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EASY PROOFS OF RIEMANN'S FUNCTIONAL EQUATION FOR $\zeta(s)$ AND OF LIPSCHITZ SUMMATION

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Abstract. We present a new, simple proof, based upon Poisson summation, of the Lipschitz summation formula. A conceptually easy corollary is the functional relation for the Hurwitz zeta function. As a direct consequence we obtain a short, motivated proof of Riemann's functional equation for $\zeta(s)$.

Introduction

We present a short and motivated proof of Riemann's functional equation for Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, initially defined in the half plane $\mathrm{Re}(s) > 1$. In fact we prove the slightly more general functional relation for the Hurwitz zeta function

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

a close cousin of Riemann's zeta function $\zeta(s)$. Here $0 < a \leq 1$ and $\mathrm{Re}(s) > 1$. Note that $\zeta(s,1) = \zeta(s)$. In this paper we give new, detailed proofs for clarity of exposition and for guidance to the reader who is unfamiliar with the circle of ideas related to the Riemann zeta function $\zeta(s)$.

Bernhard Riemann himself provided two proofs of his classical functional equation, which reads

$$\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s} 2 \cos\left(\frac{\pi s}{2}\right) \zeta(s). \tag{1}$$

His first proof uses the theta function and its Mellin transform. Riemann's second proof uses contour integration. Our proof uses neither technique. Rather, we employ (and prove) the Lipschitz summation formula, a tool which is also useful in the study of Eisenstein and Poincaré series in number theory. There is one more important player in this story, namely the 'periodized zeta function'

$$F(s,a) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s},$$

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also defined initially for $\mathrm{Re}(s) > 1$, with a an arbitrary real number. We first show that Lipschitz summation follows very easily from the Poisson summation formula. The main point of the paper is that Hurwitz's relation

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\left\{e^{-\frac{\pi i s}{2}}F(s,a) + e^{\frac{\pi i s}{2}}F(s,-a)\right\} \tag{2}$$

follows as a conceptually easy corollary of Lipschitz summation. Notice that when $a = 1$ in Hurwitz's relation, $F(s,1) = F(s,-1) = \zeta(s)$, so that Riemann's functional equation (1) follows directly from Hurwitz's relation (2) by using $e^{-\frac{\pi i s}{2}} + e^{\frac{\pi i s}{2}} = 2\,\cos(\frac{\pi s}{2})$. As an added bonus, the meromorphic continuation of $F(s,a)$ and of $\zeta(s,a)$ into the whole complex plane also follows from our proofs (see the Corollaries to Theorem 2).

For the sake of completeness, we first define the Fourier transform $\hat{f}$ of a function f:

$$\hat{f}(m) = \int_{-\infty}^{\infty} f(x)e^{2\pi i x m}dx.$$

If f is 'sufficiently nice', then its Fourier transform $\hat{f}$ is well defined, and is known among engineers to describe the 'frequencies of f'. Poisson summation says that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m).$$

Here we can take the words 'sufficiently nice' to mean that f is continuous and decays faster than any polynomial at infinity. The Poisson summation formula is a very useful tool in Fourier analysis, number theory, and other areas of mathematics. It is very easy to prove Poisson summation for well-behaved functions. This formula provides a highly practical window into the frequency domain, and offers a powerful symmetry between a function and its Fourier transform.

§1

We now state and prove the Lipschitz summation formula, discovered by Lipschitz in 1889 ([3]). Some proofs of this result in the literature are unnecessarily long. Here we present a short proof, which uses only Poisson summation in a simple way. Rademacher's proof in [4, p. 77] is similar in spirit to our proof below.

