ON THE COMMUTANT OF OPERATORS OF MULTIPLICATION BY UNIVALENT FUNCTIONS

B. KHANI ROBATI AND S. M. VAEZPOUR

(Communicated by Joseph A. Ball)

Abstract. Let $B$ be a certain Banach space consisting of continuous functions defined on the open unit disk. Let $\phi \in B$ be a univalent function defined on $D$, and assume that $M_\phi$ denotes the operator of multiplication by $\phi$. We characterize the structure of the operator $T$ such that $M_\phi T = TM_\phi$. We show that $T = M_\phi$ for some function $\varphi$ in $B$. We also characterize the commutant of $M_\phi$ under certain conditions.

1. Introduction

Let $B$ be a Banach space consisting of continuous functions defined on the open unit disk $D$ such that $B$ satisfies conditions (1)–(6).

1. $1 \in B$, $zB \subset B$.

2. For every $\lambda \in D$ the evaluation functional at $\lambda$, $e_\lambda : B \to C$, given by $f \mapsto f(\lambda)$, is bounded.

3. $\dim \ker(M_\lambda - \lambda)^* = 1$ for every $\lambda \in D$.

4. If $f \in B$ and $f$ has an analytic extension to a neighborhood of $\lambda \in D$, then $\frac{f(\lambda)}{z-\lambda} \in B$. Also for every $\lambda \in D$ the subspace of $B$ consisting of those functions in $B$ that have analytic extension to a neighborhood of $\lambda$ is dense in $B$.

5. For every $f \in B$ the function $\bar{f}$ defined by $\bar{f}(\lambda) = f(-\lambda)$ is in $B$ and $\|\bar{f}\| = \|f\|$.

6. If $f \in B$ and $|f(z)| > 1/2$ for every $\lambda \in D$, then $\frac{1}{f}$ is a multiplier of $B$.

Throughout this article by a Banach space of continuous functions $B$ we mean one satisfying the above conditions. A complex valued function $\phi$ defined on $D$ is called a multiplier of $B$ if $\phi B \subset B$, i.e., $\phi f$ is in $B$ for every $f$ in $B$, and the set of all multipliers of $B$ is denoted by $\mathcal{M}(B)$. As it is shown in $[B]$ each multiplier $\phi$ is bounded on $B$. Given a multiplier $\phi$, let $M_\phi$, defined by $M_\phi(f) = \phi f$, denote the operator of multiplication by $\phi$. By the closed graph theorem $M_\phi$ is bounded. The algebra of all bounded operators on $B$ is denoted by $L(B)$. Let $X \in L(B)$ be a bounded operator on $B$ and $X M_\lambda = M_\lambda X$. It is easy to see that $X = M_\phi$ for some function $\varphi \in \mathcal{M}(B)$.

Received by the editors December 16, 1999.

2000 Mathematics Subject Classification. Primary 47B35; Secondary 47B38.

Key words and phrases. Commutant, multiplication operators, Banach space of analytic functions, univalent function, bounded point evaluation.

Research of the first author was partially supported by a national grant (no. 522).

©2001 American Mathematical Society
Throughout this article $\{M_\lambda\}'$ denotes the set of all bounded linear operators $X$ on $B$ such that $M_\lambda X = X M_\lambda$, i.e., the commutant of $M_\lambda$. Assume $T \in B^*$ and $f \in B$. We denote the value of $T$ at $f$ by $\langle f, T \rangle$. We define $M_\phi : B \to B$ by $M_\phi(f) = \varphi f$. By the closed graph theorem $M_\phi$ is bounded.

In what follows we present some examples of such spaces.

**Examples.**

a) Cole and Gamelin [2] proved that if $A$ is a $T$-invariant algebra on a compact set $K$, then for each $\lambda \in K$, ran($M_\lambda - \lambda$) is dense in $\ker \lambda$. Hence dim $\ker(M_\lambda^* - \lambda) = 1$ for every $\lambda \in K$. Also they have shown that every $T$-invariant algebra satisfies condition (4). Therefore the algebra of all continuous functions defined on $D$, i.e., $C(D)$, is a Banach space of continuous functions.

b) The disk algebra $A(D)$ which is the algebra of all continuous functions on the closure of disk that are analytic on $D$.

