ON THE REGULARITY OF \( p \)-BOREL IDEALS

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Abstract. In this paper we prove Pardue’s conjecture on the regularity of principal \( p \)-Borel ideals. As a consequence we obtain an upper bound for the regularity of general \( p \)-Borel ideals.

Introduction

Let \( K \) be field and \( I \subset S \) a graded ideal in the polynomial ring \( S = K[x_1, \ldots, x_n] \). Recall that the generic initial ideal \( \text{Gin}(I) \) of \( I \) with respect to the reverse lexicographical order is Borel-fixed. This means that \( \text{Gin}(I) \) is fixed under the action of the Borel group of the upper triangular invertible matrices acting linearly on the polynomial ring. By a theorem of Bayer and Stillman [2], the regularity of \( I \) and \( \text{Gin}(I) \) coincide. This is one of the reasons why one is interested in computing the regularity of \( \text{Gin}(I) \). In characteristic zero a Borel-fixed ideal is strongly stable, and so its regularity is simply the highest degree of a minimal generator. In positive characteristic however, Borel-fixed ideals are \( p \)-Borel (see 1.1 for the definition), and these are monomial ideals with a quite difficult combinatorial structure.

Monomials \( u_1, \ldots, u_m \in I \) of a \( p \)-Borel are called Borel generators of \( I \) if \( I \) is the smallest \( p \)-Borel ideal containing \( u_1, \ldots, u_m \). In this case we write \( I = \langle u_1, \ldots, u_m \rangle \). The ideal \( I \) is called principal \( p \)-Borel if \( I \) has only one Borel generator. Pardue conjectured a formula for the regularity of a principal \( p \)-Borel ideal which only depends on the exponents of the Borel generator (see 1.4). In a paper by Aramova and Herzog [1] it was shown that Pardue’s formula indeed gives a lower bound for the regularity. Some of the results in that paper were later extended by Ene, Pfister and Popescu [5] to more general ideals. In the present paper we will show that Pardue’s formula also yields an upper bound. Our method in proving this uses a criterion of Eisenbud, Reeves and Totaro [4] for determining the regularity of \( p \)-Borel ideals.

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1. \( p \)-Borel ideals

Throughout this paper we fix a field \( K \) and let \( S = K[x_1, \ldots, x_n] \) be the polynomial in \( n \) indeterminates over \( K \).
Let \( p \) be a prime number, and \( k \) and \( l \) be non-negative integers with \( p \)-adic expansion \( k = \sum_i k_ip^i \) and \( l = \sum_i l_ip^i \). We set \( k \leq_p l \) if \( k_i \leq l_i \) for all \( i \).

**Definition 1.1.** A monomial ideal \( I \subset S \) is \( p \)-Borel if the following condition holds: for each monomial \( u \in I \), \( u = \prod_i x_i^{\mu_i} \), one has \((x_i/x_j)^u \in I \) for all \( i, j \) with \( 1 \leq i < j \leq n \) and all \( \nu \leq_p \mu_j \).

The significance of \( p \)-Borel principal ideals is given by

**Proposition 1.2** (Pardue). Suppose \( \text{char } K = p \), and let \( I \subset S \) be a monomial ideal. Then \( I \) is \( p \)-Borel if and only if \( I \) is \( p \)-Borel.

We denote by \( G(I) \) the unique minimal set of monomial generators of a monomial ideal \( I \). It is easy to see (cf. [1]) that \( I \) is \( p \)-Borel if the conditions of [1.1] are satisfied for all \( u \in G(I) \).

A principal \( p \)-Borel ideal can be explicitly described. We use the following standard notation: If \( J \) is a monomial ideal we let \( J[p^n] \) be the ideal generated by all monomials \( u^p \) with \( u \in G(J) \). The ideal \( J[p^n] \) is called the \( p^n \)th Frobenius power of \( J \). Note that we define the \( p \)-th Frobenius power of monomial ideals in any characteristic.

**Proposition 1.3** (Pardue). Let \( u = \prod_i x_i^{\mu_i} \), and let \( \mu_i = \sum_i \mu_{ij}p^j \) be the \( p \)-adic expansion of \( \mu_i \) for \( i = 1, \ldots, n \). Then
\[
\langle u \rangle = \prod_{i=1}^n \prod_j ((x_1, \ldots, x_i)^{\mu_{ij}})[p^j].
\]

In particular, \( \langle u \rangle = \prod_{i=1}^n \langle x_i^{\mu_i} \rangle \).

