THE FEFFERMAN-STEIN TYPE INEQUALITY
FOR THE KAKEYA MAXIMAL OPERATOR

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Abstract. Let \( K_\delta, 0 < \delta << 1 \), be the Kakeya maximal operator defined as the supremum of averages over tubes of the eccentricity \( \delta \). We shall prove the so-called Fefferman-Stein type inequality for \( K_\delta \),

\[
\|K_\delta f\|_{L^p(\mathbb{R}^d;w)} \leq C_{d,p} \left( \frac{1}{\delta} \right)^{d/p-1} (\log \left( \frac{1}{\delta} \right))^{\alpha(d)} \|f\|_{L^p(\mathbb{R}^d;K_\delta w)},
\]

in the range \( 1 < p \leq \frac{(d^2-2)(2d-3)}{d-1} \), \( d \geq 3 \), with some constants \( C_{d,p} \) and \( \alpha(d) \) independent of \( f \) and the weight \( w \).

1. Introduction

The purpose of this note is to investigate the so-called Fefferman-Stein type inequality for the Kakeya maximal operator. Throughout this note \( 0 < \delta \ll 1 \) will be a small parameter. For \( f \) a locally integrable function on \( \mathbb{R}^d, d \geq 2 \), define

\[
(K_{h,\delta} f)(x) = \sup_T \frac{1}{|T|} \int_T |f(y)| dy,
\]

where the supremum is taken over all tubes \( T \) containing \( x \in \mathbb{R}^d \) with the length \( h \) and the radius of the cross section \( h\delta \). We define the Kakeya maximal operator \( K_\delta \) by

\[
(K_\delta f)(x) = \sup_{h>0} (K_{h,\delta} f)(x).
\]

We call a non-negative Borel measurable function \( w \) a weight if it is a locally integrable function on \( \mathbb{R}^d \). By \( w(A) \) we mean the \( w(x)dx \) measure of a set \( A \).

It is verified for \( d = 2 \) that in the range \( 1 < p \leq d \) the Fefferman-Stein type inequality

\[
\left( \int_{\mathbb{R}^d} (K_\delta f(x))^p w(x) dx \right)^{1/p} \leq C_{d,p,\epsilon} \left( \frac{1}{\delta} \right)^{d/p-1+\epsilon} \left( \int_{\mathbb{R}^d} |f(x)|^p (K_\delta w)(x) dx \right)^{1/p}
\]

holds for all \( \epsilon > 0 \) (Müller and F. Solia, [MS]). But in higher dimensions this fact has been verified only in the range \( 1 < p \leq (d+1)/2 \) (A. M. Vargas, [Va]). The main difficulty of this problem lies in making the exponent \( p \) as high as possible.

Bourgain proved that an unweighted version of (1.1) (putting \( w \equiv 1 \)) holds in the range \( 1 < p \leq d \), where \( (d+1)/2 < p < (d+2)/2 \) is some exponent given by
a recursive formula starting from $p_3 = 7/3$ [Bo1]. Wolff improved this result [Wo]. He proved that an unweighted version of (1.1) holds in the range $1 < p \leq (d+2)/2$. Recently, in higher dimensions Bourgain improved it further to $1 < p \leq (1/2 + c)d$ ($c > 0$ independent of $d$) [Bo2].

A different approach to this problem (an unweighted version) was given by Igari. He investigated the most difficult case $p = d$. He proved that an unweighted version of (1.1) with $p = d$ holds for a special basis [Ig]. He restricted the bases for taking the supremum to only tubes $T$ of which the axis intersects a fixed line. The author proved the weighted version of this restricted result [Ta2]. In this note we shall improve Vargas’s result by using this restricted estimates.

The main theorem of this note is the following.

**Theorem 1.** Let $d \geq 3$. There exist constants $C_{d,p}$ and $\alpha(d)$ independent of $\delta, f,$ and $w$ such that

$$\|K_\delta f\|_{L^p(\mathbb{R}^d;w)} \leq C_{d,p}(\frac{1}{\delta})^{d/p - 1}(\log(\frac{1}{\delta}))^{\alpha(d)}\|f\|_{L^p(\mathbb{R}^d,K_{\delta}w)}$$

holds in the range $1 < p \leq (d^2 - 2)/(2d - 3)$.

By using sieve arguments and three-points interpolation lemma our result can be reduced to the discrete analogue as stated in the following theorem. (See [MS], [Va], and also [Ta2].)

Let $Q = (-1/2, 1/2)^d$ and $\tilde{Q} = (-2, 2)^d$. We divide $\tilde{Q}$ into $\delta$-tubes, $Q_i$ centered at $i \in I$, where $I$ is the set of lattice points with the $\delta$-separation.

