SUFFICIENT CONDITIONS FOR A LINEAR FUNCTIONAL TO BE MULTIPLICATIVE

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ABSTRACT. A commutative Banach algebra \( A \) is said to have the \( P(k,n) \) property if the following holds: Let \( M \) be a closed subspace of finite codimension \( n \) such that, for every \( x \in M \), the Gelfand transform \( \hat{x} \) has at least \( k \) distinct zeros in \( \Delta(A) \), the maximal ideal space of \( A \). Then there exists a subset \( Z \) of \( \Delta(A) \) of cardinality \( k \) such that \( \hat{M} \) vanishes on \( Z \), the set of common zeros of \( M \). In this paper we show that if \( X \subset \mathbb{C} \) is compact and nowhere dense, then \( R(X) \), the uniform closure of the space of rational functions with poles off \( X \), has the \( P(k,n) \) property for all \( k, n \in \mathbb{N} \). We also investigate the \( P(k,n) \) property for the algebra of real continuous functions on a compact Hausdorff space.

1. INTRODUCTION

The well known Gleason-Kahane-Zelazko theorem ([6] [10]) states that if \( A \) is any commutative unital Banach algebra and \( M \) is a closed subspace with \( \text{codim}(M)=1 \) such that \( M \) does not contain invertible elements, then \( M \) is a maximal ideal in \( A \). In other words if \( \varphi \in A^* \) is such that \( \varphi(a) \in \sigma(a) \) for every \( a \in A \), then \( \varphi \) is multiplicative. This theorem has been generalized to higher codimensions (see [1] [12] [13] [7]). To state these generalizations we first need a definition.

Let \( A \) be a commutative complex Banach algebra with identity. We say that \( A \) satisfies the \( P(k,n) \) property if the following holds: Let \( M \) be a closed subspace of \( A \) of finite codimension \( n \). Suppose that for every \( x \in M \), the Gelfand transform \( \hat{x} \) has at least \( k \) distinct zeros in \( \Delta(A) \), the maximal ideal space of \( A \). Then there exists a subset \( Z \) of \( \Delta(A) \) of cardinality \( k \) such that \( \hat{x} \) vanishes on \( Z \) for every \( x \in M \). We call \( Z \) a set of common zeros of \( M \). In fact the hypothesis states that if every \( x \in M \) is contained in \( k \) maximal ideals \( I_1, I_2, \ldots, I_k \) (depending on \( x \)), then there exist \( k \) maximal ideals \( I_1, \ldots, I_k \) such that every \( x \in M \) is contained in \( I_1 \), \( I_2, \ldots, I_k \).

Note that if for \( k > 1 \), the \( P(k,n) \) holds for \( C(X) \) the space of continuous complex valued functions on a compact Hausdorff space \( X \), then every point of \( X \) is a \( G_\delta \) set. We now discuss some background history of the subject.

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In [12] Warner and Whitely conjectured that if $X$ is a compact Hausdorff space such that each point of $X$ is a $G_δ$ set, then $C(X)$ satisfies the $P(k, n)$ property. This conjecture was settled by K. Jarosz [7]. Garimella and Rao [5] showed that $C^n[a, b]$ and $L^1(\mathbb{R})$ satisfy $P(k, n)$. Chen and Cohen [2] have proved that a selfadjoint joint regular $n$-point spectral Banach algebra satisfies $P(k, n)$. Rao [11] removed the spectrality condition in the Chen and Cohen theorem, thus making the result applicable to $C^n[a, b]$. He generalized the above result to any selfadjoint, regular commutative Banach algebra with identity. We would also like to investigate the $P(k, n)$ property. Extensions of the Gleason-Kahane-Zelazko theorem in other directions can be found in [9].

In §2 we generalize some results in [7] and [5] and show that if $X$ is a compact and nowhere dense subset of the plane, then $R(X)$, the uniform closure of the algebra of rational functions with poles off $X$, has the $P(k, n)$ property for all $k, n \in \mathbb{N}$.

