A SHORT PROOF THAT HYPERSPACES OF PEANO CONTINUA ARE ABSOLUTE RETRACTS

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Abstract. We give a short proof of Wojdyslawski’s famous theorem.

Theorem (Wojdyslawski [6]). Let X be a Peano continuum. Then the hyperspace 2^X of all nonempty compact subsets of X is an absolute retract for metric spaces.

This result is an essential step in the proof of the Curtis-Schori-West Hyperspace Theorem to the effect that 2^X is a Hilbert cube for any Peano continuum X (see, e.g., the book of van Mill [5, §8.4]). Wojdyslawski’s original proof is rather complicated [6]. A simpler proof was suggested later on by Kelley [4], which is, however, based on a difficult Lefschetz-Dugundji characterization of metric ANR’s (see [5, Theorem 5.2.1]). Yet another proof, also based on the Lefschetz-Dugundji characterization, can be found in [5, §5.3]. Our proof is elementary and it does not rely on the Lefschetz-Dugundji criterion.

Proof. Let d be any compatible metric on X and let d_H be the Hausdorff metric on 2^X. Assume that (Y, ρ) is a metric space, A is a closed subset of Y and f : A → 2^X is a continuous map. Following [3], choose a canonical cover ω of Y \ A in Y, that is to say: (1) ω is an open cover of Y \ A, locally finite in Y \ A; (2) for each neighborhood V of a point a ∈ A in Y there exists a neighborhood S of a in Y contained in V, such that every element U ∈ ω which meets S is contained in V. We note that the second condition implies that every neighborhood of any boundary point of A in Y contains infinitely many open sets in ω (see [2, Ch. III, §1]).

Let N(ω) denote the nerve of ω endowed with the CW topology. We will denote by p_U the vertex of N(ω) corresponding to U ∈ ω. Then according to [X], there exist a Hausdorff space Z and a continuous map μ : Y → Z with the following properties:

(a) Z as a set coincides with the disjoint union A ∪ N(ω);
(b) A is closed in Z and the restriction μ|A is the identical homeomorphism;
(c) Z \ A=N(ω) is taken with its CW topology and μ(Y \ A) ⊆ Z \ μ(A);
(d) a base of neighborhoods of a ∈ A in Z is determined by selecting a neighborhood W of a in Y and taking in Z the set W ∩ A together with the closed

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star of every vertex \( p_U \) of \( \mathcal{N}(\omega) \) corresponding to a set \( U \in \omega \) with \( U \subset W \). This neighborhood is denoted by \( \tilde{W} \).

It is sufficient to prove that \( f \) extends to a continuous map \( F : Z \to 2^X \); then the map \( \Phi = F_H : Y \to 2^X \) will be the desired extension of \( f \).

Let \( \mathcal{N}_k(\omega) \) denote the \( k \)-skeleton of \( \mathcal{N}(\omega) \). First we extend \( f \) to a map \( f_0 : A \cup \mathcal{N}_0(\omega) \to 2^X \) as follows: in every set \( U \in \omega \) we select a point \( x_U \) and then choose a point \( a_U \in A \) such that \( p(x_U, a_U) < 2p(x_U, A) \). Set \( f_0(p_U) = f(a_U) \) and \( f_0(a) = f(a) \) for \( a \in A \). It is readily seen that \( f_0 \) is continuous. Now we will extend \( f_0 \) over each simplex of \( \mathcal{N}(\omega) \) and thus we obtain the desired map \( F \). Since \( 2^X \) is a Peano continuum [5, Proposition 5.3.10], it is path-connected and locally path-connected by a well-known result of Mazurkiewicz (see [5, Theorem 5.3.13]). For any two points \( B, C \in 2^X \) we select a path \( l_{B,C} : [0, 1] \to 2^X \) such that \( l_{B,C}(0) = B, \ l_{B,C}(1) = C \) and

\[
diam l_{B,C}([0, 1]) < 2 \inf \{diam \gamma([0, 1]) : \gamma \text{ is a path from } B \text{ to } C\}.
\]