c) The Bergman space of analytic functions defined on the unit disk $L_1^2(D)$ for $1 \leq p \leq \infty$.

d) The spaces $D_\alpha$ of all functions $f(z) = \sum f(n)z^n$, holomorphic in $D$, for which

$$\|f\|^2 = \sum (n+1)^\alpha |f(n)|^2 < \infty$$

for every $\alpha \geq 1$ or $\alpha < 0$.

e) The analytic Lipschitz spaces $A_\alpha$ for $0 < \alpha < 1$, i.e., the space of all analytic functions defined on $D$ that satisfy a Lipschitz condition of order $\alpha$.

f) The subspace $A^\alpha$ of $A_\alpha$ consisting of functions $f$ in $A_\alpha$ for which

$$\lim_{z \to w} \frac{|f(z) - f(w)|}{|z - w|^{1/\alpha}} = 0.$$ 

g) The classical Hardy spaces $H^p$ for $1 \leq p \leq \infty$.

Shields and Wallen [6] studied the commutant of the operator $M_z$ on the Hilbert spaces of analytic functions. By a slight change in their methods one can obtain the commutant of $M_z$ on the Banach spaces of analytic functions. The commutant of a Toeplitz operator on certain Hilbert spaces of functions was studied by many mathematicians. See for example [1, 7, 8]. Cuckovic in [3] investigated the commutant of $M_{z^n}$ on the Bergman space $L_2^2(D)$. Seddighi and Vaezpour [4] have shown that under certain conditions on the reproducing kernels of a functional Hilbert spaces every operator $S$ essentially commuting with $M_z$ and commuting with $M_{z^n}$ for some $n > 1$ is a multiplication operator. Also the commutant of $M_z$ on a Banach space of analytic functions and the commutant of $M_{z^n}$ on a certain Hilbert space of functions were studied in [4]. In section 2 of this article we characterize the commutant of $M_\phi$ for a univalent function $\phi \in M(B) \cap A(D)$ on a Banach space of continuous functions and we investigate the commutant of $M_\phi^2$ under certain conditions.

2. The main results

**Lemma 2.1.** If $\phi \in M(B)$ and $T \in \{M_\phi\}'$, then $T^*(e_\lambda) \in \ker(M_\phi - \phi(\lambda))^*$ for every $\lambda \in D$.

**Proof.** Let $f \in B$. We have $\langle f, M_\phi^* T^*(e_\lambda) \rangle = \langle f, T^* M_\phi^*(e_\lambda) \rangle = \langle M_\phi T(f), e_\lambda \rangle = \phi(\lambda) T(f) = \phi(\lambda) (T(f), e_\lambda) = \phi(\lambda) (f, T^*(e_\lambda)) = (f, \phi(\lambda) T^*(e_\lambda)) = (f, \phi(\lambda) T^*(e_\lambda))$; hence $M_\phi^* T^*(e_\lambda) = \phi(\lambda) T^*(e_\lambda)$ which implies that $T^*(e_\lambda) \in \ker(M_\phi - \phi(\lambda))^*$.

\[\Box\]
Theorem 2.2. Let \( \phi \in \mathcal{M}(B) \cap A(D) \) be a univalent map. If \( T \in \{ M_\phi \}' \), then \( T = M_\psi \) for some function \( \psi \in \mathcal{M}(B) \).

**Proof.** Let \( \lambda \in D \). We show that \( \text{ran}(M_\phi - \phi(\lambda)) = \ker e_\lambda \). It is easy to see that \( \text{ran}(M_\phi - \phi(\lambda)) \subset \ker e_\lambda \).