Theorem 1.

$$\sum_{n=1}^{\infty}(n-\alpha)^{s-1}e^{2\pi i\tau(n-\alpha)} = \frac{\Gamma(s)}{(-2\pi i)^s}\sum_{m\in\mathbb{Z}}\frac{e^{2\pi i\alpha m}}{(\tau+m)^s}, \tag{3}$$

where $\mathrm{Re}(s) > 1,\ \tau \in H$*, the complex upper half plane, and* $0 \le \alpha < 1$*.*

Proof. First we extend the sum on the left-hand side of (1) to the full group of integers by multiplying by the characteristic function of the interval $[\alpha, \infty)$. Equivalently, define

$$f(x) = \begin{cases} (x-\alpha)^{s-1}e^{2\pi i\tau(x-\alpha)} & \text{for } x > \alpha, \\ 0 & \text{for } x \le \alpha. \end{cases}$$

Because $0 \le \alpha < 1$, it is evident that extending the summand by zero in this way does not affect the discrete sum $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty}(n-\alpha)^{s-1}e^{2\pi i\tau(n-\alpha)}$. Furthermore, note that the appearance of the factor $(x-\alpha)^{s-1}$ makes $f(x)$ continuous at

$x = \alpha$ in the range $\mathrm{Re}(s) > 1$. This is in fact the reason we must first restrict s to lie in this range. We now apply the Poisson summation formula to the left-hand side of (1):

$$\begin{aligned}
\sum_{n=1}^{\infty}(n-\alpha)^{s-1}e^{2\pi i\tau(n-\alpha)} &= \sum_{n\in\mathbb{Z}} f(n) \\
&= \sum_{m\in\mathbb{Z}} \hat{f}(m) \\
&= \sum_{m\in\mathbb{Z}} \int_{\alpha}^{\infty} e^{2\pi imx}(x-\alpha)^{s-1}e^{2\pi i\tau(x-\alpha)}dx \\
&= \sum_{m\in\mathbb{Z}} \int_{0}^{\infty} e^{2\pi im(x+\alpha)}x^{s-1}e^{2\pi i\tau x}dx \\
&= \sum_{m\in\mathbb{Z}} e^{2\pi im\alpha} \int_{0}^{\infty} x^{s-1}e^{2\pi i(\tau+m)x}dx \\
&= \frac{1}{(-2\pi i)^s} \sum_{m\in\mathbb{Z}} \frac{e^{2\pi im\alpha}}{(\tau+m)^s} \int_{0}^{\infty} y^{s-1}e^{-y}dy \\
&= \frac{\Gamma(s)}{(-2\pi i)^s} \sum_{m\in\mathbb{Z}} \frac{e^{2\pi im\alpha}}{(\tau+m)^s},
\end{aligned}$$

and we are done.

To justify the steps, note that we can employ Poisson summation because $f \in L^1(\mathbb{R})$ and furthermore the proof shows that $\hat{f} \in L^1(\mathbb{R})$ when $\mathrm{Re}(s) > 1$ (the integral converges). We used the change of variable $y = -2\pi i(\tau + m)x$ in the penultimate equality. The fact that $\tau \in H$ implies $\mathrm{Re}(y) > 0$. Furthermore, the integral along the real line equals the integral along the complex ray $\{(x, y)|y = -2\pi i(m + \tau)x, \text{and } x \geq 0\}$.

We expand on this point for the reader who is not familiar with this trick from complex analysis. Consider a piece of 'pie' formed by the first ray $\{x|x > 0\}$ going away from the origin, then around a piece of a circle of radius R, and then back toward the origin along the second ray $\{(x, y)|y = -2\pi i(m + \tau)x, \text{and } x \geq 0\}$. When we let $R \to \infty$, we observe that the contribution at ∞ from the integral is zero because the integrand decays exponentially at infinity. Since the integrand is an entire function of z, Cauchy's theorem implies that the integral along the first ray minus the integral along the second ray is zero. □

Note that if we rewrite the Lipschitz summation formula in the following way, a suggestive symmetry between s and $1 - s$ becomes apparent:

$$e^{-2\pi i\tau\alpha} \sum_{n=1}^{\infty} \frac{e^{2\pi i\tau n}}{(n-\alpha)^{1-s}} = \frac{\Gamma(s)}{(-2\pi i)^s} \sum_{m\in\mathbb{Z}} \frac{e^{2\pi i\alpha m}}{(\tau+m)^s}.$$

§2

We prove the Hurwitz functional relation in a straightforward way from Lipschitz summation, relegating to a Lemma the technical details of taking a limit inside an infinite sum. We adopt the standard convention that a sum over the integers which omits 0 is written as $\sum^*$.

