REAL GROUPS TRANSITIVE
ON COMPLEX FLAG MANIFOLDS

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Abstract. Let $Z = G/Q$ be a complex flag manifold. The compact real form $G_u$ of $G$ is transitive on $Z$. If $G_0$ is a noncompact real form, such transitivity is rare but occasionally happens. Here we work out a complete list of Lie subgroups of $G$ transitive on $Z$ and pick out the cases that are noncompact real forms of $G$.

0. THE PROBLEM

Let $Z = G/Q$ be a complex flag manifold where $G$ is a complex connected semisimple Lie group and $Q$ is a parabolic subgroup. Let $G_0$ be a real form of $G$. If $G_0$ is the compact real form, then it is transitive on $Z$. On a number of occasions the question has come up as to whether any noncompact real form of $G$ can be transitive on $Z$. Here I'll record the answer. The rough answer is “yes, but just a few.” The precise answer, Corollaries 1.7 and 2.3 below, follows from a more general classification, Theorems 1.6 and 2.2. This more general classification uses a technique of D. Montgomery [M], together with some results of [W1] that depend in an essential way on a classification [O1] of A. L. Onishchik.

After this paper was written I learned of Onishchik’s book [O2]. There is some overlap for compact groups, but there are no inclusions.

1. THE SOLUTION FOR IRREDUCIBLE FLAGS

We formulate the problem in terms of transitive subgroups. Let $G_u$ be the compact real form of $G$, so $Z = G_u/(G_u \cap Q)$ and $G_u \cap Q$ is the compact real form of the reductive part of $Q$. Let $A \subset G$ be a closed subgroup that is transitive on $Z$. The identity component $A^0$ of $A$ is transitive on $Z$, because $Z$ is connected, so a maximal compact subgroup $B^0 \subset A^0$ already is transitive on $Z$, according to Montgomery [M]. We may replace $A$ by a conjugate and assume $B = A \cap G_u$. So

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now we have several expressions:
\[
Z = G/Q = G_u/(G_u \cap Q) = A/(A \cap Q) = B/(B \cap Q)
\]
\[
= A^0/(A^0 \cap Q) = B^0/(B^0 \cap Q).
\]

According to [11 Prop. 3.1] there are just a few possibilities for a homogeneous almost–hermitian manifold \(Z\) to have distinct expressions such as \(G_u/L_u\) and \(B^0/(B^0 \cap L_u)\), where \(G_u\) is the identity component of the group of all almost–hermitian isometries, \(G_u\) is simple, \(L_u\) is the centralizer of a torus subgroup of \(G_u\), and \(B^0 \subset G_u\) with \(B^0\) connected. They are:

\[
Z = \mathbb{P}^{2n-1}(\mathbb{C}) = SU(2n)/U(2n-1) = Sp(n)/(Sp(n-1) \cdot U(1)), \text{ complex projective space,}
\]
\[
Z = SO(2r + 2)/U(r + 1) = SO(2r + 1)/U(r), \text{ unitary structures on } \mathbb{R}^{2r+2},
\]
\[
Z = SO(7)/(SO(5) \cdot SO(2)) = G_2/U(2), \text{ 5–dimensional complex quadric, and}
\]
\[
Z = SO(8)/(SO(6) \cdot SO(2)) = \{Spin(7)/Z_2\}/U(3), \text{ 6–dimensional complex quadric.}
\]

This applies in our situation because \(L_u = G_u \cap Q\) is the centralizer of a torus subgroup of \(G_u\), and \(Z\) has a \(G_u\)-invariant hermitian metric.

Now return to the expression \(Z = G/Q\). \(G\) (and thus \(G_u\)) is simple. Let \(A \subset G\) be a closed subgroup that is transitive on \(Z\) and let \(B\) be its maximal compact subgroup. We may assume \(B = A \cap G_u\). Then \(B \subset G_u\), \(B^0\) is transitive on \(Z\), and the expression \(Z = G_u/L_u = B^0/(B^0 \cap L_u)\) is given above. In each case the group \(B^0\) is simple, so \(A^0\) has Levi decomposition \(A^0 = A^0_{ss}A^0_{rad}\) into semisimple part and solvable radical, where \(B^0\) is a maximal compact subgroup of \(A^0_{ss}\). We run through the 4 possibilities listed above.