Note that the GKZ theorem does not hold for real Banach algebras. For example let $A = \text{Re}C[0, 1]$ and $\varphi(f) = \frac{1}{2}(f(0) + f(1))$ for all $f \in A$. By the intermediate value theorem $\varphi(f) \in \sigma(f)$ for all $f \in A$ but $\varphi$ is not multiplicative. This example, in turn, shows that the proof of the GKZ theorem depends heavily on complex analysis techniques. But the GKZ theorem is true for $\text{Re}C(X)$, $X$ compact Hausdorff, if and only if $X$ is assumed to be totally disconnected (see Farnum and Whitely [4]). We may use the definition of the $P(k, n)$ property without any difficulty for real Banach algebras and in §3 we show that if $X$ is a compact Hausdorff and totally disconnected space, then $\text{Re}C(X)$ has the $P(1, n)$ property for all $n > 1$ and also has the $P(k, n)$ property for all $k, n \in \mathbb{N}$, if we further assume that the points of $X$ are $G_δ$. We also show that the $P(1, n)$ property holds for all unital commutative real Banach algebras with a totally disconnected maximal ideal space.

2. Algebras of rational functions

To prove the $P(1, n)$ property for $C(X)$ where $X$ is a compact Hausdorff space, Jarosz proves a lemma [7, Lemma 1] stating that if $X$ is a compact subset of the real line and $p_1, \ldots, p_n$ are polynomials such that each linear combination of them has a zero in $X$, then there is a common zero for $p_1, \ldots, p_n$ in $X$. Garimella and Rao [5] generalized this result for those closed subsets $X$ of $\mathbb{C}$ that have an empty interior and called this result the polynomial lemma. The assumption that $X$ has an empty interior is essential as the following example shows:

Let $p_1 = (iz + 1)^2$ and $p_2 = (z + i)^2$ on $\overline{D}$ the closed unit disk. We show that every linear combination of $p_1$ and $p_2$ has a zero in the disk. To see this note that the Möbius transformation $\frac{iz + 1}{i + 1}$ sends $D \setminus \{-i\}$ to the lower closed half plane and so the function $h = \frac{(iz + 1)^2}{(i + 1)^2}$ sends it to the whole plane. Consequently, every linear combination of $p_1$ and $p_2$ has a zero in the disk while there is no common zero for $p_1$ and $p_2$ in the disk.

This example also shows that the assumption of being finite codimensional is essential in the generalizations of GKZ. See also Jarosz [7, §2].

In the proof of Jarosz and Garimella and Rao for the polynomial lemma, the method of several variables is used. Here we prove a generalization of this lemma that uses only elementary complex analysis and needs no assumption of closedness. We only assume that the closure of $X$ in $\mathbb{C}$ has an empty interior, i.e. $X$ is nowhere dense in $\mathbb{C}$. But first we need a lemma.
Lemma 2.1. Let \( X \) be a nowhere dense subset of \( \mathbb{C} \) and \( f \) be a holomorphic function on a neighborhood of \( X \). Then \( f(X) \) has an empty interior in \( \mathbb{C} \). Consequently, there are infinitely many complex numbers outside \( f(X) \).

Proof. Denote by \( \overline{X} \) the closure of \( X \) in \( \mathbb{C} \). By definition, for each \( x \in X \) there is a neighborhood of \( x \) such that \( f \) is holomorphic there. In this proof we restrict ourselves to those neighborhoods \( V \) such that \( f \) is holomorphic on a neighborhood of \( \overline{V} \). Note that \( f \) need not be defined on \( \overline{X} \).