Lemma. (a) *Suppose* $0 \le a < 1$ *and* $\mathrm{Re}(s) < 0$. *Then*

$$\lim_{\tau \to 0} \sum_{n=1}^{\infty} (n-a)^{s-1} e^{2\pi i n\tau} = \sum_{n=1}^{\infty} (n-a)^{s-1} = \zeta(1-s, 1-a).$$

(b) *Let* $0 \le a < 1$ *and* $y > 0$. *Put*

$$S_y(s) = \sum\nolimits_{-\infty}^{*\ \infty} e^{2\pi i a m} \left\{ (m+iy)^{-s} - m^{-s} + siym^{-s-1} \right\},$$

with the argument convention that $-\pi < arg(w) \le \pi$ *for* $w \ne 0$. *(Thus* $0 < arg(m+iy) < \pi$ *and* $arg(m) = 0$ *or* π*, according as* $m > 0$ *or* $m < 0$.*)*

Then:

(i) $S_y(s)$ *converges absolutely for* $\mathrm{Re}(s) > -1$*;*

(ii) $S_y(s)$ *is holomorphic in* s *for* $\mathrm{Re}(s) > -1$*;*

(iii) *for fixed* s *with* $\mathrm{Re}(s) > -1$,

$$\lim_{y \to 0^+} S_y(s) = 0.$$

Remark. Note that both (a) and (b) hold in the strip $-1 < \mathrm{Re}(s) < 0$.

Proof. (a) The proof is completely straightforward since the right-hand side converges absolutely. It suffices to prove uniform convergence of the left-hand side for $y = \mathrm{Im}(\tau) \ge 0$. However,

$$\sum_{n=1}^{\infty} |(n-a)^{s-1} e^{2\pi i n\tau}| = \sum_{n=1}^{\infty} (n-a)^{\sigma-1} e^{-2\pi n y} \le \sum_{n=1}^{\infty} (n-a)^{\sigma-1} < \infty,$$

for $y \ge 0, \sigma < 0$. By the Weierstrass M-test the convergence is uniform, and the result follows.

(b) The proof here is slightly more elaborate since absolute convergence of $S_y(s)$ for $\mathrm{Re}(s) > 1$ requires verification. For $m \ne 0$, we have $(m+iy)^{-s} = m^{-s}(1+\frac{iy}{m})^{-s}$, valid by our argument convention. Thus, by the binomial expansion we have

$$(m+iy)^{-s} = m^{-s} \left\{ 1 - s\frac{iy}{m} + \sum_{j=2}^{\infty} \binom{-s}{j} \left(\frac{iy}{m}\right)^j \right\},$$

for $|m| > y$. Writing $s = \sigma + it$, we have

$$\left|(m+iy)^{-s} - m^{-s} + s(iy)m^{-s-1}\right| \le e^{\pi|t|}|m|^{-2-\sigma} \sum_{j=2}^{\infty} \left|\binom{-s}{j}\right| \frac{y^j}{|m|^{j-2}}. \tag{4}$$

It is elementary that $\left|\binom{-s}{j}\right| \le |s|^j + 1$, so the right-hand side of (4) is bounded by

$$e^{\pi|t|}|m|^{-2-\sigma} \left\{ |s|^2 y^2 \sum_{j=0}^{\infty} \left(\frac{|s|y^j}{|m|}^j\right) + y^2 \sum_{j=0}^{\infty} \left(\frac{y}{|m|}\right)^j \right\}.$$

Assuming $|m| > 2\max(y, |s|y)$, we can sum the geometric series to obtain

$$\left|(m+iy)^{-s} - m^{-s} + iysm^{-s-1}\right| \le 2e^{\pi|t|}|m|^{-2-\sigma} y^2(|s|^2+1).$$

It follows that

$$\sum_{-\infty}^{\infty}{}^{*} \left| e^{2\pi iam} \left\{ (m+iy)^{-s} - m^{-s} + iysm^{-s-1} \right\} \right| \le K(y,s)\zeta(2+\sigma) < \infty,$$

where $K(y,s) > 0$, because $\sigma > -1$. This proves (i).