For (1.2), \(G = SL(2n; \mathbb{C})\) and \(B^0 = Sp(n)\). The semisimple Lie groups with maximal compact subgroup \(Sp(n)\) are \(Sp(n), Sp(n; \mathbb{C}), \text{ the quaternionic linear group } SL(n; \mathbb{H})\), and, for \(n = 4\), the real group \(F_4, C_2\). But \(F_4\) does not have a representation of degree 8, in other words \(F_4 \not\subset G\), so now \(A^0_{ss}\) is one of \(Sp(n), Sp(n; \mathbb{C})\) and \(SL(n; \mathbb{H})\). Each of them is irreducible on \(\mathbb{C}^{2n}\), so the unipotent radical of the algebraic hull of \(A^0\) acts trivially on \(\mathbb{C}^{2n}\) and the center of the reductive part of \(A^0\) acts by scalars. As \(G\) acts effectively and by transformations of determinant 1 on \(\mathbb{C}^{2n}\) now \(A^0_{ss} = A^0\), so \(A^0\) is one of \(Sp(n), Sp(n; \mathbb{C})\) and \(SL(n; \mathbb{H})\). If \(g \in G\) normalizes \(A^0\), then some element \(g' \in gA^0\) centralizes \(A^0\), because \(A^0\) has no rational outer automorphism. As \(A^0\) is irreducible on \(\mathbb{C}^{2n}\) now \(g'\) is scalar (and thus acts trivially on \(Z\)). Thus \(A = A^0F\) where \(F\) can be any subgroup of the center \(\{e^{2\pi i k/2n}I \mid 0 \leq k < 2n\}\) of \(G\).

For (1.3), \(G = SO(2r + 2; \mathbb{C})\) and \(B^0 = SO(2r + 1)\). The semisimple Lie groups with maximal compact subgroup \(SO(2r + 1)\) are \(SO(2r + 1), SO(2r + 1; \mathbb{C}), SO(2r + 1; \mathbb{R})\), and \(SL(2r + 1; \mathbb{R})\). But \(A^0_{ss} = SL(2r + 1; \mathbb{R})\) would give \(SL(2r + 1; \mathbb{C}) \subset SO(2r + 2; \mathbb{C})\), so the respective dimensions would satisfy \(4r^2 + 4r \leq 2r^2 + 3r + 1\), forcing \(r = 0\) and \(Z = \{\text{point}\}\). Thus \(A^0_{rad} \neq SL(2r + 1; \mathbb{R})\). Now \(A^0_{ss}\) is one of \(SO(2r + 1), SO(2r + 1; \mathbb{C})\), and \(SO(1, 2r + 1)\). The last one acts irreducibly on \(\mathbb{C}^{2r+2}\), and there \(A^0 = A^0_{ss}\) as above. For the first two, recall that \(SO(2r + 1)\) is absolutely irreducible on the tangent space \(so(2r + 2)/so(2r + 1)\) of the sphere \(S^{2r+1}\), so \(A^0_{rad}\) has Lie algebra reduced to 0, and again \(A^0_{ss} = A^0\). Now \(A^0\) is one of \(SO(2r + 1), SO(2r + 1; \mathbb{C})\), and \(SO(1, 2r + 1)\). If \(g \in G\) normalizes \(A^0\), then some

\[\text{1 The author thanks the referee for a comment that improved and clarified his treatment of this } SL(2r + 1; \mathbb{R}) \text{ case.}\]
element \( g' \in gA^0 \) centralizes \( A^0 \), because \( A^0 \) has no rational outer automorphism. Thus either \( A = A^0 \) or \( A/A^0 \) has order 2 where \( A \) is one of \( O(2r + 1), O(2r + 1; \mathbb{C}) \), and \( SO(1, 2r + 1) \cdot \{ \pm I \} \).

For (1.4), \( G = SO(7; \mathbb{C}) \) and \( B^0 = G_2 \). The semisimple Lie groups with maximal compact subgroup \( G_2 \) are \( G_2 \) and its complexification \( G_{2, \mathbb{C}} \). They are irreducible on \( \mathbb{C}^7 \) and have no rational outer automorphisms, so, as before, \( A^0 \) is either \( G_2 \) or \( G_{2, \mathbb{C}} \), and if \( g \in G \) normalizes \( A^0 \), then some element \( g' \in gA^0 \) centralizes \( A^0 \). This forces \( g' \) to be central in \( SO(7; \mathbb{C}) \), so \( g' = 1 \) and \( A = A^0 \). Thus \( A \) is either \( G_2 \) or \( G_{2, \mathbb{C}} \).