Let \( x \) be in \( X \). Two cases are possible:

(i) \( f'(x) \neq 0 \). In this case there exists a bounded open set \( V \) containing \( x \) such that \( f \) is one-to-one on \( \overline{V} \). Thus \( \overline{V} \) is compact and \( f_1 = f|_{\overline{V}} \) is invertible and if \( f_1(V \cap \overline{V}) = U \) or equivalently \( f_1^{-1}(U) = \overline{V} \cap \overline{X} \), then \( U \) has an empty interior in \( \mathbb{C} \) since otherwise \( f_1^{-1} \), being holomorphic and nonconstant on \( U \), sends open sets to open sets, and so \( \overline{V} \cap \overline{X} \) and hence \( \overline{X} \) would contain an open set, which contradicts the hypothesis. Therefore \( f(\overline{V} \cap \overline{X}) = f_1(\overline{V} \cap \overline{X}) \) has an empty interior in \( \mathbb{C} \).

(ii) \( f'(x) = 0 \). In this case again two cases may occur. First, there exists a neighborhood \( V \) of \( x \) such that \( f' = 0 \) on \( \overline{V} \). Then \( f \) is constant on \( V \) and hence \( f(\overline{V} \cap \overline{X}) \) is a singleton and so is closed and has an empty interior. The second possibility is that there exists a neighborhood \( V \) such that \( f' \) has a single zero \( x \) in \( V \). In this case we may suppose that \( V \) is an open ball with center \( x \). Choose closed annuli \( V_n \) such that \( \overline{V} \setminus \{x\} = \bigcup_{n=1}^{\infty} V_n \). By (i), for all \( n = 1, 2, 3, \ldots \), \( f(V_n \cap \overline{X}) \) has an empty interior in \( \mathbb{C} \) and therefore \( f(\overline{V} \cap \overline{X}) = \bigcup_{n=1}^{\infty} f(V_n \cap \overline{X}) \cup \{f(x)\} \) has an empty interior in \( \mathbb{C} \) by the Baire category theorem.

Therefore for each \( x \in X \), there exists a suitable neighborhood \( V \) of \( x \) such that \( f(\overline{V} \cap \overline{X}) \) has an empty interior in \( \mathbb{C} \). But \( X \) can be covered with a countable union of such suitable neighborhoods \( V \). Since for such a neighborhood \( V \), \( f(\overline{V} \cap \overline{X}) \) is closed and has an empty interior in \( \mathbb{C} \), their countable union also has an empty interior by the Baire category theorem. Hence, \( f(X) \), being a subset of this countable union, has an empty interior in \( \mathbb{C} \).

We also need the following well-known facts from linear algebra.

Lemma 2.2. (i) If a finite dimensional complex or real linear space is a countable union of subspaces, then one of these subspaces must contain the others.

(ii) If a complex or real linear space is a finite union of subspaces, then one of these subspaces must contain the others.

Theorem 2.3. Suppose \( X \) is a nowhere dense subset of the plane and \( h_0, \ldots, h_n \) are functions analytic on a neighborhood of \( X \) such that each linear combination of them has a zero in \( X \). If one of these \( h_j \), \( 0 \leq j \leq n \), has countably many zeros in \( X \), then there exists a common zero for \( h_0, \ldots, h_n \) in \( X \).

Proof. Suppose \( h_0 \) has countably many zeros in \( X \). Set \( Z = \{x \in X : h_0(x) = \theta\} \). For a linear combination \( h \) of \( h_0, \ldots, h_n \) consider the function \( f = h/h_0 \) on \( X' = X \setminus Z \). Since \( X' \) is also nowhere dense in \( \mathbb{C} \), by Lemma 2.1 there exists a complex number \( \beta \neq 0 \) such that \( f(x) \neq \beta \) for all \( x \in X' \). But the equation \( h(x) = \beta h_0(x) \) has a solution in \( X \) and this solution is not in \( X' \), so that it is in \( Z \). Let \( M \) denote the complex linear space of linear combinations of \( h_0, \ldots, h_n \), i.e. \( M = [h_0, \ldots, h_n] \). We have shown that every element in \( M \) has a zero in \( Z \). For a point \( z \in Z \) let \( M_z = \{f \in M : f(z) = \theta\} \), so that \( M = \bigcup_{z \in Z} M_z \). But \( Z \) is a
Corollary 2.4. Under the same hypothesis, if each linear combination of \( h_0, \ldots, h_n \) has at least \( k \) zeros in \( X \), then there exist \( k \) common zeros for \( h_0, \ldots, h_n \) in \( X \).