(ii) For this it suffices to prove that the convergence is uniform on compact subsets of $\{s|\sigma = \mathrm{Re}(s) > -1\}$. Suppose C is such a compact set. Then there exist positive constants K_c, ϵ_c such that for all $s \in C, |s| \le K_c$ and $\sigma \ge -1+\epsilon_c$. For s in C and $|m| > 2\max(y, K_c y)$, we find that the left-hand side of (4) is bounded by $2e^{\pi K_c}|m|^{-1-\epsilon_c}y^2(K_c^2+1)$. Since $\zeta(1+\epsilon_c) < \infty$, the Weierstrass M-test implies uniform convergence on C.

(iii) For y sufficiently small, $2\max(y, |s|y) < 1$, so

$$|S_y(s)| \le 2e^{\pi|t|}y^2(|s|^2+1)\zeta(2+\sigma).$$

Now let $y \to 0^+$ to get the result. □

§3

We next prove the Hurwitz relation (2). This proof gives us some additional information about the analytic continuation of the periodized zeta function defined above by $F(s,a) = \sum_{n=1}^{\infty} \frac{e^{2\pi ina}}{n^s}$ for $\mathrm{Re}(s) > 1$.

Theorem 2.

$$e^{-\frac{\pi is}{2}}F(s,a) + e^{\frac{\pi is}{2}}F(s,-a)$$

can be continued analytically into $\mathrm{Re}(s) > -1$. *For s in the vertical strip* $-1 < \mathrm{Re}(s) < 0$, *and* $0 < a \le 1$, *we have the Hurwitz relation*

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\frac{\pi is}{2}}F(s,a) + e^{\frac{\pi is}{2}}F(s,-a)\}.$$

Proof. Beginning with the Lipschitz summation formula (3), we subtract the first two terms of the binomial expansion on the right-hand side:

$$(\tau+m)^{-s} = m^{-s} - \tau sm^{-s-1} + \cdots .$$

(This approach is inspired by the elegant paper of Harold Stark [6].) We obtain

$$\begin{aligned}&\sum_{m\in\mathbb{Z}}{}^{*} e^{2\pi i\alpha m}\left[(\tau+m)^{-s} - m^{-s} + sm^{-s-1}\tau\right] + \frac{1}{\tau^s}\\ &= \frac{(-2\pi i)^s}{\Gamma(s)}\sum_{n=1}^{\infty}(n-\alpha)^{s-1}e^{2\pi i\tau(n-\alpha)} - \sum_{m\in\mathbb{Z}}{}^{*}\frac{e^{2\pi i\alpha m}}{m^s} + \tau s\sum_{m\in\mathbb{Z}}{}^{*}\frac{e^{2\pi i\alpha m}}{m^{s+1}},\end{aligned}$$

for any $0 \le \alpha < 1$ and $\mathrm{Re}(s) > 1$. Note that

$$\sum_{m\in\mathbb{Z}}{}^{*}\frac{e^{2\pi i\alpha m}}{m^s} = F(s,\alpha) + e^{-\pi is}F(s,-\alpha)$$

and put $\tau = iy, y > 0$ to rewrite the above as

$$(5)\qquad \sum_{m\in\mathbb{Z}}{}^{*} e^{2\pi i\alpha m}\{(iy+m)^{-s} - m^{-s} + iysm^{-s-1}\} + \frac{1}{(iy)^s}$$
$$= \frac{(-2\pi i)^s}{\Gamma(s)}\sum_{n=1}^{\infty}(n-\alpha)^{s-1}e^{-2\pi y(n-\alpha)} - \{F(s,\alpha) + e^{-\pi i s}F(s,-\alpha)\}$$
$$+iys\{F(s+1,\alpha) - e^{-\pi i s}F(s+1,-\alpha)\}.$$

