ON ALGEBRAIC POLYNOMIALS WITH RANDOM COEFFICIENTS

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Abstract. The expected number of real zeros and maxima of the curve representing algebraic polynomial of the form $a_0\left(\frac{n-1}{2}\right)^{1/2} + a_1\left(\frac{n-1}{1}\right)^{1/2}x + a_2\left(\frac{n-1}{2}\right)^{1/2}x^2 + \cdots + a_n\left(\frac{n-1}{n-1}\right)^{1/2}x^{n-1}$ where $a_j, j = 0, 1, \ldots, n-1$, are independent standard normal random variables, are known. In this paper we provide the asymptotic value for the expected number of maxima which occur below a given level. We also show that most of the zero crossings of the curve representing the polynomial are perpendicular to the $x$ axis. The results show a significant difference in mathematical behaviour between our polynomial and the random algebraic polynomial of the form $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ which was previously the most studied.

1. Introduction

The random algebraic polynomial is commonly defined as

$$Q(x) \equiv Q_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega)x^j,$$

where $(\Omega, \mathcal{A}, \Pr)$ is a fixed probability space, and $\{a_j(\omega)\}_{j=0}^{n-1}$ is a sequence of independent random variables defined on $\Omega$. The previous works on $Q(x)$ mainly assume identical distribution for the coefficients $a_j$’s. They include the pioneer works of Littlewood and Offord [7] and [8] and recent works of Wilkins [9] and Farahmand [5]. It is known that for identical standard normally distributed coefficients and $n$ sufficiently large the expected number of real zeros of $Q(x)$ is asymptotic to $(2/\pi) \log n$. However, there is little known about random polynomials with non-identical coefficients. Motivated by their close relation with physics, reported by Edelman and Kostland [1], as well as their mathematical interest, we assume that the coefficients $a_j, j = 0, 1, 2, \cdots, n-1$, have means zero and non-identical variances $(\frac{n-1}{j})$. This is the same as considering polynomials of the form

$$P(x) \equiv P_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega)\left(\frac{n-1}{j}\right)^{1/2}x^j$$
with the same assumption of identically normal standard distribution for $a_j$'s as above. Let $N(a, b)$ be the number of real zeros and $M_u(a, b)$ the number of maxima of $P(x)$ which occurs below a level $u$, in the interval $(a, b)$. In [6] it is shown that for $n$ sufficiently large $EN(-\infty, \infty) \sim EM_\infty(-\infty, \infty) \sim \sqrt{n}$. This is interesting as it shows that the curve representing $P(x)$ has a significantly larger expected number of real zero crossings and therefore oscillates more frequently than $Q(x)$. Also, unlike $Q(x)$, since $EN(-\infty, \infty) \sim EM_u(-\infty, \infty)$ asymptotically all the oscillations of $P(x)$ are between two zero crossings. Therefore, in order to obtain a better understanding of the mathematical behaviour of $P(x)$, it is of special interest to study the number of local maxima which occurs below a given level as well as the slope or the nature of crossings. To this end, as in [4] and [5], we define $S_u(a, b)$ as the number of up-crossings of $P(x)$ which possess a slope greater than $u$ or down-crossings with a slope less than $-u$. We define these crossings as $u$-sharp. Theorem 1 shows that there is no significant number of maxima which occurs below the $x$-axis while Theorem 2 gives a high level below which asymptotically all the maxima occur. By letting $u \to \infty$ as $n \to \infty$, Theorem 3 shows that asymptotically all the crossings are sharp. We prove the following theorems:

**Theorem 1.** For all sufficiently large $n$, the expected number of maxima which occurs below the $x$-axis is

$$EM_0(-\infty, \infty) \sim O(1).$$

**Theorem 2.** For $f_n \equiv f$ as any function of $n$ such that $f \to \infty$ as $n \to \infty$ and $u$ such that $u/f^2n\sqrt{n} \to \infty$ as $n \to \infty$

$$EM_u(-\infty, \infty) \sim \frac{\sqrt{n}}{2}.$$

**Theorem 3.** For any $u$ such that $u/\sqrt{n} \to 0$ as $n \to \infty$

$$ES_u(-\infty, \infty) \sim \frac{\sqrt{n}}{2}.$$

2. **Moments**

In order to be able to obtain estimates for the expected number of maxima below a level and the expected number of sharp crossings we need the following identities ((2.1) is well known and (2.2)-(2.6) are easily derived by consecutive differentiation of (2.1)):

$$A^2 = \text{var}\{P(x)\} = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{2j} = (x^2 + 1)^{n-1},$$

$$B^2 = \text{var}\{P'(x)\} = \sum_{j=0}^{n-1} j^2 \binom{n-1}{j} x^{2j-2} = (n-1)(x^2 + 1)^{n-3}(nx^2 - x^2 + 1),$$

$$C^2 = \text{var}\{P''(x)\} = \sum_{j=0}^{n-1} j^4 \binom{n-1}{j} x^{2j-4} = \cdots.$$
\[ C^2 = \var{P''(x)} = \sum_{j=0}^{n-1} j^2 (j-1)^2 \binom{n-1}{j} x^{2j-4} \]