It follows from [1.3] that \( \langle x^{\mu} \rangle = \langle x_1^{\mu_1} \rangle \langle x^n/x_1^{\mu_1} \rangle \), so that \( \text{reg} \langle x^{\mu} \rangle = \mu_1 + \text{reg} \langle x^n/x_1^{\mu_1} \rangle \). Therefore, if we are interested in the regularity of the \( p \)-Borel principal ideal \( \langle x^{\mu} \rangle \) we may assume that \( x_1 \) does not divide \( x^n \).

Denote by \( \lfloor \cdot \rfloor \) the greatest integer function, and for \( 1 \leq k \leq n \) and \( j \geq 0 \) define
\[
d_{kj}(\mu) = \sum_{i=1}^k \lfloor \frac{\mu_i}{p^j} \rfloor.
\]

For each \( k \) such that \( \mu_k \neq 0 \), let \( s_k = \lfloor \log_p \mu_k \rfloor \), and set
\[
D_k = d_{ks_k}(\mu)p^{s_k} + (k - 1)(p^{s_k} - 1).
\]

**Conjecture 1.4** (Pardue). If \( x_1 \) does not divide \( x^n \), then
\[
\text{reg} \langle x^{\mu} \rangle = \max_{k: \mu_k \neq 0} \{ D_k \}.
\]

In the following we will express the right-hand side of Conjecture 1.4 in different ways. The following easy fact can be found in [11].

**Proposition 1.5.** Let \( S = \{ s_k : \mu_k \neq 0 \} \), let \( q_j = \max \{ k : s_k = j \} \) for each \( j \in S \), and set \( E_j = D_{q_j} \). Then:
\[
\text{(i)} \ E_j = \sum_{i=j}^n \left( \sum_{k=2}^i \mu_k \right) p^i + (q_j - 1)(p^j - 1) \text{ for all } j \in S;
\]
\[
\text{(ii)} \ \max \{ D_k : \mu_k \neq 0 \} = \max \{ E_j : j \in S \}.
\]
We shall need still another reformulation of Pardue’s formula for the regularity of a principal $p$-Borel ideal. Set $s = \max\{s_k : \mu_k \neq 0\}$, and for each $t$ with $1 \leq t \leq s$ let $m_t = \max\{k : \mu_{kt} \neq 0\}$. Finally set

$$F_t = \sum_{i=t}^{s} \left( \sum_{k=2}^{n} \mu_{ki} \right) p^i + (m_t - 1)(p^t - 1) \text{ for all } t = 1, \ldots, s.$$ 

**Proposition 1.6.** With the notation introduced we have

$$\max_{1 \leq t \leq s} F_t = \max_{j \in S} E_j.$$ 

**Proof.** It is clear that $m_j \geq q_j$ for all $j \in S$, so that $\max_{1 \leq t \leq s} F_t \geq \max_{j \in S} E_j$.

In order to show the opposite inequality we first prove the following claim: Let $S_t = \{j \in S : j \geq t\}$ and $Q_t = \{q_j : j \in S_t\}$. (Note that $S_t \neq \emptyset$, since $s \in S_t$.) Let $e \in S_t$ such that $q_e = \max\{q_j \in Q_t\}$. Then we claim that $F_e \geq F_t$.

Indeed, we have

$$F_e - F_t = - \sum_{i=t}^{e-1} \left( \sum_{k=2}^{n} \mu_{ki} \right) p^i + (m_e - 1)(p^e - 1) - (m_t - 1)(p^t - 1).$$

Since we assume that $q_e$ is maximal in $Q_t$ it follows that $\mu_{kt} = 0$ for $k > q_e$ and $i \geq t$. Thus $m_e = q_e \geq \max\{k : \mu_{kt} \neq 0\} = m_t$, because again $\mu_{kt} = 0$ for $k > q_e$.