Clearly the exponential sum on the right-hand side of (5) is entire in s. The Lemma (part (b), ii) implies that the left-hand side of (5) is meromorphic in $\mathrm{Re}(s) > -1$, with at most a simple pole at $s = 0$. Therefore the remainder

$$-\frac{\Gamma(s)}{(-2\pi i)^s}\{F(s,\alpha) + e^{-\pi i s}F(s,-\alpha)\} + \frac{iys\Gamma(s)}{-2\pi i)^s}\{F(s+1,\alpha) - e^{-\pi i s}F(s+1,-\alpha)\}$$

is meromorphic in $\mathrm{Re}(s) > -1$, with at most a simple pole at $s = 0$. Since this is true for any $y > 0$, it follows easily that

$$-\frac{\Gamma(s)}{(-2\pi i)^s}\{F(s,\alpha) + e^{-\pi i s}F(s,-\alpha)\}$$

is meromorphic in $\mathrm{Re}(s) > -1$, with at most a simple pole of order 1 at $s = 0$.

Now restrict s to the vertical strip $-1 < \mathrm{Re}(s) < 0$, let $y \to 0+$, and apply the Lemma to (5). For $0 \le \alpha < 1$ this yields

$$(6)\qquad \zeta(1-s,1-\alpha) - \frac{\Gamma(s)}{(-2\pi i)^s}\{F(s,\alpha) + e^{-\pi i s}F(s,-\alpha)\} = 0,$$

valid in the strip $-1 < \mathrm{Re}(s) < 0$. Putting $a = 1 - \alpha$ in (6) gives the the Hurwitz formula (2) for $-1 <$ $\mathrm{Re}(s) < 0$ and $0 < a \le 1$, because $F(s, 1-a) = F(s,-a)$ and $F(s,-1+a) = F(s,a)$. This completes the proof of Theorem 2. □

Corollary 1. *For* $0 < a \le 1$,

$$F(0,-a) + F(0,a) = -1.$$

Proof. Simply fix $y > 0$ and let $s \to 0^-$ in the relation (5) above. Because $\Gamma(s)$ has a pole at $s = 0$, the relation becomes $0 + 1 = 0 - F(0,\alpha) - F(0,-\alpha)$, for $0 \le \alpha < 1$. On the left-hand side we have taken the limit as $s \to 0^-$ inside the infinite sum by virtue of Lemma(b, ii). The result follows if we put $\alpha = 1 - a$ with $0 < a \le 1$.

We have taken for granted that $\lim_{s\to 0^-} F(s,a)$ is finite in the above remarks. To see this note that $\lim_{s\to 0^-}\{F(s,a) + e^{-\pi i s}F(s,-a)\} = -1$, and that the left-hand side of the last limit is independent of a. Thus it follows that

$$\lim_{s\to 0^-}\{F(s,-a) + e^{-\pi i s}F(s,a)\} = -1,$$

and multiplying the former equality by $e^{\pi i s}$ and subtracting from the latter equality gives

$$\lim_{s\to 0^-}\left(e^{\pi i s} - e^{-\pi i s}\right)F(s,a) = 0.$$

Since $e^{\pi i s} - e^{-\pi i s}$ has a *simple* zero at $s = 0$, $F(s,a)$ cannot have a pole at $s = 0$. Thus $\lim_{s\to 0^-} F(s,a)$ is finite and the proof is complete. □

Corollary 2. (a) *For $0 < a \leq 1$, the Hurwitz zeta function $\zeta(s,a)$ is holomorphic in $\mathbb{C}$ except for a simple pole at $s = 1$, with residue 1.*

