rightarrow$ (3), it is sufficient to take a new equivalent norm $|||x||| = \sup_{n\geq 0} \|A^n x\|$. For (3) $\Rightarrow$ (4), we consider the commutative diagram

$$\begin{tikzcd}
(X, \|\cdot\|) \arrow[r, "A"] \arrow[d, "S^{-1}=Id"'] & (X, \|\cdot\|) \\
(X, |||\cdot|||) \arrow[r, "A_1"] & (X, |||\cdot|||) \arrow[u, "S=Id"']
\end{tikzcd}$$

where $(X, |||.|||)$ is the Banach space X with the new norm. Then $A = SA_1S^{-1}$.

Proposition 10. *Let $A \in \mathcal{B}(X)$ be invertible. Then the following statements are equivalent.*

(1) $B_A = \mathcal{B}(X)$.
(2) $K(X) \subset B_A$, where $K(X)$ is the ideal of compact operators.
(3) A is similar to a non-zero scalar multiple of an invertible isometry.

Proof. Implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are trivial. For (2) $\Rightarrow$ (3), we use ideas similar to J. P. Williams [13, Theorem 1] and Lemma 9 to conclude the proof.

We finish this paper with some lifting problems in the spirit of Stampfli's paper.

Problem 1. Let $a \in \mathcal{C}(H)$ with $a \in \Omega_I$ (Ω_I is the connected component of I in the group of invertible elements of the Calkin algebra $\mathcal{C}(H)$). Does there exist an invertible operator $A \in \mathcal{B}(H)$, satisfying $a = \pi(A)$, such that $\pi(B_A) = B_a$, where $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/K(H)$ is the canonical surjective mapping?

Problem 1′. Assume that A is invertible in $\mathcal{B}(H)$; does $\pi(B_A) = B_{\pi(A)}$?

Problem 2. Assume that A is Fredholm. Let $B_A^e = \{T \in \mathcal{B}(H) : \pi(T) \in B_{\pi(A)}\}$. Then

(i)$K(H) \subset B_A^e$ and
(ii) if A is invertible, then $B_A + K(H) \subset B_A^e$.

Does $B_A + K(H) = B_A^e$?

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