Now it follows that

$$F_e - F_t = - \sum_{i=t}^{e-1} \left( \sum_{k=2}^{n} \mu_{ki} \right) p^i + (q_e - 1)(p^t - 1) - (m_t - 1)(p^t - 1).$$

Finally, since $\sum_{i=t}^{e-1} \left( \sum_{k=2}^{n} \mu_{ki} \right) p^i = \sum_{k=2}^{q_e} \left( \sum_{i=t}^{e-1} \mu_{ki} p^i \right) \leq (q_e - 1)(p^e - p^t)$, we get $F_e - F_t \geq (q_e - 1)(p^e - 1) - (m_t - 1)(p^t - 1) = (q_e - m_t)(p^t - 1) \geq 0$. This concludes the proof of the claim.

Continuing with the proof of the opposite inequality, we let $t \leq s$ be the maximal number for which $F_t = \max_{1 \leq r \leq s} F_r$. Let $e \in S_t$ be chosen such that $q_e$ is maximal in $Q_t$. Then, according to our claim, we have $F_e \geq F_t$. By the choice of $t$ this implies that $e = t$, so that, in particular, $t \in S_t$. Since $q_e$ is maximal in $Q_t$ it now follows that $\mu_{kt} = 0$ for $i \geq t$ and $k > q_e$. Consequently, $m_t = q_e$, and so $F_t = E_t$. \hfill $\Box$

**Remark 1.7.** Using the methods of [1] the following result was proved in [3]: Let $(I_t)_{1 \leq t \leq s}$ be some stable ideals and $I = \prod_{t=1}^{s} I_t^{[p^t]}$ for some integers $0 \leq r_1 < \cdots < r_s$. If $I_j$ contains $x^{p^t_j p^{j-t}_j - 1}$ for all $1 \leq j < s$ (we set $m(u) = \max\{j : x_j | u\}$ for a monomial $u$) and $m(I_{j+1}) = \max\{m(u) : u \in G(I_{j+1})\}$, then $\text{reg}(I) = \text{pa}(I)$, where

$$\text{pa}(I) = \max_{1 \leq t \leq s} \left\{ \sum_{i=t}^{s} p^{r_i} \text{max}(I_i) + \max_{u \in G(I_t)} [p^{r_t} \deg(u) + (m(u) - 1)(p^{r_t} - 1)] \right\}.$$ 

Moreover if $I_t$ has the form $I_t = \prod_{i=2}^{n} (x_1, \ldots, x_i)^{p_{it}}$ with $0 \leq \mu_{it} < p$ for all $t < s$, the above result gives $\text{reg}(I) = \max_{1 \leq t \leq s} F_t$. Hence the Pardue Conjecture holds in a special case, which can also be obtained directly from [1]. Trying to extend the equality $\text{reg}(I) = \text{pa}(I)$ for general products of $p^t$th Frobenius powers of stable ideals one must consider first the following example which shows how tight Pardue’s Conjecture is: Let $n = 3$, $p = 2$, $I_1 = (x_1, x_2)^2$, $I_2 = (x_1, x_2, x_3)$ and $I = I_1 I_2^{[2]}$. Then $\text{pa}(I) = 4$, but $\text{reg}(I) > 4$, because $I$ is not stable (see Proposition 2.1 below).
2. The proof of Pardue’s conjecture

In [1] it is shown that if \( x^\mu \in S \) is a monomial which is not divisible by \( x_1 \), then

\[
\operatorname{reg}(x^\mu) \geq \max_{k : \mu_k \neq 0} \{ D_k \}.
\]

In this section we will prove the opposite inequality. Our proof is based on the following result [3]:

**Proposition 2.1** (Eisenbud, Reeves, Totaro). Let \( I \) be a \( p \)-Borel ideal with \( \max(I) = d \), and let \( e \geq d \) be the smallest integer such that \( I_{\geq e} \) is stable. Then \( \operatorname{reg}(I) = e \).

Proposition 2.1 needs some explanation: \( \max(I) = \max\{ \deg u : u \in G(I) \} \), and \( I_{\geq e} \) is the ideal generated by all monomials \( u \in I \) with \( \deg u \geq e \). Finally, recall that, according to Eliahou and Kervaire [3], a monomial ideal \( I \) is stable if for all monomials (or equivalently all generators) \( u \) of \( I \) one has \( (x_1/x_m(u))u \in I \) for all \( i \leq m(u) \), where \( m(u) = \max\{ j : x_j | u \} \).