(b) *The relation (2) holds in all of $\mathbb{C}$ and has the alternative form*

$$F(s,a) = \frac{(2\pi)^s e^{-\frac{\pi i s}{2}}}{2i\Gamma(s)\sin(\pi s)}\{\zeta(1-s,a) - e^{\pi i s}\zeta(1-s,1-a)\}.$$

(c) *For $a \notin \mathbb{Z}$, the periodized zeta function $F(s,a)$ is an entire function of s.*

Proof. (a) We begin with relation (2) in the strip $-1 < \mathrm{Re}(s) < 0$. The left-hand side of (2) is clearly holomorphic for $\mathrm{Re}(s) < 0$, while Theorem 2 shows that the right-hand side of (2) is holomorphic in $\mathrm{Re}(s) > -1$ except for a possible simple pole at $s = 0$ arising from $\Gamma(s)$.

Identity (2) in the strip $-1 < \mathrm{Re}(s) < 0$ provides a meromorphic continuation of both $\zeta(1-s,a)$ and the right-hand side of (2) into all of $\mathbb{C}$, with (2) of course continuing to hold in the entire plane. The form of (2) shows that $\zeta(1-s,a)$ is holomorphic in $\mathbb{C} - \{0\}$.

By Corollary 1 the right-hand side of (2), without the Gamma factor, is -1 at $s = 0$. On the other hand, $\Gamma(s)$ has a simple pole with residue -1 at $s = 0$. Thus $\zeta(s,a)$ has a simple pole with residue 1 at $s = 1$.

To prove (b), compare (2) with its alternative form

$$\zeta(1-s,1-a) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\frac{\pi i s}{2}}F(s,-a) + e^{\frac{\pi i s}{2}}F(s,a)\}$$

to obtain

$$F(s,a) = \frac{(2\pi)^s e^{-\frac{\pi i s}{2}}}{2i\Gamma(s)\sin(\pi s)}\{\zeta(1-s,a) - e^{\pi i s}\zeta(1-s,1-a)\}.$$

(c) The above relation shows that $F(s,a)$ is holomorphic in $\mathbb{C} - \{0,1\}$ and has at most simple poles at $s = 0$ and 1. To see that these two potential poles do not actually occur for $a \notin \mathbb{Z}$, first note that we have $F(1,a) = \sum_{n=1}^{\infty} \frac{e^{2\pi i a n}}{n}$, a convergent Fourier series. (Alternatively, one can prove the convergence of $\sum_{n=1}^{\infty} \frac{e^{2\pi i a n}}{n}$ by a simple application of partial summation.) Therefore $F(1,a)$ is finite and there is no pole there. Finally, existence of a pole for $F(s,a)$ at $s = 0$ is impossible, for we showed in the proof of Corollary 1 that $F(0,a)$ is finite. Hence $F(s,a)$ is an entire function of s. □

Remark 1. Riemann's proof of the functional equation for $\zeta(s)$ using the theta function $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2}$ is conceptually more difficult than this proof (and requires taking the Mellin transform to boot). For, here we apply the Lipschitz summation formula with its *linear* exponential damping factor $e^{\pi i \tau n}$, as opposed to the *quadratic* exponential damping factor $e^{\pi i \tau n^2}$ occuring in $\theta(\tau)$.

Remark 2. One motivation for the proof of Theorem 2 is Harold Stark's paper [6]. Using the binomial expansion and subtracting the first few terms, he shows that the special values of Dirichlet L-functions and the Hurwitz zeta function are easily obtained at the nonpositive integers. The class number formula of Dirichlet is also recovered quickly in this highly recommended paper.

Remark 3. The functional equation for the Dirichlet L-functions also follows quite easily from the Hurwitz formula (2), as Apostol shows in [1]. Thus our approach simplifies the theory for the Dirichlet L-functions as well as the Hurwitz zeta function.

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