THE FEFFERMAN-STEIN TYPE INEQUALITY
FOR THE KAKEYA MAXIMAL OPERATOR

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Abstract. Let \( K_\delta, 0 < \delta \ll 1, \) be the Kakeya maximal operator defined as the supremum of averages over tubes of the eccentricity \( \delta. \) We shall prove the so-called Fefferman-Stein type inequality for \( K_\delta, \)

\[
\|K_\delta f\|_{L^p(\mathbb{R}^d;w)} \leq C_{d,p} \left( \frac{1}{\delta} \right)^{d/p-1} (\log \frac{1}{\delta})^{\alpha(d)} \|f\|_{L^p(\mathbb{R}^d;K_\delta w)},
\]

in the range \( 1 < p \leq (d^2 - 2)/(2d - 3), \) \( d \geq 3, \) with some constants \( C_{d,p} \) and \( \alpha(d) \) independent of \( f \) and the weight \( w. \)

1. Introduction

The purpose of this note is to investigate the so-called Fefferman-Stein type inequality for the Kakeya maximal operator. Throughout this note \( 0 < \delta \ll 1 \) will be a small parameter. For \( f \) a locally integrable function on \( \mathbb{R}^d, d \geq 2, \) define

\[
(K_h f)(x) = \sup_T \frac{1}{|T|} \int_T |f(y)| dy,
\]

where the supremum is taken over all tubes \( T \) containing \( x \in \mathbb{R}^d \) with the length \( h \) and the radius of the cross section \( h\delta. \) We define the Kakeya maximal operator \( K_\delta \) by

\[
(K_\delta f)(x) = \sup_{h>0} (K_{h\delta} f)(x).
\]

We call a non-negative Borel measurable function \( w \) a weight if it is a locally integrable function on \( \mathbb{R}^d. \) By \( w(A) \) we mean the \( w(x)dx \) measure of a set \( A. \)

It is verified for \( d = 2 \) that in the range \( 1 < p \leq d \) the Fefferman-Stein type inequality

\[
\left( \int_{\mathbb{R}^d} (K_\delta f)(x)^p w(x) dx \right)^{1/p} \leq C_{d,p,\epsilon} \left( \frac{1}{\delta} \right)^{d/p-1+\epsilon} \left( \int_{\mathbb{R}^d} |f(x)|^p (K_\delta w)(x) dx \right)^{1/p}
\]

holds for all \( \epsilon > 0 \) (Müller and F. Solia, [MS]). But in higher dimensions this fact has been verified only in the range \( 1 < p \leq (d+1)/2 \) (A. M. Vargas, [V]). The main difficulty of this problem lies in making the exponent \( p \) as high as possible.

Bourgain proved that an unweighted version of (1.1) (putting \( w \equiv 1 \)) holds in the range \( 1 < p \leq p_d, \) where \( (d+1)/2 < p_d < (d+2)/2 \) is some exponent given by

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a recursive formula starting from $p_3 = 7/3$ [Bo1]. Wolff improved this result [Wo]. He proved that an unweighted version of (1.1) holds in the range $1 < p \leq (d+2)/2$. Recently, in higher dimensions Bourgain improved it further to $1 < p \leq (1/2 + c)d$ ($c > 0$ independent of $d$) [Bo2].

A different approach to this problem (an unweighted version) was given by Igari. He investigated the most difficult case $p = d$. He proved that an unweighted version of (1.1) holds for a special basis [Ig]. He restricted the bases for taking the supremum to only tubes $T$ of which the axis intersects a fixed line. The author proved the weighted version of this restricted result [Ta2]. In this note we shall improve Vargas's result by using this restricted estimates.

The main theorem of this note is the following.

**Theorem 1.** Let $d \geq 3$. There exist constants $C_{d,p}$ and $\alpha(d)$ independent of $\delta$, $f$, and $w$ such that

$$
\|K_{\delta}f\|_{L^p(\mathbb{R}^d, w)} \leq C_{d,p}(\frac{1}{\delta})^{d/p-\frac{1}{p}}\left(\log\left(\frac{1}{\delta}\right)\right)^{\alpha(d)}\|f\|_{L^p(\mathbb{R}^d, K_{\delta}w)}
$$

holds in the range $1 < p \leq (d^2-2)/(2d-3)$.

By using sieve arguments and three-points interpolation lemma our result can be reduced to the discrete analogue as stated in the following theorem. (See [MS], [Va], and also [Th2].)

Let $Q = (-1/2, 1/2)^d$ and $\tilde{Q} = (-2, 2)^d$. We divide $\tilde{Q}$ into $\delta$-tubes, $Q_i$, centered at $i \in \mathcal{I}$, where $\mathcal{I}$ is the set of lattice points with the $\delta$-separation.