Recall from Section 1 that \( (x^\mu) = \prod_t I_t^{p^{\mu_t}} \) where \( I_t = \prod_{i=2}^n (x_1, \ldots, x_{k_t})^{\mu_{ti}} \) with \( 0 \leq \mu_i < p \) and the following result [4]:

**Theorem 2.2.** For given integers \( 0 \leq r_1 < \cdots < r_s \), and integers \( 0 \leq a_k < p^{r_1+\cdots+\lambda_1} \) for \( t = 1, \ldots, s \) and \( k = 1, \ldots, m_t \) let \( I_t = \prod_{k=2}^{m_t} (x_1, \ldots, x_{k})^{a_{tk}} \) and \( I = \prod_{t=1}^s I_t^{p^{r_t}} \). Let \( \delta_t = \sum_{i=1}^m p^{r_t} \max(I_i) + (m_t - 1)(p^{r_t} - 1) \) and \( d = \max\{ \delta_t | 1 \leq t \leq s \} \). Then \( I_{\geq d} \) is stable.

The proof needs some preparation.

**Lemma 2.3.** Let \( J = \prod_{k=2}^n (x_1, \ldots, x_{k})^{a_k} \) with \( 0 \leq a_k < p^{r_1} - 1 \), and let \( \eta \in J \) be a monomial such that \( \deg \eta \geq 1 + \max\{ m(\eta)(p^{r_1} - 1), \max(J) + p^{r_1} - 1 \} \). Then there exists an integer \( t \) such that \( \eta \in x_1^{p^t} J \).

**Proof.** We reduce the problem to the case where \( m(\eta) \leq m \). Since \( J \) is stable, \( \eta \) has the following Eliahou-Kervaire decomposition: \( \eta = vw \) for monomials \( v \) and \( w \) with \( v \in G(J) \) and \( \min(w) \geq m(v) \). We may assume that \( w \notin (x_1^{p^t})^t \) for all \( t \). Then \( w = w'x_{m+1}^{\beta_{m+1}} \cdots x_{m(n)}^{\beta_{m(n)}} \) with \( \beta_i \leq p^r - 1 \). Thus the element \( \eta' = vw' \) has degree \( \deg \eta - (m(\eta) - m)(p^{r_1} - 1) \), \( m(\eta')(p^{r_1} - 1) - 1 \). Since \( m(\eta') \leq m \) and \( \max(J) \leq (m - 1)(p^{r_1} - 1) - 1 \), we may replace \( \eta \) by \( \eta' \), and thus may as well suppose that \( m(\eta) \leq m \).

Let \( \eta = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \). We apply induction on \( \max(J) \), and may assume that \( \alpha_m \neq 0 \). If \( \max(J) = 1 \), then \( J = (x_1, \ldots, x_m) \). If \( m(\eta) = 1 \), then \( \eta = x_1^{\alpha_1} \) with \( \alpha_1 \geq \max(J) + p^t = p^t + 1 \). In that case, \( \eta = x_1^{p^t} \eta' \) with \( \eta' = x_1^{\alpha_1 - p^t} \in J \). Suppose now that \( m(\eta) \geq 2 \). Then, since \( \deg \eta \geq m(\eta)(p^{r_1} - 1) + 1 \), it follows that \( \eta \in (x_1^{p^t})^t \) for a certain \( t \leq m(\eta) \), and so \( \eta = x_1^{p^t} \eta' \) where \( \eta' \) is a monomial of degree \( \geq (m(\eta) - 1)(p^{r_1} - 1) - 1 \). Hence \( \eta' \in J \), and so \( \eta \in x_1^{p^t} J \).

Now suppose that \( \max(J) > 1 \). We will distinguish several cases. In the first case suppose that \( \alpha_m(\eta) \geq p^t \). Let again \( \eta = vw \) be the Eliahou-Kervaire decomposition of \( \eta \). Then \( \deg w - \deg \eta = \deg v \geq p^t > 0 \). Hence, since \( m(v) \leq \min(w) \), it follows that \( x_1^{p^t} \) divides \( w \), and we are done.