Proof. There exists a common zero for \( h_0, \ldots, h_n \). Remove it from \( X \) and continue if necessary.

Corollary 2.5. Suppose \( A \) is a linear space of functions on a nowhere dense subset \( X \) of the plane each of which is analytic on a neighborhood of \( X \), and has at least \( k \) zeros in \( X \). Furthermore assume that one element of \( A \) has finitely many zeros in \( X \). Then there exist \( k \) common zeros for \( A \) in \( X \).

For example, \( A \) can be a space of polynomials or rational functions.

Proof. Let \( p \) be a polynomial in \( A \) have only finitely many zeros in \( X \). The set \( \{ x \in X : p(x) = 0 \} \) is finite and has cardinality \( \geq k \). For an arbitrary element \( q \in A \), let \( h = q/p \), so by Lemma 2.1, \( h(X \setminus Z) \) has empty interior in \( C \). Thus there exists \( \beta \neq 0 \) such that \( h(x) \neq \beta \) for all \( x \in X \setminus Z \). But by the hypothesis, \( q - \beta p \) is in \( A \) and has \( k \) zeros in \( X \) and none of the zeros are in \( X \setminus Z \); hence all are in \( Z \). Therefore \( q \) has at least \( k \) zeros in \( Z \). For a subset \( K \) of \( Z \) of cardinality \( k \), set

\[ A_K = \{ f \in A : f(x) = 0 \text{ for all } x \in K \}. \]

Thus \( A = \bigcup_K A_K \) where \( K \) runs over all subsets of \( Z \) of cardinality \( k \). But \( Z \) is a finite set and has only finitely many subsets of cardinality \( k \). By Lemma 2.2 (ii) one of these \( A_K \) must contain the others, i.e. \( A \) has \( k \) common zeros in \( X \).

Now the \( P(k, n) \) property for the algebra of rational functions over a compact nowhere dense subset of the plane follows.

Theorem 2.6. Let \( R(X) \) be the uniform closure of the algebra of rational functions \( \text{Rat}(X) \) with poles off a compact nowhere dense subset \( X \) of the plane. If \( M \) is a finite codimensional subspace of \( R(X) \) such that every element of it has at least \( k \) zeros in \( X \), then there are \( k \) common zeros for \( M \) in \( X \). That is, the \( P(k, n) \) property holds for \( R(X) \).

Proof. Note that the maximal ideal space of \( R(X) \) is \( X \). Let \( M \) be a finite codimensional subspace of \( R(X) \) such that each element of \( M \) has at least \( k \) zeros in \( X \). Let \( M' = M \cap \text{Rat}(X) \). Since \( M \) is finite codimensional in \( R(X) \), \( M' \) would be dense in \( M \). Also each element of \( M' \) has at least \( k \) zeros in \( X \) and hence by Corollary 2.5, \( M' \) has \( k \) common zeros in \( X \). Now \( M \), being the uniform closure of \( M' \), has \( k \) common zeros in \( X \).

It follows that the \( P(k, n) \) property holds for \( C(X) \) whenever \( X \subset C \) is compact and nowhere dense with a connected complement (especially whenever \( X \subset R \)) since rational functions are dense in \( C(X) \) in this case.

The above proof is also applicable to \( C^0[a, b] \) and to \( C^\infty[a, b] \), although the latter has no Banach algebra norm, since polynomials are dense in these algebras.
3. What happens for Re\(C(X)\)?

As stated in the introduction, the GKZ theorem does not hold in general for real Banach algebras and it is true for Re\(C(X)\), \(X\) compact, if and only if \(X\) is totally disconnected [3]. In fact, if \(U\) is a nontrivial connected component of \(X\), then we have an abundance of examples of continuous linear functionals \(\varphi\) with \(\varphi(f) \in \text{Im}(f)\) for all \(f \in \text{Re}C(X)\) such that \(\varphi\) is not multiplicative.