To show the converse, let \( \phi - \phi(\lambda) = (z - \lambda)g(z) \) by the properties of \( B, \ g \in B \). Since \( \phi \) is univalent, \( g(z) \neq 0 \) on \( D \) and hence \( \frac{1}{g} \) is in \( \mathcal{M}(B) \). Now assume that \( f \in \ker e_\lambda \) so \( f(\lambda) = 0 \). Since the subspace of \( B \) consisting of functions which are analytic in a neighborhood of \( \lambda \) is dense in \( B \), it follows that there is a sequence \( \{ f_n \} \) of functions in this subspace such that \( f_n \) tends to \( f \). Now assume that \( f_n - f_n(\lambda) = (z - \lambda)g_n \) by property (4) of \( B, \ g_n \in B \). Hence
\[
\frac{f_n - f_n(\lambda)}{g_n(z)} = (\phi - \phi(\lambda))\frac{g_n(z)}{g(z)},
\]
which implies that \( (f_n - f_n(\lambda)) \in \text{ran}(M_\phi - \phi(\lambda)) \). Since \( f_n - f_n(\lambda) \) tends to \( f \) in \( B \), it follows that \( f \in \text{ran}(M_\phi - \phi(\lambda)) \). Now since \( (M_\phi - \phi(\lambda))^+(e_\lambda) = (M_\phi - \phi(\lambda))^+(e_\lambda) = 0 \) and \( \dim \ker(M_\phi - \phi(\lambda))^+ = 1 \), we conclude that \( T^+(e_\lambda) = \psi(\lambda)e_\lambda \) for some constant \( \psi(\lambda) \). Therefore, we have
\[
T(f)(\lambda) = \langle T(f), e_\lambda \rangle = \langle f, T^+(e_\lambda) \rangle = \psi(\lambda)\langle f, e_\lambda \rangle = \psi(\lambda)f(\lambda).
\]
Hence \( T(f) = \psi f \) for every \( f \in B \) and the proof is complete. \( \square \)

In the remainder of this section we investigate the commutant of \( M_\phi \) for some univalent function \( \phi \).

**Corollary 2.3.** If \( \phi \in \mathcal{M}(B) \cap A(D) \) is a univalent map such that \( \phi(D) \) has no distinct points which are symmetric with respect to the origin, then \( \{ M_\phi \}' = \{ M_\psi : \psi \in \mathcal{M}(B) \} \). In particular if \( |\lambda| > 1 \), then \( \{ M_{\frac{1}{\lambda} - \lambda} \}' = \{ M_\psi : \psi \in \mathcal{M}(B) \} \).

**Remark.** In the proofs of Lemma 2.1 and Theorem 2.2 we did not use property (5) of Banach space \( B \). Also \( D \) can be replaced by every bounded open set \( G \).

**Lemma 2.4.** Let \( \phi \) be a univalent odd function in \( \mathcal{M}(B) \cap A(D) \), \( S \in L(B) \) and \( SM_\phi = -M_\phi S \). Then there exists \( \psi \in \mathcal{M}(B) \) such that \( S = M_\psi \).

**Proof.** Since \( T^*M_\phi = -M_\phi T^* \) by a similar argument as in the proof of Lemma 2.1, we have \( M_\phi T^*(e_\lambda) = -\phi(\lambda)T^*(e_\lambda) \); hence \( (M_\phi + \phi(\lambda))^+(T^*(e_\lambda)) = 0 \) which yields \( T^*(e_\lambda) \in \ker(M_\phi + \phi(\lambda))^+ \). By the proof of Theorem 2.2, \( e_{-\lambda} \) spans \( \ker(M_\phi + \phi(\lambda))^+ \) so \( T^*(e_\lambda) = \psi(\lambda)e_{-\lambda} \) for some constant \( \psi(\lambda) \). Now
\[
\langle T(f), e_\lambda \rangle = \langle f, T^*(e_{0}) \rangle = \psi(\lambda)\langle f, e_{-\lambda} \rangle = \psi(\lambda)f(-\lambda)
\]
which implies that \( T = M_\psi \). Also \( T(f) = \psi f \); hence \( \psi \in \mathcal{M}(B) \). \( \square \)

**Theorem 2.5.** Let \( \phi \in \mathcal{M}(B) \cap A(D) \) be an odd univalent map. Let \( S \in \{ M_\phi \}' \) and \( SM_\phi - M_\phi S \) be a compact operator. Then there is some \( \Psi \in \mathcal{M}(B) \) such that \( S = M_\Psi \).