(2.3) \[ D = \cov(P(x), P'(x)) = \sum_{j=0}^{n-1} j \binom{n-1}{j} x^{2j-1} = (n-1)x(x^2+1)^{n-2}, \]

(2.4) \[ E = \cov(P(x), P''(x)) = \sum_{j=0}^{n-1} j(j-1) \binom{n-1}{j} x^{2j-2} \]

(2.5) \[ F = \cov(P'(x), P''(x)) = \sum_{j=0}^{n-1} j^2(j-1) \binom{n-1}{j} x^{2j-3} \]

(2.6) \[ (n-1)(n-2)x(x^2+1)^{n-3}, \]

and

Then as is seen in [3] and [2]

\[ EM_a(a, b) = \int_a^b \int_{-\infty}^0 \int_{-\infty}^0 |z|p_x(t, 0, z) \, dz \, dt \, dx, \]

where \( p_x(t, y, z) \) denotes the three-dimensional density function for \( P(x), P'(x) \) and \( P''(x) \). Since from (2.1)-(2.6) the determinant of the covariance matrix of the above density function is

\[ |\Sigma| = A^2B^2C^2 - A^2E^2 - B^2E^2 - C^2D^2 + 2DEF \]

(2.7) \[ = 2(n-1)^2(n-2)(x^2+1)^{3n-9} \]

from [3] and [2] we have

(2.8) \[ EM_a(a, b) = \frac{1}{\pi} \int_a^b \frac{d}{2\sqrt{2a}} \left\{ \Phi \left( u\sqrt{2a} \right) - \frac{b}{2\sqrt{a}} \Phi \left( \frac{ub}{\sqrt{2a}} \right) \exp \left( -\frac{aeu^2}{c} \right) \right\} \, du \]

where from (2.1)-(2.7)

(2.9) \[ a = \frac{B^2C^2 - F^2}{2|\Sigma|} = \frac{2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2}{4(x^2+1)^{n-1}}, \]

(2.10) \[ b = \frac{DF - B^2E}{|\Sigma|} = \frac{x^2}{2(x^2+1)^{n-3}}, \]

(2.11) \[ c = \frac{A^2B^2 - D^2}{2|\Sigma|} = \frac{1}{4(n-1)(n-2)(x^2+1)^{n-5}}, \]

\[ e = c - \frac{k^2}{4a} \]

(2.12) \[ = \frac{nx^2 - x^2 + 1}{2(n-1)(n-2)(x^2+1)^{n-5}(2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2)}, \]

and

(2.13) \[ d = \frac{1}{\sqrt{|\Sigma|}.} \]
In the next section we will use (2.8) together with (2.9)-(2.13) to find the asymptotic estimate for $EM_u(-\infty, \infty)$. As far as the expected number of sharp crossings is concerned, from \[4\] we have

$$ES_u(a, b) = \int_a^b \Delta \exp \left( \frac{u^2 A^2}{2 \Delta^2} \right) dx$$

(2.14)

where from (2.1)-(2.3)

$$\Delta^2 = A^2 - D^2 = (n - 1)(x^2 + 1)^{2n-4}.$$  

(2.15)

We give the asymptotic value for $ES_u(1;1)$ in the final part of the paper.

Without loss of generality we only consider the interval $(0,1)$. However, since in all above identities the power of $x$'s are even we obtain our results for the entire real line.

3. Maxima below a level

(i). Level zero

Here we obtain the expected number of maxima below the $x$-axis. From (2.8) we can easily show

$$EM_0(0, \infty) = \frac{\sqrt{n - 2}}{4\pi} \int_0^{\infty} \frac{\sqrt{2nx^2 - 2x^4 + n^2x^4 - 3nx^4 + 2x^4 + 2}}{(x^2 + 1)(nx^2 - x^2 + 1)} dx.$$  

(3.1)

