Lipα HARMONIC APPROXIMATION ON CLOSED SETS

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(Communicated by J. Marshall Ash)

Abstract. In this paper the Lipα harmonic approximation (0 < α < 1/2) on relatively closed subsets of a domain in the complex plane is characterized under the same conditions given by S. Gardiner for the uniform case. Thus, the result of P. Paramonov on Lipα harmonic polynomial approximation for compact subsets is extended to closed sets. Moreover, the problem of uniform approximation with continuous extension to the boundary for harmonic functions and similar questions in Lipα harmonic approximation are also studied.

1. Introduction

Let \( F \) be a relatively closed subset of a domain \( G \) of the complex plane \( \mathbb{C} \). A. Roth proved in [25] that if \( f \) is a uniform limit on \( F \) of holomorphic or meromorphic functions, then it is possible to select the approximating function \( m \) in such a way that the difference function \( f - m \) can be extended continuously to the closure of \( F \), including the points of \( \partial F \cap \partial G \) for which \( f \) itself has no continuous extension. Furthermore if \( f \) is a Lipα limit of holomorphic or meromorphic functions (0 < α < 1), it is proved in [7] that it is possible to choose the approximating function \( m \) such that \( f - m \) belongs to \( \text{lip}(\alpha, G) \).

Roth’s result was extended in [8] to solutions of certain partial differential equations. Also, the main result obtained by M. Goldstein and W. Ow in [15], concerning the problem of uniform approximation with continuous extension to the boundary by harmonic functions, was improved in [9] by removing most of the unnecessary conditions assumed in their work. However, a mild restriction on \( G \) remains, namely that \( G \) is not dense in \( \mathbb{C} \). In this paper, this restriction is eliminated and we obtain analogous results to those in [7] and [9] for Lipα harmonic approximation. Another improvement of [18] was done by Gardiner [14] considering that the function to be approximated extends continuously to the closure of \( F \) in \( \mathbb{R}^n \cup \{ \infty \} \).

On the other hand, the characterization of the Lipα harmonic approximation on relatively closed subsets of a domain in the complex plane for 0 < α < 1/2 is perhaps the main question dealt with in this paper. We prove that, under the same conditions given by S. Gardiner [12] for the uniform case, not only is this approximation always possible, but also, as in the holomorphic and meromorphic cases, we can choose the approximating function \( m \) such that \( f - m \) belongs to \( \text{lip}(\alpha, F) \). Thus, we extend to closed sets a theorem of P. Paramonov about harmonic polynomial approximation.
approximation for compact subsets [22] and we obtain a result of decomposition of \( \text{Lip} \alpha \) approximable functions on \( F \) by harmonic functions on \( G \). This result is in the spirit of the papers of A. Stray ([26]) and S. Gardiner ([15]) dealing with the holomorphic and harmonic case respectively.

2. Preliminaries

Denote by \( h(G) \) the set of harmonic functions on \( G \) and by \( h(F) \) the class of harmonic functions on a neighbourhood of \( F \) relative to \( G \). Following [10], a function \( u \) is said to be \( \Delta \)-meromorphic on \( G \) if \( u \) is harmonic on \( G \), except for isolated singularities and if in a neighborhood of any such singularity \( y \), \( u \) can be represented in the form

\[
u(x) = s_y(x) + u_y(x)
\]

where \( u_y \) is harmonic at \( y \) and \( s_y(x) \) is a finite linear combination of \( E(x-y) \) and its derivatives, \( E \) being a fundamental solution of the Laplacian \( \Delta \). Such a singularity is named a pole. We denote by \( m(G) \) the set of \( \Delta \)-meromorphic functions on \( G \), by \( m(F) \) the set of \( \Delta \)-meromorphic functions on a neighbourhood of \( F \), and by \( m_F(G) \) the set of \( \Delta \)-meromorphic functions on \( G \) and having no poles on \( F \).

Let \( f \) be a bounded complex function on \( F \). As usual, the modulus of continuity \( w_f \) of \( f \) is given by

\[
w_f(r) = \sup \{|f(z) - f(w)| : z, w \in F, |z - w| \leq r\}.
\]

For \( 0 < \alpha < 1 \), consider the seminorm

\[\|f\|_{\alpha,F} = \sup \left\{ \frac{w_f(r)}{r^{\alpha}} : r > 0 \right\}\]

and the function spaces

\[\text{Lip}(\alpha,F) = \{f : F \to \mathbb{C} : \|f\|_{\alpha,F} < \infty\}\]

and

\[\text{lip}(\alpha,F) = \{f \in \text{Lip}(\alpha,F) : r^{-\alpha}w_f(r) \to 0, \text{ as } r \to 0^+\}\].

If \( f \) is defined on \( F \), we will say that \( f \) belongs to \( \text{lip}_{\text{loc}}(\alpha,F) \) if \( f \in \text{lip}(\alpha,K) \) for every compact subset \( K \) of \( F \). The \( \text{Lip} \alpha \) norm is defined by

\[\|f\|_{\alpha,F}^* = \|f\|_{\alpha,F} + \|f\|_{\infty,F}\]

where \( \|f\|_{\infty,F} \) is the supremum norm.