**Proof.** We have \( (SM_\phi - M_\phi S)M_\phi = (SM_\phi - M_\phi SM_\phi) = M_\phi S - M_\phi SM_\phi = -M_\phi (SM_\phi - M_\phi S) \). Hence, by Lemma 2.4 there exists some \( \psi \in B \) such that \( SM_\phi - M_\phi S = M_\psi \). Now we show that \( M_\psi \) is compact. Let the operator \( T \) be
defined by $T(f) = \hat{f}$; it is obvious that $T$ is continuous. Now we have $M_{\psi}(f) = M_{\psi}T(f)$ for every $f \in \mathcal{B}$ and so $M_{\psi}$ is compact and by the Fredholm alternative theorem $\psi = 0$. This implies that $M_{\psi}S = SM_{\psi}$; hence $S \in \{M_{\phi}\}'$ and we conclude that $S = M_{\Psi}$ for some $\Psi \in \mathcal{M}(\mathcal{B})$.

**Theorem 2.6.** Let $\phi \in \mathcal{M}(\mathcal{B}) \cap A(D)$ be an odd univalent map. Suppose $T$ is an operator in $\{M_{\phi}\}'$ and let $TM_{\phi} + M_{\phi}T$ be a compact operator. Then there exists a $\Psi \in \mathcal{M}(\mathcal{B})$ such that $T = M_{\Psi}$.

**Proof.** We have $(TM_{\phi} + M_{\phi}T)M_{\phi} = M_{\phi}(TM_{\phi} + M_{\phi}T)$; hence by Theorem 2.2, there is a function $\psi \in \mathcal{M}(\mathcal{B})$ such that $TM_{\phi} + M_{\phi}T = M_{\psi}$. Since $TM_{\phi} + M_{\phi}T$ is compact, we have $\psi = 0$, so $M_{\psi}T = -M_{\phi}T$. Now, by Lemma 2.4, there is $\Psi \in \mathcal{M}(\mathcal{B})$ such that $T = M_{\Psi}$. 

**Theorem 2.7.** Let $\phi \in \mathcal{M}(\mathcal{B}) \cap A(D)$ be a univalent map of $D$ onto $D$ such that $f \circ \phi$ and $f \circ \phi^{-1}$ are in $\mathcal{B}$ for every $f \in \mathcal{B}$. Let $S \in \{M_{\phi}\}'$. If polynomials are dense in $\mathcal{B}$, then $S(f) = \Phi f + \psi \frac{f + (f \circ \phi^{-1}) \circ \Phi}{2\phi}$, where $S(1) = \Phi$ and $(SM_{\phi} - M_{\phi}S)(1) = \psi$.

**Proof.** We define $T : \mathcal{B} \rightarrow \mathcal{B}$ by $T(f) = f \circ \phi^{-1}$. Clearly $T \in L(\mathcal{B})$ with inverse $T^{-1}(f) = f \circ \phi$. Since $M_{\phi}T = TM_{\phi}$, by induction we have $M_{\phi^n}T = TM_{\phi^n}$ for every positive integer $n$. Since $SM_{\phi^n} = M_{\phi^n}S$, it follows that $S(T^{-1}M_{\phi^n}T = T^{-1}M_{\phi^n}ST$ and so $TST^{-1} \in \{M_{\phi^n}\}'$. Now by a similar argument as in the proof of [1] Theorem 2.6] we have

$$TST^{-1}(f) = TST^{-1}(1)f + (TST^{-1}M_{\phi} - M_{\phi}TST^{-1})(1)\frac{f + \hat{f}}{2\phi}.$$ If $S(1) = \Phi$ and $(SM_{\phi} - M_{\phi}S)(1) = \psi$, then $TST^{-1}(1) = \Phi \circ \phi^{-1}$. Since

$$T(SM_{\phi} - M_{\phi}S)T^{-1} = TST^{-1}M_{\phi}TST^{-1},$$

it follows that $(TST^{-1}M_{\phi} - M_{\phi}TST^{-1})(1) = \psi \circ \phi^{-1}$. Hence

$$S(f) = T^{-1}TST^{-1}T(f) = \Phi f + \psi \frac{f + (f \circ \phi^{-1}) \circ \Phi}{2\phi}.$$ 

**Remark.** If $\phi$ in Theorem 2.7 is an odd function, then $S(f) = \Phi f + \psi \frac{f + \hat{f}}{2\phi}$.

**REFERENCES**


Department of Mathematics, Shiraz University, Shiraz 71454, Iran
E-mail address: Khani@math.susc.ac.ir

Department of Mathematics, Yazd University, Yazd, Iran