Now we consider the case that \( \alpha_m(\eta) \leq p^t - 1 \), \( m(\eta) \geq 3 \) and \( \alpha_m(\eta) \leq \sum_{i=1}^m a_i \). We choose the maximal integer \( t, m(\eta) \leq t \leq m \), such that \( \alpha_m(\eta) \leq \sum_{i=1}^m a_i \), and
write $\alpha_{m(n)} = \sum_{i=t+1}^{m} a_i + b_t$ with $1 \leq b_t \leq a_t$. Now set $\varphi = \eta/x^{\alpha_{m(n)}}$. Observe that for all monomials $\rho \in J$ with $m(\rho) \leq m$ one has

$$\rho/x_{m(n)} \in \prod_{k=2}^{m-1} (x_1, \ldots, x_k)^{a_k}(x_1, \ldots, x_m)^{a_{m-1}}.$$  

Applying (1) successively we see that

$$J'' = \prod_{k=2}^{t-1} (x_1, \ldots, x_k)^{a_k}(x_1, \ldots, x_t)^{a_{t-1}}.$$  

We have

$$\deg \varphi = \deg \eta - \alpha_{m(n)} \geq \deg \eta - (p^r - 1) \geq 1 + (m(\eta) - 1)(p^r - 1) \geq 1 + m(\varphi)(p^r - 1),$$

and

$$\deg \varphi = \deg \eta - \alpha_{m(n)} \geq \max(J) + p^r - \alpha_{m(n)} = \max(J'') + p^r.$$  

Hence we may apply our induction hypothesis and conclude that there exists an integer $q \leq m(\varphi)$ such that $\varphi \in x_q^{p^r}J''$. It follows that $\eta \in x_q^{p^r}J$.

Next we consider the case $\alpha_{m(n)} \leq p^r - 1$, $m(\eta) \geq 3$, and $\alpha_{m(n)} > \sum_{i=m(n)}^{m} a_i$. Using (1) again we see that $\eta = x_{m(n)}^{\alpha_{m(n)}} \eta'$ with $\eta' \in \tilde{J}$ where

$$\tilde{J} = \prod_{k=2}^{m-n} (x_1, \ldots, x_k)^{a_k}.$$  

Note that for any monomial $\rho \in J$ with $m(\rho) > m$ it follows that $\rho/x_{m(n)} \in J$. Applying this successively to $\eta'$ we see that $\varphi = \eta/x_{m(n)}^{\alpha_{m(n)}}$ belongs to $\tilde{J}$. As in the second case it follows that $\deg \varphi \geq 1 + m(\varphi)(p^r - 1)$. Since on the other hand $\max(J') \leq (m(\eta) - 2)(p^r - 1)$, it also follows that $\deg \varphi \geq 1 + (m(\eta) - 1)(p^r - 1) \geq \max(J) + p^r$. Applying the induction hypothesis to $\varphi$ and $\tilde{J}$ yields the desired conclusion for $\eta$.

It remains to consider the case $\alpha_{m(n)} \leq p^r - 1$ and $m(\eta) \leq 2$. If $m(\eta) = 1$, then $a_1 \geq p^r$, a contradiction. Therefore $\eta = x_{\alpha_1}^{a_1}x_2^{a_2}$ with $a_2 \neq 0$ and $a_1 + a_2 \geq \max(2p^r - 1, \max(J) + p^r)$. It follows that $a_1 \geq p^r$. Then the element $\eta' = x_{\alpha_1-p^r}x_2^{a_2}$ belongs to $(x_1, x_2)^{\max(J)}$ which is contained in $J$, and so $\eta = x_1^{p^r} \eta' \in x_1^{p^r}J$.

**Corollary 2.4.** Let $J = \prod_{k=2}^{m} (x_1, \ldots, x_k)^{a_k}$ where $0 \leq a_k \leq p^r - 1$ for $k = 2, \ldots, m$, and let $q$ be a positive integer and $\eta \in J$ a monomial with $m(\eta) < q$ and $\deg \eta \geq 1 + \max\{(q-1)(p^r - 1), \max(J)\}$. Then there exists $t \leq m(\eta)$ such that $x_q^{p^r-1} \eta \in x_q^{p^r}J$.

**Proof.** Let $\eta' = x_q^{p^r-1} \eta$. We have $\deg \eta' \geq 1 + \max\{m(\eta')(p^r - 1), \max(J) + p^r\}$ since $m(\eta') = q$. Thus by Lemma 2.3 there exists an integer $t \leq m(\eta') = q$ such that $\eta' \in x_t^{p^r}J$, and hence $x_q^{p^r-1} \eta \in x_t^{p^r}J$. Since $p^r - 1$ is the maximal power of $x_q$ which divides $\eta'$, we have $t \neq q$ and so $t \leq m(\eta)$.