Finally, (1.5) is obtained from the case \( r = 3 \) of (1.3) by applying the triality automorphism, so it does not give us anything more.

In summary,

**Theorem 1.6.** Consider a complex flag manifold \( Z = G/Q \). Suppose that \( Z \) is irreducible, i.e., that \( G \) is simple. Then the closed subgroups \( A \subset G \) transitive on \( Z \), \( G \neq A \neq G \), are precisely those given as follows:

1. \( Z = SU(2n)/U(2n - 1) = P^{2n-1}(\mathbb{C}) \) complex projective \((2n - 1)\)-space; \( G = SL(2n; \mathbb{C}) \) and \( A = A^0 F \) where \( A^0 \) is one of \( Sp(n) \), \( Sp(n; \mathbb{C}) \) and \( SL(n; \mathbb{H}) \), and \( F \) is any subgroup of the center \( \{ e^{2\pi ik/2n} I \mid 0 \leq k < 2n \} \) of \( G \). Here \( F \) acts trivially on \( Z \), so \( A \) and \( A^0 \) have the same action on \( Z \).

2. \( Z = SO(2r + 2)/U(r + 1) \), unitary structures on \( \mathbb{R}^{2r+2} \); \( G = SO(2r + 2; \mathbb{C}) \) and \( A = A^0 F \) where \( A^0 \) is one of \( SO(2r + 1), SO(2r + 1; \mathbb{C}) \), and \( SO(1, 2r + 1) \), and where \( F \) is any subgroup of the center \( \{ \pm I \} \) of \( G \). Here \( F \) acts trivially on \( Z \), so \( A \) and \( A^0 \) have the same action on \( Z \).

3. \( Z = SO(7)/(SO(5) \cdot SO(2)), 5\)-dimensional complex quadric; \( G = SO(7; \mathbb{C}) \) and \( A \) is either the compact connected group \( G_2 \) or its complexification \( G_{2, \mathbb{C}} \).

Picking out the cases where \( A \) is a real form of \( G \) we have

**Corollary 1.7.** Consider a complex flag manifold \( Z = G/Q \). Suppose that \( Z \) is irreducible, i.e., that \( G \) is simple. Then the (connected) noncompact real forms \( G_0 \subset G \) transitive on \( Z \) are precisely those given as follows:

1. \( Z = SU(2n)/U(2n - 1) = P^{2n-1}(\mathbb{C}) \) complex projective \((2n - 1)\)-space; \( G = SL(2n; \mathbb{C}) \) and \( G_0 \) is the quaternion linear group \( SL(n; \mathbb{H}) \), which has maximal compact subgroup \( Sp(n) \).

2. \( Z = SO(2r + 2)/U(r + 1) \), unitary structures on \( \mathbb{R}^{2r+2} \); \( G = SO(2r + 2; \mathbb{C}) \) and \( G_0 \) is the Lorentz group \( SO(1, 2r + 1) \), which has maximal compact subgroup \( SO(2r + 1) \).

2. **The solution for flag manifolds in general**

We complete the solution of the problem by reducing it to the case where \( Z \) is irreducible.

**Proposition 2.1.** Decompose \( G = \prod G_i \), the local direct product of complex connected simple Lie groups. Thus \( Z = \prod Z_i \), the product of irreducible flag manifolds \( Z_i = G_i/Q_i \) where \( Q_i = Q \cap G_i \). Then \( A^0 = \prod A^0_i \) with \( A^0_i = A^0 \cap G_i \) and \( B^0 = \prod B^0_i \) with \( B^0_i = B^0 \cap G_i \). The groups \( A^0_i \) and \( B^0_i \) are connected, simple, and transitive on \( Z_i \).