**Theorem 2.** Let $d \geq 3$. For a measurable set $A \subset Q$ and $0 < \lambda \leq 1$ define

$$
I = \{i \in \mathcal{I} : (K_{1,\delta\chi_A})(i) > \lambda\}.
$$

Then

$$
\sum_{i \in I} w(Q_i) \leq C_d(\frac{1}{\delta})^{d-p(d)}\left(\frac{1}{\lambda}\right)^{p(d)}\left(\log\left(\frac{1}{\delta}\right)\right)^{\beta(d)}(K_{\delta}w)(A),
$$

where

$$
p(d) = (d^2-2)/(2d-3)
$$

and

$$\beta(d) = (d+1)(d-2)/(2d-3).$$

In the following $C$'s will denote constants which may be different in each occasion but depend only on the dimension $d$.

2. **Proof of Theorem**

2.1. **Preliminaries.** We summarize some known results for later use.

Given any line $L$ in $\mathbb{R}^d$ define

$$
(K_{L,\delta}^Tf)(x) = \sup_{T} \frac{1}{|T|} \int_T |f(y)||dy,
$$

where the supremum is taken over all $\delta$-tubes $T$ which contain $x$ and of which the axis intersects $L$. Here, $\delta$-tube is the tube with the length 1 and the radius of the cross section $\delta$. 
Lemma 3 (Theorem 2 in [Ta2]). Let $d \geq 3$. Let $\lambda > 0$ and $L$ be any line in $\mathbb{R}^d$. Then
\[
w(\{x \in \mathbb{R}^d : (K_{1,\delta}^L)(x) > \lambda\}) \leq C(\frac{1}{\lambda})^d(\log(\frac{1}{\lambda}))^{d+1}\|f\|_{L^d_{w}(\mathbb{R}^d;K_{1,\delta}w)}.
\]

Let $B_{1/\delta}$ be the class of all rectangles in $\mathbb{R}^d$ which satisfy
\[
1 \leq \frac{\text{(the length of longest sides)}}{\text{(the length of shortest sides)}} \leq \frac{1}{\delta}.
\]

The corresponding maximal operator associated to this base $B_{1/\delta}$ will be denoted by $K_{1,\delta}^w$.

Lemma 4 (Theorem 3 in [Ta1]). Let $d \geq 2$. There exist constants $C_1$ and $C_2$ depending only on $d$ such that
\[
C_1(K_{\delta}f)(x) \leq (K_{1/\delta}f)(x) \leq C_2(K_{\delta}f)(x)
\]
holds for every $x \in \mathbb{R}^d$.

2.2. Main argument. Write $W = K_{\delta}w$. Fix $A \subset Q$ and $0 < \lambda \leq 1$. Recall $I = \{i \in I : (K_{1,\delta}^A)(i) > \lambda\}$. Then for every $i \in I$ we can select a $\delta$-tube $T_i$, which contains $i$, such that
\[
|A \cap T_i| > C\delta^{d-1}\lambda.
\]

Then, it follows from (2.1) and the Schwarz inequality that
\[
(C\delta^{d-1}\lambda \sum_{i \in I} w(Q_i))^2 \leq (\sum_{i \in I} w(Q_i)|A \cap T_i|)^2 = (\int_A \sum_{i \in I} w(Q_i) \chi_{T_i})^2 \leq \{ \int_A (\sum_{i \in I} w(Q_i) \chi_{T_i})^2 W^{-1} \} W(A) \leq \{ \sum_{i \in I} w(Q_i) \sum_{j \in I} w(Q_j) W^{-1}(T_i \cap T_j) \} W(A) \leq \max_{i \in I}(\sum_{j \in I} w(Q_j) W^{-1}(T_i \cap T_j)) \cdot (\sum_{i \in I} w(Q_i)) W(A).
\]

Hence
\[
N^{-1} \sum_{i \in I} w(Q_i) \leq C(\frac{1}{\delta})^{2(d-1)}(\frac{1}{\lambda})^2 W(A),
\]

which corresponds to low multiplicity of Wolff (see the proof of Lemma 3.1 in [Wo]), where
\[
N = \max_{i \in I}(\sum_{j \in I} w(Q_j) W^{-1}(T_i \cap T_j)).
\]

By the fact that $p(d) > 2$ and $\frac{1}{d} \geq 1$, we may assume that
\[
C\delta^{2(d-1)} \leq N.
\]

The following proposition, corresponding to high multiplicity of Wolff, will be proven later.
Proposition 5. With previous setup we have

(2.4) \[ N\{\sum_{i \in I} w(Q_i)\}^{(d-2)/(d-1)} \leq C \left( \frac{1}{\delta} \right)^{-d} \left\{ (\log(\frac{1}{\delta}))^{d+1} \left( \frac{1}{\lambda} \right)^d W(A) \right\}^{(d-2)/(d-1)}. \]

Multiplying both sides of (2.2) and (2.4) together, we obtain the desired inequality (1.2).