**Lemma 2.5.** Let $J = \prod_{k=2}^{m} (x_1, \ldots, x_k)^{a_k}$ with $0 \leq a_k \leq p^r-1$ for $k = 2, \ldots, m$ and integers $0 \leq e < r$. Let $I = J^{[p^e]}$ and $\eta \in I$ be a monomial such that $\deg \eta \geq 1 + \max\{m(\eta)(p^r - 1), \max(I) + p^r - p^e + m(\eta)(p^r - 1)\}$. Then there exists $t \leq m(\eta)$ such that $\eta \in x_t^{p^e}I$. 
Proof. We may write \( \eta = \sigma^p \psi, \psi \in G(J) \) and \( w = \sigma^p \sigma_0, \) where \( \sigma_0 \) and \( \sigma_1 \) are monomials, and \( \sigma_0 \notin (x_1^\alpha, \ldots, x_m^\beta). \) Thus \( \deg \sigma_0 \leq m(\eta)(p^e - 1) \) and the monomial \( \eta' = \psi \sigma_1 \) belongs to \( J. \) Since \( \eta = \eta' \sigma_0 \) it follows that
\[
\deg \eta' = \deg \eta - \deg \sigma_0 \geq \deg \eta - m(\eta)(p^e - 1) \\
\geq 1 + \max\{ m(\eta)(p^e - p^e), p^e \max(J) + p^e - p^e \}.
\]
Therefore \( \deg \eta' \geq (1/p^e) + \max\{ m(\eta)(p^e - 1), \max(J) + p^e - 1 \}. \) Since \( \deg \eta' \) is an integer we get \( \deg \eta' \geq \max\{ m(\eta)(p^e - 1), \max(J) + p^e - 1 \}. \) Note that \( m(\eta') \leq m(\eta). \) Therefore by Lemma 2.6, there exists an integer \( t, t \leq m(\eta') \leq m(\eta), \) such that \( \eta' \in x_t^{p^e} J. \) Thus \( \eta \in x_t^{p^e} I. \)

Applying Lemma 2.8 recursively we get

**Corollary 2.6.** With the hypotheses of Lemma 2.5 suppose in addition that \( \deg \eta \geq cp^e + 1 + \max\{ m(\eta)(p^e - 1), \max(I) + p^e - p^e + m(\eta)(p^e - 1) \} \) for some integer \( c \geq 0. \) Then there exists a monomial of degree \( c + 1 \) such that \( m(\sigma) \leq m(\eta) \) and \( \eta \in \sigma \psi. \)

**Lemma 2.7.** Let \( J = \prod_{k=2}^m(x_1, \ldots, x_k)^{a_k} \) with \( 0 \leq a_k \leq p^e - 1 \) for \( k = 2, \ldots, m \) and integers \( 0 \leq e < r. \) Let \( I = J^{[p^e]} \), and let \( q \) be a positive integer and \( \eta \in J \) a monomial with \( m(\eta) < q \) and \( \deg \eta \geq cp^e + 1 + \max\{ (q-1)(p^e - 1), \max(I) + (q-1)(p^e - 1) \} \) for some integer \( c \geq 0. \) Then there exists a monomial of degree \( c + 1 \) such that \( x_{q^{p^e - 1}} \eta \in \sigma \psi \) and \( m(\sigma) \leq m(\eta). \)

**Lemma 2.8.** Let \( I_t = \prod_{k=2}^m(x_1, \ldots, x_k)^{a_k} \) with \( 0 \leq a_k \leq p^e + 1 - e_i - 1, 1 \leq t \leq s, \) and integers \( 0 \leq e_1 < \cdots < e_s < e_{s+1}. \) Let \( I_t = \prod_{s=1}^s \{ I_t^{[p^e]} \}, \) and let \( q \) be a positive integer and \( \eta \in I \) a monomial with \( m(\eta) < q \) and \( \deg \eta \geq cp^e + 1 + \max_{1 \leq i \leq s}(\sum_{t=1}^i p^e \max(I_t) + (q-1)(p^e - 1)) \) for some integer \( c \geq 0. \) (Here \( \sum_{t=1}^i p^e \max(I_t) = 0 \) for \( s+1 \).) Then there exists a monomial of degree \( c + 1, \) \( m(\sigma) \leq m(\eta) \) and \( x_{q^{p^e - 1}} \eta \in \sigma \psi. \)