Let \(x\) and \(y\) be two distinct points of \(U\) and \(\varphi(f) = \frac{f(x) + f(y)}{2}\). Then for all \(f \in \text{Re}C(X)\), \(\varphi(f) \in \text{Im}(f)\) by the intermediate value theorem and obviously \(\varphi\) is not multiplicative. If we consider \(U\) as a subset of the dual space of \(\text{Re}C(X)\) (the points of \(U\) will then be the point mass measures at points of \(U\)), a deeper observation reveals that each element \(\varphi\) of \(\overline{\text{Re}C(U)}\), the closure of the convex hull of \(U\) in the dual space of \(\text{Re}C(X)\), has the property \(\varphi(f) \in \text{Im}(f)\) for all \(f \in \text{Re}C(X)\).

Now we show that if \(X\) is a totally disconnected compact Hausdorff space, then \(P(1, n)\) is true for \(\text{Re}C(X)\) for all \(n \in \mathbb{N}\).

**Theorem 3.1.** If \(X\) is a totally disconnected compact Hausdorff space, then \(\text{Re}C(X)\) has the \(P(1, n)\) property for all \(n \in \mathbb{N}\).

**Proof.** Let \(M\) be a finite codimensional subspace of \(\text{Re}C(X)\) consisting only of non-invertible elements. Since \(X\) is totally disconnected, \(X\) can be considered as a closed subspace of a discrete product \(\{0, 1\}^J\) where \(J\) is some index set. For a finite subset \(J_0\) of \(J\) denote

\[
M(J_0) = \{f \in M : f \text{ depends only on the coordinates from } J_0\}
\]

and

\[
X(J_0) = \{x \in X : f(x) = 0 \text{ for all } f \in M(J_0)\}.
\]

Clearly \(M(J_0)\) can be considered as a subspace of \(\mathbb{R}^{2^{J_0}}\) and since each element of \(M(J_0)\) is non-invertible; then \(X(J_0)\) would be closed (and hence compact) and nonvoid by Lemma 2.2 (i).

Now for a family \(J_1, \ldots, J_m\) of finite subsets of \(J\) let \(J_0 = \bigcup_{i=1}^{m} J_i\). Thus \(X(J_0) \subseteq \bigcap_{i=1}^{m} X(J_i)\) and since \(X(J_0)\) is not empty, then it follows that the class \(\{X(J_0) : J_0 \text{ is a finite subset of } J\}\) has the finite intersection property and therefore has a nonempty intersection \(Z\) which is compact and is a zero set for the subspace

\[
M' = \{f \in M : f \text{ depends only on a finite number of coordinates from } J\}.
\]

But the linear subspace

\[
Y = \{f \in \text{Re}C(X) : f \text{ depends only on a finite number of coordinates from } J\}
\]

is a dense subspace of \(\text{Re}C(X)\) and \(M\) is finite codimensional in \(\text{Re}C(X)\), so that \(M' = Y \cap M\) is dense in \(M\) and therefore each element of \(M\) is zero on \(Z\); that is, \(M\) has at least a common zero in \(X\).

Now we are ready to prove the \(P(k, n)\) property for \(\text{Re}C(X)\) if we further assume that the points of \(X\) are \(G_{\delta}\). In general, all the points of the space \(\{0, 1\}^J\) are not \(G_{\delta}\). For example, if \(J = [0, 1]\), then the point \(\emptyset\) in \(\{0, 1\}^J\), which is zero in all of its coordinates, is not a \(G_{\delta}\) set.
Theorem 3.2. If $X$ is a totally disconnected compact Hausdorff space such that each point of $X$ is a $G_δ$, then the $P(k,n)$ property holds for $\mathbb{R}C(X)$ for all $k, n \in \mathbb{N}$.