2.3. Proof of Proposition 5. Take some \( i_0 \in I \) so that

(2.5) \[ N = \sum_{j \in I} w(Q_j) W^{-1}(T_{i_0} \cap T_j). \]

Let

(2.6) \[ I_0 = \{ j \in I : T_{i_0} \cap T_j \neq \emptyset \} \]

and

(2.7) \[ s_0 = \inf_{y \in T_{i_0}} W(y). \]

By the geometric observation of Córdoa [Co] one sees that

(2.8) \[ |T_{i_0} \cap T_j| \leq C \frac{\delta^d}{\delta + \operatorname{dist}(T_{i_0}, T_j)}. \]

From (2.5)–(2.8) we have

(2.9) \[ N \leq C(s_0)^{-1} \sum_{j \in I_0} w(Q_j) \frac{\delta^d}{\delta + \operatorname{dist}(T_{i_0}, T_j)}. \]

Define the subset of \( I_0 \) as

\[ \sigma_k = \{ j \in I_0 : (k-1)\delta \leq \operatorname{dist}(T_{i_0}, j) < k\delta, \quad k = 1, 2, \ldots, \} \]

and rewrite

\[ (s_0)^{-1} \sum_{j \in I_0} w(Q_j) \frac{\delta^d}{\delta + \operatorname{dist}(T_{i_0}, T_j)} = (s_0)^{-1} \sum_k \sum_{j \in \sigma_k} w(Q_j) \frac{\delta^d}{\delta + \operatorname{dist}(T_{i_0}, T_j)}. \]

Then

(2.10) \[ N \leq C(s_0)^{-1} \delta^{d-1} \sum_k \sum_{j \in \sigma_k} \frac{w(Q_j)}{k}. \]

It follows for some \( k_0 \) to be specified later that

\[ \delta^{d-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} \frac{w(Q_j)}{k} = \delta^{d-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \left( \sum_{l=k}^{k_0} \frac{1}{l(l+1)} + \frac{1}{k_0+1} \right) \]

\[ = \{ \delta^{d-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \left( \sum_{l=k}^{k_0} \frac{1}{l(l+1)} \right) \} + \{ \frac{\delta^{d-1}}{k_0+1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \} \]

\[ = I + II. \]
By reversing the order of summation we have
\[
I = \delta^{d-1} \sum_{l=1}^{k_0} \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / ((l+1))
\]
\[
\leq C \delta^{d-2} \sum_{l=1}^{k_0} l \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / ((l\delta)^{d-1}).
\]

By using Lemma 4 we see that
\[
(s_0)^{-1} (\sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / ((l\delta)^{d-1}) \leq C (s_0)^{-1} \int_{R_l} w / |R_l| \leq C,
\]
where
\[
R_l = \{ x \in \mathbb{R}^d : \text{dist}(T_{i_0}, x) \leq l\delta \}.
\]
Hence
\[
(s_0)^{-1} I \leq C \delta^{d-2} \sum_{l=1}^{k_0} l \sum_{j \in \sigma_k} w(Q_j) / ((l\delta)^{d-1}) \leq C \delta^d (k_0\delta)^{d-2}.
\]
Combining these inequalities we obtain
\[
(2.11) \quad N \leq C \delta^d \{(k_0\delta)^{d-2} + (k_0\delta)^{-1} (s_0)^{-1} \sum_{j \in I_0} w(Q_j) \}.
\]

Now, we can choose some \(k_0\) so that
\[
(k_0\delta)^{d-1} \sim (s_0)^{-1} \sum_{j \in I_0} w(Q_j)
\]
by (2.3) and (2.10). Then the two terms in the right-hand side of (2.11) balance and hence
\[
(2.12) \quad N \leq C \delta^d \{(s_0)^{-1} \sum_{j \in I_0} w(Q_j) \}^{(d-2)/(d-1)}.
\]

Applying Lemma 3 with \(L = \text{the axis of } T_{i_0}\) and \(f = \chi_A\), we clearly obtain
\[
(2.13) \quad \sum_{j \in I_0} w(Q_j) \leq C (\frac{1}{\chi})^d (\log(\frac{1}{\delta}))^{d+1} W(A).
\]

Thus, from (2.12) and (2.13) we have
\[
(2.14) \quad N (s_0)^{(d-2)/(d-1)} \leq C \delta^d \{(\frac{1}{\chi})^d (\log(\frac{1}{\delta}))^{d+1} W(A) \}^{(d-2)/(d-1)}.
\]

Finally, again by Lemma 3 we observe that
\[
(2.15) \quad \sum_{i \in I} w(Q_i) \leq C \frac{w(Q)}{|Q|} \leq C' s_0.
\]
Thus, from (2.14) and (2.15) we obtain
\[
(2.16) \quad N \{ \sum_{j \in I} w(Q_j) \}^{(d-2)/(d-1)} \leq C \delta^d \{(\frac{1}{\chi})^d (\log(\frac{1}{\delta}))^{d+1} W(A) \}^{(d-2)/(d-1)}.
\]
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