**Proof.** The solvable radical of \( A^0 \) is contained in a Borel subgroup of \( G \), and thus has a fixed point on \( Z \). It is normal in the transitive group \( A^0 \) so it fixes every point. Thus \( A^0 \) is semisimple. Similarly \( B^0 \) is semisimple.
Let $\pi_i : G \to G_i$ denote the projection. The compact connected group $\pi_i(B^0)$ is transitive on $Z_i$. So it must be the compact real form $G_{a,i} = G_i \cap G_a$ of $G_i$ or one of the compact connected transitive groups described in (1.2), (1.3) or (1.4). (Recall that (1.5) is in fact a special case of (1.3).) In all cases, $\pi_i(B^0)$ is nontrivial and simple. Now $\pi_i$ annihilates all but one of the simple factors of $B^0$. Obviously no simple factor of $B^0$ is annihilated by every $\pi_i$. So now $B^0 = \prod B^0_\alpha$ where the $B^0_\alpha$ are simple and where the index set $I$ for $G = \prod_i G_i$ is a disjoint union of subsets $I_\alpha$ with $B^0_\alpha \subset \prod_{i \in I_\alpha} G_i$. The proof of Proposition 2.1 is reduced to the case where $B^0$ (and thus also $A^0$) is simple, and there it is reduced to the proof that $G_u$ is simple.

We may now assume $B^0$ simple. Suppose that $G_u$ is not simple. Projecting to $G_1 \times G_2$ we may assume $G = G_1 \times G_2$. View the isomorphisms $\pi_i : B^0 \cong \pi_i(B^0)$ as identifications. Denote $E_i = \pi_i(B^0)$, the complexification of the image of $B^0$ in $G_i$. Denote $E_{u,1} = \pi_i(B^0)$, the compact real form of $E_i$. Denote $P_i = E_i \cap Q_i$, the parabolic subgroup of $E_i$ that is its isotropy subgroup in $Z_i$, so $Z_i = E_i/P_i$. Now $B^0_\alpha = \{(e,e) | e \in E_i\}$, $B^0_C \cap Q = \{(p,p) | p \in (P_1 \cap P_2)\}$, and $Z = B^0_1/(B^0_1 \cap Q) \cong E_1/(P_1 \cap P_2)$. In particular $P_1 \cap P_2$ is a parabolic subgroup of $E_1$. Compute complex dimensions: dim $E_1 - \text{dim}(P_1 \cap P_2) = \text{dim} B^0 - \text{dim}(B^0 \cap Q) = \text{dim} Z = \text{dim} Z_1 + \text{dim} Z_2 = (\text{dim} E_1 - \text{dim} P_1) + (\text{dim} E_1 - \text{dim} P_2)$. On the Lie algebra level this says dim $\mathfrak{e}_1 = \text{dim} \mathfrak{p}_1 + \text{dim} \mathfrak{p}_2 - \text{dim}(\mathfrak{p}_1 \cap \mathfrak{p}_2)$, in other words $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{e}_1$. As $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is a parabolic subalgebra of $\mathfrak{e}_1$ we have a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{s}$ with $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$. In the root order such that $\mathfrak{s}$ is the sum of $\mathfrak{h}$ and the negative root spaces, no parabolic containing $\mathfrak{s}$ can contain the root space for the maximal root. This contradicts $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{e}_1$. The contradiction proves $G_u$ simple and completes the proof.

Combining Proposition 2.1 with Theorem 1.6 we have

Theorem 2.2. Let $Z = G/Q$, the complex flag manifold, where $G$ is a complex connected semisimple Lie group acting with finite kernel on $Z$. Then the closed subgroups $A \subset G$ transitive on $Z$ are precisely those given as follows. Decompose $G = \prod G_i$ with $G_i$ simple, so $Z = \prod Z_i$ with $Z_i = G_i/(Q \cap G_i)$. Then $A = A^0 F$ where $A^0 = \prod A_i$ with $A_i = (A \cap G_i)^0$, and $A_i$ is equal to $G_i$, or to its compact real form $G_{a,i}$, or to one of the three types listed in Theorem 1.6, and $F$ is any subgroup of the center of $G$. Here $F$ acts trivially on $Z$, so $A$ and $A^0$ have the same action on $Z$.

Picking out the cases where $A$ is a real form of $G$ we have, as in Corollary 1.7

Corollary 2.3. Let $Z = G/Q$, the complex flag manifold, where $G$ is a complex connected semisimple Lie group acting with finite kernel on $Z$. Then the (connected) real forms $G_0 \subset G$ transitive on $Z$ are precisely those given as follows. Decompose $G = \prod G_i$ with $G_i$ simple, so $Z = \prod Z_i$ with $Z_i = G_i/(Q \cap G_i)$. Then $A = \prod A_i$ where $A_i = A \cap G_i$ either is the compact real form $G_{a,i}$ of $G_i$ or is one of the two types listed in Corollary 1.7.

References


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