First we assume $x > \epsilon$ where $\epsilon = g_n/\sqrt{n}$ and $g_n$ is any function of $n$ such that $g_n \to \infty$ as $n \to \infty$. Then since with the above assumption $nx^2 \to \infty$ as $n \to \infty$ the first term that appears in the integrand of (3.1) is

$$\sqrt{2nx^2 - 2x^4 + n^2x^4 - 3nx^4 + 2x^4 + 2} = \sqrt{(n-1)(n-2)x^4 + 2(nx^2 - x^2 + 1)} = \sqrt{(n-1)(n-2)x^2 + Y}$$

where $Y$ is a function of $x$ and $n$ satisfies the relation $2(nx^2 - x^2 + 1) = Y^2 + 2Y \sqrt{(n-1)(n-2)x^2}$ and therefore tends to unity as $n \to \infty$. Hence from (3.1) letting $x = \tan \theta$ and since $\cot \arctan \epsilon = 1/\epsilon$ we obtain

$$EM_0(\epsilon, \infty) \sim \frac{\sqrt{n - 2}}{4\pi} \int_\epsilon^{\infty} \frac{dx}{(x^2 + 1)(nx^2 - x^2 + 1)}$$

$$\sim \frac{1}{4\pi \sqrt{n}} \int_\epsilon^{\infty} \frac{dx}{x^2(x^2 + 1)}$$

$$= \frac{1}{4\pi \sqrt{n}} \int_{\arctan \epsilon}^{\pi/2} \cot^2 \theta d\theta$$

$$\sim \frac{1}{4\pi \sqrt{n}} \sim \frac{1}{4\pi g_n}.$$  

(3.2)

Now we find the expected number of maxima in the interval $(0, \epsilon)$. To this end for all sufficiently large $n$ we have

$$EM_0(0, \epsilon) < \frac{\sqrt{n - 2}}{4\pi} \int_0^{\epsilon} \frac{\sqrt{nx^4 + 2nx^4 + 2 - x^2} \sqrt{(n-1)(n-2)}}{(x^2 + 1)(nx^2 - x^2 + 1)} dx$$

$$= \frac{\sqrt{n - 2}}{4\pi} \int_0^{\epsilon} \frac{(nx^2 + 1)^2 + 1 - x^2 \sqrt{(n-3/2)^2 - 1/4}}{(x^2 + 1)(nx^2 - x^2 + 1)} dx.$$  

(3.3)
Now we can show that both terms appearing in the numerator of the integrand in (3.3) can be written as
\[ \sqrt{(nx^2 + 1)^2 + 1} = (nx^2 + 1) + X_1 \]
and
\[ \sqrt{\left( n - \frac{3}{2} \right)^2 - \frac{1}{4}} = \left( n - \frac{3}{2} \right) + X_2 \]
where \( X_1 \) and \( X_2 \) satisfy relations
\[ X_1^2 + 2X_1(nx^2 + 1) - 1 = 0 \quad \text{and} \quad X_2^2 + 2X_2(n - 3/2) + 1/4 = 0. \]
Therefore \( X_1 < 1 \) and \( X_2 > -1/n \) are functions of \( n \) and \( x \). Hence from (3.3) letting \((\sqrt{n} - 1)x = \tan \theta \) we obtain
\[
EM_0(0, \epsilon) < \frac{\sqrt{n - 2}}{4\pi} \int_0^{\epsilon} \frac{2 \, dx}{(nx^2 - x^2 + 1)(x^2 + 1)}
\]
\[
< \frac{\sqrt{n - 2}}{2\pi} \int_0^{\epsilon} \frac{dx}{(n - 1)x^2 + 1}
\]
\[
\sim \frac{1}{2\pi} \int_0^{\arctan \epsilon/\sqrt{n-1}} d\theta
\]
\[
= O(1).
\]
This together with (3.2) completes the proof of Theorem 1.

(ii). High level
Since the second integrand that appears in (2.8) is positive we can obtain an upper limit for \( EM_u(0, \infty) \) as
\[
EM_u(0, \infty) < \int_0^{\infty} \frac{\sqrt{(n - 2)(2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2)}}{2\pi(1 + x^2)(nx^2 - x^2 + 1)} \times \Phi\left( \frac{unx^2}{\sqrt{n(x^2 + 1)^{n+1}/2}} \right) \, dx.
\]
Now in order to evaluate the above integral we divide the positive line into two subintervals \((0, \epsilon)\) and \((\epsilon, \infty)\) where \( \epsilon = \sqrt{f_n}/n \) and \( f_n \) is any function of \( n \) smaller than \( o(\sqrt{n}) \) such that \( f_n \to \infty \) as \( n \to \infty \). Then since in \((\epsilon, \infty)\)
\[ \frac{nx^2}{(1 + x^2)^{n-1}} \sim nf^{-2n}, \]
by the assumptions of Theorem 2 the term inside \( \Phi \) function tends to infinity as \( n \to \infty \). Hence from (3.4) we have
\[
EM_u(\epsilon, \infty) < \frac{\sqrt{n - 2}}{2\pi} \int_\epsilon^{\infty} \frac{\sqrt{2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2}}{(1 + x^2)(nx^2 - x^2 + 1)} \, dx
\]
\[
\sim \frac{\sqrt{n - 2}}{2\pi} \int_\epsilon^{\infty} \frac{dx}{1 + x^2}
\]
\[
\sim \frac{\sqrt{n - 2}}{4}.
\]
Also from (2.8) we obtain