Proof. Let $M$ be a subspace of $\mathbb{R}C(X)$ of codimension $n$ such that each element of $M$ has at least $k$ zeros in $X$. By Theorem 3.1, there is at least one common zero for $M$ in $X$. Let $Z = \{x_1, \cdots, x_m\}$ denote the set of common zeros for $M$ in $X$. Since the multiplicative functionals on a commutative Banach algebra are linearly independent, it follows that $m \leq n$. If $k \leq m$ we are done. If $k > m$, put

$$M_k^+ = [δ_1, \cdots, δ_m, μ_1, \cdots, μ_m']$$

where $m + m' = n$ and $δ_j$ denotes the point mass measure at $x_j$ for $1 \leq j \leq m$. Let

$$M' = \{f \in \mathbb{R}C(X) : μ_j(f) = \int f dμ_j = 0, \text{ for all } j, 1 \leq j \leq m'\}.$$

We may suppose, by adding suitable linear combination of $δ_1, \cdots, δ_m$ to each $μ_j$, that $|μ_j|(Z) = 0$ for $1 \leq j \leq m'$.

Since there is no common zero for $M'$, by Theorem 3.1 there is a function $f \in M'$ such that $f \neq 0$ everywhere on $X$. By using the duality notions in the subalgebra

$$\{f \in \mathbb{R}C(X) : f(Z) = 0\},$$

of $\mathbb{R}C(X)$, we may choose $f_j$ such that $f_j$ is zero on $Z$ and $\int f_j dμ_j = δ_{ij}, 1 \leq i, j \leq m'$. Let $ε > 0$ be such that $g = |f| - ε \sum_j |f_j|$ is a strictly positive function. Set $s = \sup_{x \in X} g(x)$. Since each $μ_j$ is regular and zero on $Z$, there exists an open neighborhood $V$ of $Z$ such that $|μ_j|(V) < ε/s$.

Since $Z$ is a $G_δ$ set, we may choose $h$ in $\mathbb{R}C(X)$ such that $0 \leq h \leq 1$ on $X$, $h$ is 1 only on $Z$ and is zero outside $V$. Let $u = gh(\text{sgn} f)$ (since $f \neq 0$ everywhere on $X$, $\text{sgn} f$ is continuous). Then

$$|\int udμ_j| \leq |μ_j|(V) \sup_{x \in X} g(x) < (\frac{ε}{s})s = ε.$$

If we consider the function $w = f - u + \sum_j μ_j(u)f_j$, then a simple computation shows that $w$ is zero on $Z$ and $\int wdμ_j = 0, 1 \leq j \leq m'$. Thus $w$ belongs to $M$. Furthermore, $|w| \geq |f| - |u| - ε \sum_j |f_j| = (1 - h)g > 0$, outside $Z$. Hence $w \in M$ has exactly $m$ zeros which is absurd. \hfill \square

Corollary 3.3. If $X$ is a totally disconnected compact Hausdorff space with points $G_δ$ and $M$ is a subspace of codimension $n$ in $\mathbb{R}C(X)$ such that each element of $M$ has at least $n$ zeros in $X$, then $M$ is an ideal. \hfill \square

In [11] Theorem (0.4)], Rao proved that a complex unital commutative Banach algebra $A$ has the $P(k,n)$ property if and only if its corresponding semisimple algebra $\mathcal{B}$ has this property where $\mathcal{B}$ denotes the Jacobson radical of $A$. The proof there is also applicable to real Banach algebras as well. Therefore we conclude the following:

Theorem 3.4. Let $A$ be a unital real commutative Banach algebra with a totally disconnected maximal ideal space $X$. If $M$ is a finite codimensional subspace of $A$ such that the Gelfand transform of each element of $M$ has a zero in $X$, then $M$ is contained in a maximal ideal.
Proof. We can assume that \( \mathcal{A} \) is semisimple. Let \( \Lambda \) be the Gelfand map from \( \mathcal{A} \) into \( \text{Re}C(X) \). Then \( N = \overline{\Lambda}(M) \) is finite codimensional in \( \text{Re}C(X) \) and each element of \( N \) is noninvertible in \( \text{Re}C(X) \). So by Theorem 3.1 there exists a common zero for \( N \) in \( X \) and this is a common zero for \( M \). Consequently \( M \) is contained in a maximal ideal.

