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BORREL COMPLEXITY OF THE SPACE OF PROBABILITY MEASURES

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ABSTRACT. Using a technique developed by Louveau and Saint Raymond, we find the complexity of the space of probability measures in the Borel hierarchy: if X is any non-Polish Borel subspace of a Polish space, then $P(X)$, the space of probability Borel measures on X with the weak topology, is always *true* $\mathbf{\Pi}^0_\xi$, where ξ is the least ordinal such that X is $\mathbf{\Pi}^0_\xi$.

1. INTRODUCTION

For X a separable metric space, let $P(X)$ be the space of probability Borel measures on X with the usual topology of weak convergence, so that $P(X)$ is also a separable metrizable space. Many results relating the descriptive complexities of X and $P(X)$ are classical and standard. For example (see [1], [5]), X is compact-metrizable (resp. Polish) iff $P(X)$ is compact-metrizable (resp. Polish), X is Borel (resp. projective) iff $P(X)$ is Borel (resp. projective), etc. Beyond the domain of Borel sets we have: X is analytic (resp. co-analytic) iff $P(X)$ is analytic (resp. co-analytic), which follows from results of Kechris [2], and this also extends to every level of the projective hierarchy under additional set-theoretic hypotheses. Shreve's theorem ([6]) establishes the same equivalence for each level of the C-hierarchy of Selivanovski.

In this note we prove a similar result for the Borel hierarchy. It is known ([2]) that, for $\alpha \geq 2$, $P(X)$ is $\mathbf{\Pi}^0_\alpha$ iff X is $\mathbf{\Pi}^0_\alpha$; hence if X is $\mathbf{\Sigma}^0_\alpha$, then $P(X)$ is $\mathbf{\Pi}^0_{\alpha+1}$. This suggests the question: *Is this the best bound?* More generally, for Γ and Γ' any two intrinsic Borel pointclasses, we ask: *If X is in Γ, is $P(X)$ in Γ'?* It is also easy to find, for each $\alpha \geq 3$, an example of a $P(X)$ which is true $\mathbf{\Pi}^0_\alpha$: just take X to be any space which is true $\mathbf{\Pi}^0_\alpha$; since X is always embedded in $P(X)$ as a closed subset, $P(X)$ cannot be $\mathbf{\Sigma}^0_\alpha$ lest X be $\mathbf{\Sigma}^0_\alpha$. This suggests the question: *For which $\alpha \geq 3$ can we find X such that $P(X)$ is true $\mathbf{\Sigma}^0_\alpha$ (or true $\mathbf{\Delta}^0_\alpha$)?*

Theorem 3.1 below answers these questions, and determines uniquely the true Borel class of $P(X)$ from the true Borel class of X; it says: $P(X)$ is Polish if X is Polish, and for every $\alpha \geq 3$, $P(X)$ is true $\mathbf{\Pi}^0_\alpha$ if X is true $\mathbf{\Pi}^0_\alpha$, $P(X)$ is true $\mathbf{\Pi}^0_{\alpha+1}$

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if X is true $\mathbf{\Sigma^0_\alpha}$, and $P(X)$ is true $\mathbf{\Pi^0_\alpha}$ if X is true $\mathbf{\Delta^0_\alpha}$. (These cases are mutually exclusive and exhaustive for Borel X.)

In particular, if X is Borel, then either $P(X)$ is Polish or $P(X)$ is true $\mathbf{\Pi^0_\alpha}$ for some (unique) $\alpha \geq 3$, so that if $\alpha \geq 3$, there is no X such that $P(X)$ is true $\mathbf{\Sigma^0_\alpha}$ or true $\mathbf{\Delta^0_\alpha}$.

2. Terminology

We use the notation of [1] and [4] for the Borel pointclasses: the additive, multiplicative, and ambiguous classes of level α are denoted, respectively, by $\mathbf{\Sigma^0_\alpha}$, $\mathbf{\Pi^0_\alpha}$, and $\mathbf{\Delta^0_\alpha}$, with $\mathbf{\Sigma^0_1}$ denoting the pointclass of open sets. Γ is a Borel pointclass if Γ is one of $\mathbf{\Sigma^0_\alpha}$, $\mathbf{\Pi^0_\alpha}$, and $\mathbf{\Delta^0_\alpha}$. If X is a Polish space, we use the notation $\mathbf{\Sigma^0_\alpha}\restriction X$ to denote the $\mathbf{\Sigma^0_\alpha}$ subsets of X, and similarly for $\mathbf{\Pi^0_\alpha}$ and $\mathbf{\Delta^0_\alpha}$. Note that for $\alpha \geq 2$ the pointclass $\mathbf{\Pi^0_\alpha}$ is intrinsic,[1] and for $\alpha \geq 3$ the pointclasses $\mathbf{\Sigma^0_\alpha}$ and $\mathbf{\Delta^0_\alpha}$ are also intrinsic. For an intrinsic pointclass Γ, we can speak (unambiguously) of a separable metrizable space X being Γ, without mentioning any Polish space in which X is embedded. The notion of "the true Borel class" of a Borel set is defined in the usual way.

$\boldsymbol{\omega^\omega}$ denotes the Baire space, i.e., the space of irrationals. If X, Y are Polish spaces, $B \subseteq Y$, and $\mathcal{C}$ is a collection of subsets of X, we say that B *is* $\mathcal{C}$*-hard* if for all $C \in \mathcal{C}$ there is a continuous $f\colon X \to Y$ such that $C = f^{-1}(B)$.