**Proof.** We apply induction on \( s. \) The case \( s = 1 \) is given in Lemma 2.7. Let \( d_j = \sum_{i=1}^s p^e \max(I_t) + (q-1)(p^e - 1), 1 \leq j \leq s+1, \) and let \( t \leq s \) be a maximal integer such that \( d_t = \max\{ d_j : 1 \leq j \leq s \}. \) Then we have \( d_t < d_k \) for \( t < k \leq s \) and \( d_t < d_j \) for \( j < t. \)

We now distinguish two cases. In case \( t > 1, \) write \( \eta = \psi \prod_{i=t}^{s+1} \psi_i^{e_i}, \) with \( \psi_i \in G(I_t) \) and \( \psi_i \in I' = \prod_{i=1}^{s+1} \{ I_t^{[p^e]} \}. \) We have \( \deg(\eta') = \deg(\eta) - \sum_{i=t}^s p^e \max(I_t) \geq cp^e + 1 + d_i - \sum_{i=t}^s p^e \max(I_t) = cp^e + 1 + (q-1)(p^e - 1). \) Choose the maximal integer \( \varepsilon \geq 0 \) such that \( \deg(\eta') \geq cp^e + 1 + \varepsilon(p^e - 1). \) As \( d_t = \max\{ d_i : 1 \leq i \leq s \}, \) we see that \( \eta' \) satisfies the necessary inequalities and so by induction hypothesis there exists a monomial \( \tau \) with \( \deg(\tau) = cp^e + \varepsilon + 1, m(\tau) \leq m(\eta') \leq m(\eta) \) and \( x_{q^{p^e - 1}} \tau = \tau \eta' \) for some \( \eta' \in I'. \) Note that by the choice of \( \varepsilon, \) we have \( \deg(\eta') \leq cp^e + 1 + \varepsilon(p^e - 1), \) and so \( p^e \deg(\tau) + \deg(\eta') \leq cp^e + 1 + \varepsilon(p^e - 1). \) Hence \( \deg(\eta'') \leq 1 - p^e + (q + 1)(p^e - 1) = q(p^e - 1). \)
Set \( \rho = \prod_{i=1}^{s-1} u_{i}^{p^{e_{i}}-1} \); then \( \rho \in \tilde{I} = \prod_{i=1}^{s} I_{i}^{[p^{e_{i}}]} \), and \( p^{r_{s}} \deg(\rho) = \deg(x_{q}^{e_{s}-1}) \). Therefore \( \deg(\rho) \geq cp^{r_{s}} + 1 \). We may apply the induction hypothesis to \( \rho, \tilde{I} \), and hence there exists a monomial \( \sigma \) with \( m(\sigma) \leq m(\rho) \leq m(\eta) \), \( \deg(\sigma) = c + 1 \) such that \( x_{q}^{e_{s}-1} \rho \in \sigma^{\rho^{r_{s}}} \tilde{I} \). Hence \( x_{q}^{e_{s}-1} \eta = x_{q}^{e_{s}-p^{r_{s}}} \rho^{r_{s}} \eta^{\prime} \in \sigma^{\rho^{r_{s}}} \).

We may suppose from now on that \( m(\rho) = m(\eta) \), \( m(\sigma) = m(\sigma^{\rho^{r_{s}}} \tilde{I}) \), and \( \deg(\sigma) = c + 1 \) such that \( x_{q}^{e_{s}-1} \psi \in p^{r_{s}} \tilde{I} \) for some monomial \( \psi \) with \( \deg(\psi) = c + 1 \) and \( m(\nu) \leq m(\psi) \leq m(\eta) \). This yields the desired conclusion.

We are now in the position to prove Theorem 2.2.