4. Some further results

In 1991, K. Jarosz [9] posed the following problems:

Problem 1. Does any commutative complex unital Banach algebra have the \( P(1, n) \) property for \( n \geq 2 \)?

Problem 2. In particular does the disk algebra have the \( P(1, n) \) property for \( n \geq 2 \)?

Problem 3. If some element \( f \) of a commutative complex Banach algebra \( \mathcal{A} \) does not belong to any regular maximal ideal of \( \mathcal{A} \), does \( \mathcal{A} \) have the \( P(1, n) \) property for \( n \geq 1 \)?

Problem 4. If the points of the maximal ideal space of such an algebra \( \mathcal{A} \) are \( G_\delta \), does \( \mathcal{A} \) have the \( P(k, n) \) property for all positive integers \( k \) and \( n \)?

However, in 1989, Rao [11] showed that \( P(2, 3) \) does not hold for \( C^1(B) \), the algebra of all continuously differentiable functions on \( B \) where \( B \) denotes the closed unit ball of \( \mathbb{R}^3 \). So the fourth problem of Jarosz is settled negatively. We give this example and modify it to give a negative answer to the third problem. However, the first two problems are still open.

Example 4.1 ([11]). Let \( \mathcal{A} = C^1(B) \). It is clear that the maximal ideal space of \( \mathcal{A} \) is \( B \). Let \( P \) and \( Q \) be arbitrary points of \( B \) with \( Q \) in the interior. Consider the three continuous linear functionals \( \nu_1, \nu_2 \) and \( \nu_3 \) defined by \( \nu_1(f) = f(P) + f_x(Q) \), \( \nu_2(f) = f(Q) \), and \( \nu_3(f) = f_x(Q) + if_y(Q) \) for all \( f \in \mathcal{A} \), where \( x, y, z \) denote the coordinates of a point in \( \mathbb{R}^3 \). We show that each element of the subspace

\[
M = \{ \nu_1, \nu_2, \nu_3 \} = \{ f \in \mathcal{A} : \nu_1(f) = \nu_2(f) = \nu_3(f) = 0 \}
\]

has at least two distinct zeros in \( B \) while \( M \) has only one common zero \( Q \).

Let \( f \in M \). If \( f(P) = f(Q) = 0 \), we are done. If \( f(P) \neq 0 \), then also by the definition of \( M \), \( f_x(Q) \) and \( f_y(Q) \) are not equal to zero. Write \( f = g + ih \) where \( g \) and \( h \) are real valued functions in \( \mathcal{A} \). Consequently \( g_x(Q) = h_y(Q) \) and \( g_y(Q) = -h_x(Q) \). Since \( f_x(Q) \neq 0 \), the previous two quantities are not equal to zero, i.e. \( g_x(Q) = h_y(Q) \neq 0 \). Therefore, by the implicit function theorem, there are two 2-dimensional \( C^1 \) manifolds passing through \( Q \) such that \( g = 0 \) on one of them and \( h = 0 \) on the other. So on their intersection, which contains at least a \( C^1 \) curve passing through \( Q \), \( f \) is identically zero.

To show that Problem 3 of Jarosz has also a negative answer, we modify the above example.

Example 4.2. We use the same notations as in Example 4.1. Put

\[
\mathcal{A}_0 = \{ f \in \mathcal{A} : f(Q) = 0 \}.
\]
It is not difficult to see that $\mathcal{A}_0$ is a non-unital commutative Banach algebra with the maximal ideal space $B' = B \setminus \{Q\}$ and there are functions in $\mathcal{A}_0$ which are nonzero on $B'$, i.e. do not belong to any regular maximal ideal of $\mathcal{A}_0$. Now put

$$M_0 = \{\nu_1, \nu_3\} = \{f \in \mathcal{A}_0 : \nu_1(f) = \nu_3(f) = 0\}$$

in $\mathcal{A}_0$. Then $M_0$ is of codimension 2 in $\mathcal{A}_0$ and as in the above we see that every element of $M_0$ has at least one zero in $B'$ but there is no common zero for $M_0$ in $B'$. Therefore $P(1,2)$ does not hold for $\mathcal{A}_0$ and the answer to the third problem would also be negative.