**Theorem 2.** Let $d \geq 3$. For a measurable set $A \subset Q$ and $0 < \lambda \leq 1$ define

$$I = \{i \in I : (K_1,\delta \chi_A)(i) > \lambda\}.$$

Then

$$\sum_{i \in I} w(Q_i) \leq C_d(\frac{1}{\delta})^{d - p(d)}(\frac{1}{\lambda})^{p(d)}(\log(\frac{1}{\delta}))^{\beta(d)}(K_{\delta}w)(A),$$

where

$$p(d) = (d^2 - 2)/(2d - 3)$$

and

$$\beta(d) = (d + 1)(d - 2)/(2d - 3).$$

In the following $C$'s will denote constants which may be different in each occasion but depend only on the dimension $d$.

2. **Proof of Theorem 2**

2.1. **Preliminaries.** We summarize some known results for later use.

Given any line $L$ in $\mathbb{R}^d$ define

$$(K^L_{1,\delta}f)(x) = \sup_T \frac{1}{|T|} \int_T |f(y)||dy,$$

where the supremum is taken over all $\delta$-tubes $T$ which contain $x$ and of which the axis intersects $L$. Here, $\delta$-tube is the tube with the length 1 and the radius of the cross section $\delta$. 

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Lemma 3 (Theorem 2 in [Ta2]). Let \( d \geq 3 \). Let \( \lambda > 0 \) and \( L \) be any line in \( \mathbb{R}^d \).

Then

\[
w(\{x \in \mathbb{R}^d : (K_{1,\lambda}^L f)(x) > \lambda\}) \leq C(1)\lambda^{d}(\log(\frac{1}{\lambda}))^{d+1}||f||_{L^d(\mathbb{R}^d;K_{\delta}w)}^{d}.
\]

Let \( \mathcal{B}_{\leq 1/\delta} \) be the class of all rectangles in \( \mathbb{R}^d \) which satisfy

\[
1 \leq \text{(the length of longest sides)}/\text{(the length of shortest sides)} \leq \frac{1}{\delta}.
\]

The corresponding maximal operator associated to this base \( \mathcal{B}_{\leq 1/\delta} \) will be denoted by \( K_{\leq 1/\delta} \).

Lemma 4 (Theorem 3 in [Ta1]). Let \( d \geq 2 \). There exist constants \( C_1 \) and \( C_2 \) depending only on \( d \) such that

\[
C_1(K_{\delta}f)(x) \leq (K_{\leq 1/\delta}f)(x) \leq C_2(K_{\delta}f)(x)
\]
holds for every \( x \in \mathbb{R}^d \).

2.2. Main argument. Write \( W = K_{\delta}w \). Fix \( A \subset Q \) and \( 0 < \lambda \leq 1 \). Recall \( I = \{ i \in I : (K_{1,\lambda}^A f)(i) > \lambda\} \). Then for every \( i \in I \) we can select a \( \delta \)-tube \( T_i \), which contains \( i \), such that

\[
|A \cap T_i| > C\delta^{d-1}\lambda.
\]

Then, it follows from (2.1) and the Schwarz inequality that

\[
(C\delta^{d-1}\lambda \sum_{i \in I} w(Q_i))^2 \leq (\sum_{i \in I} w(Q_i)|A \cap T_i|)^2 = (\int_A \sum_{i \in I} w(Q_i)\chi_{T_i})^2 \leq \{(\int_A (\sum_{i \in I} w(Q_i)\chi_{T_i})^2W^{-1})W(A) \leq \{(\sum_{i \in I} w(Q_i) \sum_{j \in I} w(Q_j)W^{-1}(T_i \cap T_j))W(A) \leq \max_{i \in I}(\sum_{j \in I} w(Q_j)W^{-1}(T_i \cap T_j)) \cdot (\sum_{i \in I} w(Q_i))W(A).
\]

Hence

\[
N^{-1} \sum_{i \in I} w(Q_i) \leq C(\frac{1}{\delta})^{2(d-1)}(\frac{1}{\lambda})^2W(A),
\]

which corresponds to low multiplicity of Wolff (see the proof of Lemma 3.1 in [W])
where

\[
N = \max_{i \in I}(\sum_{j \in I} w(Q_j)W^{-1}(T_i \cap T_j)).
\]

By the fact that \( p(d) > 2 \) and \( \frac{1}{\lambda} \geq 1 \), we may assume that

\[
(C\delta^{2(d-1)} \leq N).
\]

The following proposition, corresponding to high multiplicity of Wolff, will be proven later.
Proposition 5. With previous setup we have

\[ N\left\{ \sum_{i \in I} w(Q_i)\right\}^{(d-2)/(d-1)} \leq C\left( \frac{1}{\delta} - d\right)\left\{ (\log \left( \frac{1}{\delta} \right))^{d+1} \left( \frac{1}{\lambda} \right)^d W(A) \right\}^{(d-2)/(d-1)}. \]

Multiplying both sides of (2.2) and (2.4) together, we obtain the desired inequality (1.2).