Proof of 2.2. Let \( \rho = \prod_{i=1}^{s-1} u_{i}^{p^{e_{i}}} w, u_{i} \in G(I_{i}) \) and \( w \) is a monomial such that \( \deg(\rho) = d \). Let \( j < m(\rho) \). We must show that \( x_{j} \rho / x_{m(\rho)} \in I_{2d} \). Apply induction on \( s \), case \( s = 0 \) being trivial. If \( m(\rho) = m(\eta) \), \( \eta = \prod_{t=1}^{s-1} u_{t}^{p^{e_{t}}} w \), then we may apply the induction hypothesis because \( \deg(\eta) = d - \deg(u_{t}^{p^{e_{t}}}) \geq \max\{d_{t}^{\prime} \leq t \leq s - 1 \} \) for \( d_{t}^{\prime} = \sum_{i=1}^{t-1} p^{e_{i}} \max(I_{i}) + (m_{t}-1)(p^{r_{t}} - 1) \). By the induction hypothesis \( I^{\prime} = \prod_{t=1}^{s-1} I_{t}^{[p^{e_{i}}]} \) has \( I_{2d}^{[\deg(\eta)]} \) stable and so \( x_{j} \eta / x_{m(\eta)} \in I_{2d}^{[\deg(\eta)]} \). Hence \( x_{j} \rho / x_{m(\rho)} = (x_{j} \eta / x_{m(\eta)})u_{j}^{p^{e_{j}}} \in I_{2d}^{[\deg(\eta)]} \).

We may suppose from now on that \( m(\rho) = m(u_{s}) > m(\eta) \). Set \( q = m(u_{s}) \), \( d_{t} = \sum_{i=1}^{t-1} p^{e_{i}} \max(I_{i}) + (q - 1)(p^{r_{t}} - 1) \). We have \( d_{t} \geq d_{t}^{\prime} \) if and only if \( m_{t-1} \geq q \). In particular \( d_{s} \geq d_{s}^{\prime} \) since \( m_{s} \geq q \). If \( m_{t} < q \), then \( \max(I_{t}) \leq (q - 1)(p^{r_{t}+1} - 1) \) and \( d_{t+1} - d_{t}^{\prime} = (q - 1)(p^{r_{t+1}} - 1) - (m_{t} - 1)(p^{r_{t}} - 1) - p^{r_{t}} \max(I_{t}) > 0 \). Thus \( d_{t} < d_{t+1} \) if \( m_{t} < q \). The same argument shows that \( d_{t} \leq d_{t+1} \) if \( m_{t} < q \).
By backwards induction on $j$ we now show that

$$\max\{d_i : j \leq t \leq s\} \leq \max\{\delta'_i : j \leq t \leq s\}. \tag{2}$$

We have already seen that (2) holds for $j = s$. Suppose the inequality holds for $j + 1$. If $m_j \geq q$, then $\delta'_j \geq d_j$, and so (2) is implied by the induction hypothesis.

If $m_j < q$, then $\delta'_j < d_{j+1}$, $d_j \leq d_{j+1}$, and hence $\delta'_j < d_{j+1} \leq \max\{d_i : j + 1 \leq t \leq s\} \leq \max\{\delta'_i : j + 1 \leq t \leq s\} = \max\{\delta'_i : j \leq t \leq s\}$. Hence $\max\{\delta'_i : j \leq t \leq s\} = \max\{\delta'_i : j + 1 \leq t \leq s\}$, and equality holds if $\max\{\delta'_i : j + 1 \leq t \leq s\}$, as desired.

Now since by (2) we have $\deg(x_j \eta) = 1 + \max\{\delta'_i : 1 \leq t \leq s\} = 1 + \max\{\delta'_i : 1 \leq t \leq s\}$, for $s - 1$, we may apply Lemma 2.8 for $I_s$ is stable.

For a monomial $u$, we set $pa(u) = \max_{k: \mu_k \neq 0}\{D_k\}$ if $u$ is not a multiple of $x_1$. Otherwise $u = x_1^{d_1}v$ such that $v \not\in (x_1)$, and we set $pa(u) = \mu_1 + pa(v)$ (cf. Section 1). By our main theorem we have $\text{reg}(u) = pa(u)$. More generally we get

**Corollary 2.9.** Let $I$ be a $p$-Borel ideal with Borel generators $u_1, \ldots, u_m$. Then

$$\text{reg}(I) \leq \max\{pa(u_1), \ldots, pa(u_m)\},$$

and equality holds if $I$ is principal $p$-Borel.

**Proof.** For each $i = 1, \ldots, m$, the ideal $(u_i)_{\geq d}$ is stable for $d \geq \text{pa}(u_i)$. Thus $I_{\geq d}$ is stable for $d \geq \max\{\text{pa}(u_1), \ldots, \text{pa}(u_m)\}$. Therefore the assertion follows from Proposition 2.1. \qed

**References**


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