If X is a metrizable space, $P(X)$ denotes the space of probability measures[2] on X with the weak topology.[3]

For a separable metrizable space Y, and a Borel $X \subseteq Y$, we can (topologically) identify (see [1]) the space $P(X)$ with the subspace $P(X/Y)$ of $P(Y)$, where

$$P(X/Y) \stackrel{\text{def}}{=} \{\mu \in P(Y) \mid \mu(Y \smallsetminus X) = 0\}.$$

3. The Borel complexity of $P(X)$

Theorem 3.1. *If X is any non-Polish Borel subspace of a Polish space, then $P(X)$ is true $\mathbf{\Pi^0_\xi}$, where ξ is the least such that X is $\mathbf{\Pi^0_\xi}$.*

Proof. We will use the following two lemmas:

Lemma 3.2 (Louveau and Saint Raymond)**.** *If Y is a Polish space, α is a countable ordinal ≥ 2, $A \subseteq Y$ such that A is Borel, and $A \notin \mathbf{\Pi^0_\alpha}$, then A is $\mathbf{\Sigma^0_\alpha}\restriction\boldsymbol{\omega^\omega}$-hard, i.e. for all $B \subseteq \boldsymbol{\omega^\omega}$, if B is $\mathbf{\Sigma^0_\alpha}$, then $B = f^{-1}(A)$ for some continuous $f\colon \boldsymbol{\omega^\omega} \to Y$.*

Proof. This is an immediate consequence of [3, Theorem 3, p. 455]. $\square$

Lemma 3.3 (The δ-propagation lemma)**.** *Let Y, Z be Polish spaces, $\mathcal{C}$ a collection of subsets of Z, and X a subset of Y which is $\mathcal{C}$-hard, i.e. for every A in $\mathcal{C}$ there is a continuous map f from Z to Y such that $A = f^{-1}(X)$. Then $P(X/Y)$ is $\mathcal{C}_\delta$-hard, where $\mathcal{C}_\delta$ denotes the countable intersections of sets in $\mathcal{C}$.*

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[1] Γ *is an intrinsic pointclass* if for all Polish X, Y and $A \subseteq X$, $B \subseteq Y$, if A is a Γ-set in X and A is homeomorphic to B, then B is a Γ-set in Y.

[2] The collection of all countably additive non-negative real-valued functions μ defined on the Borel sets of X such that $\mu(X) = 1$.

[3] The weakest topology such that, for every bounded continuous real-valued function f on X, the real-valued map $\mu \to \int f d\mu$ defined on $P(X)$ is continuous.

Proof. Let X, Y, Z, and $\mathcal{C}$ be as above. To show that $P(X/Y)$ is $\mathcal{C}_\delta$-hard, let F be an arbitrary subset of Z in $\mathcal{C}_\delta$. Then there is sequence $(E_n)_{n=0}^\infty$ of subsets of Z such that $(\forall n)(E_n \in \mathcal{C})$, and

$$F = \bigcap_{n=0}^{\infty} E_n.$$

For each $n \in \omega$, choose a continuous function $f_n \colon Z \to Y$ such that

$$E_n = f_n^{-1}(X).$$

Define $\Psi \colon Z \to P(Y)$ by setting for each $x \in Z$ and each Borel $E \subseteq Y$:

$$\Psi(x)(E) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{\boldsymbol{\chi}_E(f_k(x))}{2^{k+1}},$$

where, for any set A, $\boldsymbol{\chi}_A$ denotes the characteristic function of A. In other words, for any $x \in Z$, $\Psi(x)$ is the measure

$$\Psi(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \delta_{f_k(x)},$$

where, for any $y \in Y$, δ_y denotes the probability measure on Y known as the "unit mass" at y. Ψ is well-defined and continuous. It is now easy to verify that Ψ reduces F to $P(X/Y)$. $\square$

Now let X be non-Polish Borel, and Y be any metrizable compactification of X; let α be the least such that X is $\boldsymbol{\Pi}^0_{\boldsymbol{\alpha}}$. Then $\alpha \geq 3$, as X is not $\boldsymbol{\Pi}^0_{\boldsymbol{2}}$ in Y. Also, $P(X)$ is $\boldsymbol{\Pi}^0_{\boldsymbol{\alpha}}$. It remains to show that $P(X)$ is not $\boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}$ ($\boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}$ is an intrinsic pointclass since $\alpha \geq 3$). Put $Z = \boldsymbol{\omega}^{\boldsymbol{\omega}}$, and

$$\mathcal{C} = \bigcup_{\beta<\alpha} \boldsymbol{\Sigma}^0_{\boldsymbol{\beta}} \restriction \boldsymbol{\omega}^{\boldsymbol{\omega}}.$$

By Lemma 3.2 X is $\boldsymbol{\Sigma}^0_{\boldsymbol{\beta}} \restriction \boldsymbol{\omega}^{\boldsymbol{\omega}}$-hard for every $\beta < \alpha$, and hence is $\mathcal{C}$-hard. The hypotheses of the δ-propagation lemma now hold. Therefore $P(X/Y)$ is $\mathcal{C}_\delta$-hard; but $\mathcal{C}_\delta = \boldsymbol{\Pi}^0_{\boldsymbol{\alpha}} \restriction \boldsymbol{\omega}^{\boldsymbol{\omega}}$. So $P(X)$ must be true $\boldsymbol{\Pi}^0_{\boldsymbol{\alpha}}$. $\square$

It follows from the above theorem that the Borel complexity of the space of probability measures on $\mathbb{Q}$ (the rationals) is $\boldsymbol{\mathcal{F}}_{\sigma\delta}$ but not $\boldsymbol{\mathcal{G}}_{\delta\sigma}$.

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