Finally we treat the most important type of non-unital commutative Banach algebras $C_0(X)$, the algebra of all continuous complex-valued functions on a locally compact Hausdorff space $X$ vanishing at infinity.

If $X$ is not $\sigma$-compact, then every element of $C_0(X)$ has infinitely many zeros in $X$. So, it does not make any sense to speak about the $P(k,n)$ property in this case. But if $X$ is assumed to be $\sigma$-compact the $P(1,n)$ property follows for $C_0(X)$ from Theorem 2 of [8] for all $n \in \mathbb{N}$. If we further assume that all the points of $X$ are $G_\delta$, the $P(k,n)$ property follows for $C_0(X)$ from [7] for all $k, n \in \mathbb{N}$ by only considering the natural embedding of $C_0(X)$ into $C(X_\infty)$, where $X_\infty$ denotes the one point compactification of $X$.

Here we present a simple proof of $P(1,1)$ for $C_0(X)$ that is independent of [8]:

**Theorem 4.3.** Let $X$ be a locally compact Hausdorff space that is $\sigma$-compact. Then $C_0(X)$, the space of all complex valued functions on $X$ vanishing at infinity, satisfies $P(1,1)$.

**Proof.** First we show that there exists a positive function $f_0$ in $C_0(X)$. Write $X = \bigcup_{n=1}^\infty K_n$, where $K_n$ is compact for each $n \geq 1$. By the Urysohn lemma we find a sequence $g_n \in C_c(X)$ such that $0 \leq g_n \leq 1$ and $g_n = 1$ on $K_n$. Set $f_0 = \sum_{i=1}^\infty g_i$. Then $f_0 \in C_0(X)$ and $f_0 > 0$ on $X$.

Now, let $M$ be a closed subspace of $C_0(X)$ of codimension 1 such that every $f \in M$ has a zero in $X$. Then $M^\perp = [\mu_0]$ for some regular Borel measure $\mu_0$ on $X$. Hence

$$M = \{f \in C_0(X) : \int f d\mu_0 = 0\}.$$ 

Note that $f_0 \notin M$ because every element of $M$ vanishes at some point of $X$. Let $\alpha = \int f_0 d\mu_0$. Clearly $\alpha \neq 0$. Replacing $\mu_0$ by $\alpha^{-1} \mu_0$ we obtain a new $\mu_0 \perp M$ such that $\int f_0 d\mu_0 = 1$. Now given $f \in C_0(X)$ we write $g = f - (\int f d\mu_0) f_0$. Then $\int g d\mu_0 = \int f d\mu_0 - \int f_0 d\mu_0 = 0$. Therefore $g \in M$. Hence there exists $x_0 \in X$ such that $g(x_0) = 0$, i.e. $\int f d\mu_0 = f(x_0)/f_0(x_0)$. Now if $f \geq 0$, then $\int f d\mu_0 \geq 0$; hence $\mu_0 \geq 0$.

We have shown that $\int f d\mu_0 \in \text{Im}(f/f_0)$ for every $f \in C_0(X)$. Replacing $f$ by $ff_0$ we get $\int ff_0 d\mu_0 \in \text{Im}(f)$, $f \in C_0(X)$. From this we conclude that $f_0 d\mu_0 = \delta_v$, the point mass at some $v \in X$ [3] Lemma 2.5. Write $E = X \setminus \{v\}$. Then $\int_E f_0 d\mu_0 = 0$; hence $f_0 = 0$ a.e. $\mu_0$ on $E$. Because $f_0$ is strictly positive, we conclude that $\mu_0(E) = 0$, that is, $\mu_0 = c \delta_v$ for some constant $c > 0$, so that,

$$M = \{f \in C_0(X) : f(v) = 0\}.$$ 

The proof is now complete.
REFERENCES


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