$$EM_u(0, \epsilon) < \frac{\sqrt{n - 2}}{2\pi} \int_0^\epsilon \frac{\sqrt{(nx^2 - x^2 + 1)^2 + x^4 + 1}}{(1 + x^2)(nx^2 - x^2 + 1)} \, dx$$

$$= \int_0^\epsilon \frac{\sqrt{n - 2}}{2\pi(1 + x^2)} \sqrt{1 + \frac{1 + x^4}{nx^2 - x^2 + 1}} \, dx$$

$$< \frac{\sqrt{n - 2}}{2\pi} \int_0^\epsilon \frac{dx}{1 + x^2} = \sqrt{n - 2} \epsilon = o(f_n).$$

Therefore from (3.5) and (3.6) we have \(\sqrt{n}/4\) as an upper limit for \(EM_u(0, \infty)\). In order to obtain a lower limit again from (2.8) and for our assumption of \(u\), for all sufficiently large \(n\) we can say

$$EM_u(0, \infty) > EM_u(\epsilon, \infty)$$

$$\sim \int_\epsilon^\infty \frac{\sqrt{(n - 2)(2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2)}}{(1 + x^2)(nx^2 - x^2 + 1)} \, dx$$

$$\sim \frac{\sqrt{n - 2}}{2\pi} \int_\epsilon^\infty \frac{dx}{1 + x^2}$$

$$\sim \frac{\sqrt{n - 2}}{4}.$$  

Hence from (3.7) and the above upper limit we have the proof of Theorem 2.

4. SHARP CROSSINGS

Since from (2.1), (2.2), (2.4) and (2.5) \(\Delta^2/A^2 = (n - 1)(x^2 + 1)^{n-3}\) and \(\Delta/A^2 = \sqrt{n - 1}/(x^2 + 1)\), then from (2.14) we have

$$ES_u(0, \infty) = \frac{1}{\pi} \int_0^\infty \frac{\Delta}{A^2} \exp \left( -\frac{u^2A^2}{2\Delta^2} \right) \, dx$$

$$= \frac{\sqrt{n - 1}}{\pi} \int_0^\infty \exp \left( -\frac{u^2/\sqrt{n - 1}(x^2 + 1)}{(x^2 + 1)} \right) \, dx$$

$$= \frac{\sqrt{n - 1}}{\pi} \int_0^{\pi/2} \exp \left( -\frac{u^2\cos^2\theta}{n - 1} \right) \, d\theta$$

where \(x = \tan \theta\). Now we divide the interval \((0, \pi/2)\) into two subintervals \((0, \epsilon)\) and \((\epsilon, \pi/2)\) where \(\epsilon = 1/\sqrt{n}\). Then we can evaluate the integral which appears in (4.1) as

$$\int_0^{\pi/2} \exp \left( -\frac{u^2\cos^2\theta}{n - 1} \right) \, d\theta$$

$$> \int_0^\epsilon \exp \left( -\frac{u^2}{n - 1} \right) \, d\theta + \int_\epsilon^{\pi/2} \exp \left( -\frac{u^2\cos^2\theta}{n - 1} \right) \, d\theta$$

$$\sim \epsilon \exp \left( -\frac{u^2}{n - 1} \right) + \left( \frac{\pi}{2} - \epsilon \right) \exp \left( -\frac{u^2}{n - 1} \right)$$

$$\sim \epsilon \exp \left( -\frac{u^2}{n - 1} \right) + \frac{\pi}{2} \exp \left( -\frac{u^2}{n - 1} \right) - \left( \frac{u^2}{n - 1} \right)^{(n-3)/n}$$

$$\sim \frac{\pi}{2}.$$
Hence from (4.1) we obtain $(\sqrt{n} - 1)/2$ as a lower limit for $E S_u(0, \infty)$. Therefore, since $E S_u(0, \infty) < E S_0(0, \infty) \sim (\sqrt{n} - 1)/2$, from [6] we have the proof of Theorem 3.

References


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