2.3. Proof of Proposition 5. Take some \( i_0 \in I \) so that

\[ N = \sum_{j \in I} w(Q_j) W^{-1}(T_{i_0} \cap T_j). \]

Let

\[ I_0 = \{ j \in I : T_{i_0} \cap T_j \neq \emptyset \} \]

and

\[ s_0 = \inf_{y \in T_{i_0}} W(y). \]

By the geometric observation of Córdoba one sees that

\[ |T_{i_0} \cap T_j| \leq C \frac{\delta^d}{\delta + \text{dist}(T_{i_0}, j)}. \]

From (2.5)–(2.8) we have

\[ N \leq C(s_0)^{-1} \sum_{j \in I_0} w(Q_j) \frac{\delta^d}{\delta + \text{dist}(T_{i_0}, j)}. \]

Define the subset of \( I_0 \) as

\[ \sigma_k = \{ j \in I_0 : (k - 1)\delta \leq \text{dist}(T_{i_0}, j) < k\delta \}, \quad k = 1, 2, \ldots, \]

and rewrite

\[ (s_0)^{-1} \sum_{j \in I_0} w(Q_j) \frac{\delta^d}{\delta + \text{dist}(T_{i_0}, j)} =\]

\[ = (s_0)^{-1} \sum_k \sum_{j \in \sigma_k} w(Q_j) \frac{\delta^d}{\delta + \text{dist}(T_{i_0}, j)}. \]

Then

\[ N \leq C(s_0)^{-1} \delta^{-1} \sum_k \sum_{j \in \sigma_k} w(Q_j) \frac{1}{k}. \]

It follows for some \( k_0 \) to be specified later that

\[ \delta^{-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \frac{1}{k} \]

\[ = \delta^{-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \left( \sum_{l=k}^{k_0} \frac{1}{l(l+1)} + \frac{1}{k_0+1} \right) \]

\[ = \delta^{-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \left( \sum_{l=k}^{k_0} \frac{1}{l(l+1)} \right) + \delta^{-1} \sum_{k=1}^{k_0} \sum_{j \in \sigma_k} w(Q_j) \frac{1}{k_0+1} \]

\[ = I + II. \]
By reversing the order of summation we have
\[
I = \sum_{l=1}^{k_0} \left( \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / (l(l + 1)) \right)
\]
\[
\leq C \delta^d - 2 \sum_{l=1}^{k_0} \left( \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / ((l \delta)^{d-1}) \right).
\]

By using Lemma 4, we see that
\[
(s_0)^{-1} \left( \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) / ((l \delta)^{d-1}) \right) \leq C(s_0)^{-1} \left( \int_{R_l} w \right) / |R_l| \leq C,
\]
where
\[
R_l = \{ x \in \mathbb{R}^d : \text{dist}(T_{t_0}, x) \leq l \delta \}.
\]
Hence
\[
(s_0)^{-1} I \leq C \delta^d \sum_{l=1}^{k_0} \left( \sum_{k=1}^{l} \sum_{j \in \sigma_k} w(Q_j) \right) \leq C \delta^d (k_0 \delta)^{d-2}.
\]
Combining these inequalities, we obtain
\[
(2.11) \quad N \leq C \delta^d \left\{ (k_0 \delta)^{d-2} + (k_0 \delta)^{-1} (s_0)^{-1} \sum_{j \in I_0} w(Q_j) \right\}.
\]
Now, we can choose some \(k_0\) so that
\[
(k_0 \delta)^{d-1} \sim (s_0)^{-1} \sum_{j \in I_0} w(Q_j)
\]
by (2.3) and (2.10). Then the two terms in the right-hand side of (2.11) balance and hence
\[
(2.12) \quad N \leq C \delta^d \left\{ (s_0)^{-1} \sum_{j \in I_0} w(Q_j) \right\} (d-2)/(d-1).
\]

Applying Lemma 3 with \(L = \text{the axis of } T_{t_0}\) and \(f = \chi_A\), we clearly obtain
\[
(2.13) \quad \sum_{j \in I_0} w(Q_j) \leq C \left( \frac{1}{\chi} \right)^d (\log(\frac{1}{\delta}))^{d+1} W(A).
\]
Thus, from (2.12) and (2.13) we have
\[
(2.14) \quad N(s_0)^{(d-2)/(d-1)} \leq C \delta^d \left\{ \left( \frac{1}{\chi} \right)^d (\log(\frac{1}{\delta}))^{d+1} W(A) \right\} (d-2)/(d-1).
\]
Finally, again by Lemma 3, we observe that
\[
(2.15) \quad \sum_{i \in I} w(Q_i) \leq C \frac{w(\hat{Q})}{|\hat{Q}|} \leq C' s_0.
\]
Thus, from (2.14) and (2.15) we obtain
\[
N \left( \sum_{j \in I} w(Q_j) \right)^{(d-2)/(d-1)} \leq C \delta^d \left\{ \left( \frac{1}{\chi} \right)^d (\log(\frac{1}{\delta}))^{d+1} W(A) \right\} (d-2)/(d-1).
\]
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