We now extend \( f_0 \) to a map \( f_1 : A \cup \mathcal{N}_1(\omega) \to 2^X \) by the rule: \( f_1(a) = f_0(a) \) for \( a \in A \) and \( f_1(t(p_U + (1-t)p_V)) = l_{f_0(p_U), f_0(p_V)}(t), \ 0 < t < 1 \). One needs to prove \( f_1 \) continuous only at points of \( A \). Let \( a \in A, \varepsilon > 0 \) and \( O(f(a), \delta) \) be the \( \delta \)-neighborhood of \( f(a) \) in \( 2^X \). By the local path-connectedness of \( 2^X \), there is a path-connected neighborhood \( Q \) of \( f_0(a) = f(a) \) contained in \( O(f(a), \varepsilon/8) \). By continuity of \( f_0 \), there exists a neighborhood of \( a \) in \( Z \) of the form \( W \) such that \( f_0(\tilde{W} \cap (A \cup \mathcal{N}_0(\omega))) \subset Q \). Then \( f_1(\tilde{W} \cap (A \cup \mathcal{N}_1(\omega))) \subset O(f(a), \varepsilon) \). Indeed, if \( z = t(p_U + (1-t)p_V) \in \tilde{W} \cap \mathcal{N}_1(\omega) \), then \( f_0(p_U), f_0(p_V) \) \( \in Q \); so \( Q \) contains a path \( \gamma \), connecting \( f_0(p_U) \) and \( f_0(p_V) \). Hence \( \gamma([0, 1]) < \varepsilon/2 \), which implies that \( \text{diam} \ l_{f_0(p_U), f_0(p_V)}([0, 1]) < \varepsilon/2 \). Then \( d_H(f_1(z), f_1(a)) < \varepsilon \) because \( f_1(z) \in l_{f_0(p_U), f_0(p_V)}([0, 1]) \).

Now suppose that a continuous extension \( f_k : A \cup \mathcal{N}_k(\omega) \to 2^X \) of \( f_{k-1}, k \geq 1 \) has already been constructed. We shall construct an extension \( f_{k+1} : A \cup \mathcal{N}_{k+1}(\omega) \to 2^X \) of \( f_k \). Let \( \sigma \) be any \((k+1)\)-dimensional simplex in \( \mathcal{N}(\omega) \). Let \( \mathbb{B}^{k+1} \) be the \((k+1)\)-dimensional Euclidean closed unit ball and \( S^k \) be its boundary sphere. We aim at applying the following well-known easy fact: for every \( k \geq 1 \) there exists a continuous function \( r : \mathbb{B}^{k+1} \to 2^{S^k} \) such that \( r(y) = \{y\} \) for all \( y \in S^k \) (see, e.g., [5, Proposition 5.3.11]). To this end, it is convenient to identify the pair \((\sigma, \partial \sigma)\) with \((\mathbb{B}^{k+1}, S^k)\). Then the preceding fact insures the existence of a continuous map \( r_\sigma : \sigma \to 2^{\partial \sigma} \) such that \( r_\sigma(z) = \{z\} \) for every \( z \in \partial \sigma \). The map \( g_\sigma : 2^{\partial \sigma} \to 2^X \) defined by \( g_\sigma(C) = \bigcup_{C \subset \sigma} f_k(c) \) is continuous [5, Corollary 5.3.7]. Then \( f_\sigma = g_\sigma r_\sigma : \sigma \to 2^X \) is a continuous extension of \( f_k|_{\partial \sigma} \). Now we set \( f_{k+1}(z) = f_\sigma(z) \) if \( z \in \sigma \), and \( f_{k+1}(a) = f_k(a) \) if \( a \in A \). Then \( f_{k+1} \) extends \( f_k \) and is continuous on \( \mathcal{N}_{k+1}(\omega) \).

We define the map \( F : Z \to 2^X \) as follows: \( F(z) = f_k(z) \) whenever \( z \in A \cup \mathcal{N}_k(\omega) \). Clearly, \( F \) is continuous on \( \mathcal{N}(\omega) \). Let us check its continuity at points of \( A \). Let \( a \in A \) and \( \varepsilon > 0 \). By continuity of \( f_1 \), there is a neighborhood of \( a \) in \( Z \) of the form \( \tilde{W} \) such that \( f_1(\tilde{W} \cap (A \cup \mathcal{N}_1(\omega))) \subset O(f(a), \varepsilon) \). We claim that \( F(\tilde{W}) \subset O(f(a), \varepsilon) \). We shall prove by induction on the dimension of \( \sigma \) that \( F(\sigma) \subset O(f(a), \varepsilon) \) for every simplex \( \sigma \subset \tilde{W} \). If \( \dim \sigma = 1 \), then \( F(\sigma) = f_1(\sigma) \subset O(f(a), \varepsilon) \). Assume that the claim is true for all simplices \( s \subset \tilde{W} \) with \( \dim s \leq k \). Let \( \sigma \subset \tilde{W} \), \( \dim \sigma = k + 1 \) and \( z \in \sigma \). As \( F(z) = f_{k+1}(z) = g_\sigma(r_\sigma(z)) \), we have \( F(z) = \bigcup_{C \subset \partial \sigma} f_k(c) \). But \( d_H(f_k(c), f(a)) < \varepsilon \) for all \( c \in \partial \sigma \), and in particular, for all \( c \in r_\sigma(z) \). This
yields that $d_H\left(\bigcup_{c \in \tau_r(z)} f_k(c), f(a)\right) < \varepsilon$, i.e., $d_H(F(z), f(a)) < \varepsilon$, completing the inductive step.

The reader can easily observe that the same proof serves also for Curtis’ theorem [1, Theorem 1.6] on growth hyperspaces $G \subset 2^X$, where $X$ is any connected and locally continuum-connected metrizable space.

References


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