We define

\[a_\alpha(F) = \{f \in \text{lip}_{\text{loc}}(\alpha,F) : f \in h(F^\circ)\}\]

and

\[a_{\alpha\alpha}(F) = \{f \in a_\alpha(F) : \exists g \in \text{lip}(\alpha,\overline{F}) \text{ and } g|_F = f\}\]

where \( \overline{F} \) denotes the closure of \( F \) in \( \mathbb{C} \).

If \( A \) is a class of complex functions on \( F \), we will denote \( [A]_{\alpha,F}^* \) as the set of all functions which are limits in \( \text{Lip} \alpha \) norm on \( F \) of functions belonging to \( A \), i.e., \( f \in [A]_{\alpha,F}^* \) if and only if, for each \( \varepsilon > 0 \), there exists \( h \in A \) such that \( f - h \in \text{Lip}(\alpha,\overline{F}) \) and \( \|f - h\|_{\alpha,F} < \varepsilon \). Also, we use the notation \( G^\infty = G \cup \{\infty\} \) for the one point compactification of \( G \).
**Definition 1.** The function \( f : F \rightarrow C^\infty \) is said to be \( LE \)-approximable on \( F \) by functions from \( m(G) \) (respectively \( h(G) \)) if, given \( \varepsilon > 0 \), there are functions \( m \) and \( e \) with the following properties:

i) \( m \in m(G) \) (respectively \( h(G) \)), \( e \in h(F^o) \cap \text{lip}(\alpha, F) \),

ii) \( f(z) - m(z) = e(z) \) \( (z \in F) \),

iii) \( \|e\|_{\alpha, F}^* < \varepsilon \).

The letters \( LE \) stand for “\( Lipo \) extension”. We also introduce the notation

\[
\|f\|_{Lip1, E} = \sup \left\{ \frac{|f(z) - f(w)|}{|z - w|} : z, w \in E \right\}.
\]

Finally, to unify the notation, we understand the \( Lipo \) approximation with \( lip \) extension to the boundary as the uniform approximation with continuous extension to the boundary of \( F \) referred to \( C^\infty \) and we will denote from now on the uniform norm on \( F \) by \( \|\cdot\|_{0, F} \).

3. \( LE \)-approximation by \( \Delta \)-meromorphic and harmonic functions

In this section we prove that the approximation in \( Lipo \) norm by functions from \( m_F(G) \) is equivalent to the \( LE \)-approximation by \( \Delta \)-meromorphic functions. More precisely, we shall show that both kinds of approximations are equivalent to the approximation on compact subsets in \( Lipo \) norm. For this purpose, we need the following version of the Runge’s theorem for closed sets in this norm and a fusion lemma for harmonic functions in the \( Lipo \) norm.

**Proposition 1.** Let \( F \) be a closed subset of \( C \), and let \( g \) be any function \( \Delta \)-meromorphic in a neighbourhood of \( F \) and \( 0 \leq \alpha < 1 \). Then, given any \( \varepsilon > 0 \), there exists a \( \Delta \)-meromorphic function \( r \) in \( C \) such that

\[
\|r - g\|_{\alpha, F}^* < \varepsilon
\]

and

\[
\|r - g\|_{Lip1, F} < \varepsilon.
\]

Moreover, for fixed \( u \in K \), we can choose \( r \) such that \( r(u) = g(u) \).

**Proof.** This follows from [5] Theorem 1] considering the space \( V(F) = C^1(F) \) and the operator \( L \) as the Laplacian \( \Delta \). To be more precise and with the same notation as in [5], given \( \varepsilon > 0 \) and a \( \Delta \)-meromorphic function \( g \) in a neighbourhood of \( F \), we can obtain a \( \Delta \)-meromorphic function \( r \) in \( C \) such that \( r - g \) coincides with \( f \in C^1(\mathbb{C}) \) on \( F \) and \( \|f\|_{C^1(\mathbb{C})} < \varepsilon \). Then, by [5] Proposition 3] we get the \( Lip1 \) and \( Lipo \) approximation on \( F \). Note that \( r \) can be chosen such that, for a fixed point \( u \in K \), \((r - g)(u) = 0\) just by replacing \( r \) with \( r + g(u) - r(u) \).

**Proposition 2** (Fusion Lemma). Let \( 0 \leq \alpha < 1 \). Suppose that \( K_1 \) and \( K \) are compact subsets of \( C \) and \( K_2 \) is a closed subset of \( C \) such that \( K_1 \cap K_2 = \emptyset \) and \( K_1 \cup K_2 \cup K \neq C \). If \( r_1 \) and \( r_2 \) are two \( \Delta \)-meromorphic functions in \( C \) satisfying that \( \|r_1 - r_2\|_{\alpha, K_1}^* < \varepsilon \), then there exists a constant \( C = C(K_1, K_2) \) and a \( \Delta \)-meromorphic function \( r \) in \( C \) such that

\[
(1) \quad \|r - r_i\|_{\alpha, K \cup K_i}^* < C\varepsilon \quad (i = 1, 2)
\]

and

\[
(2) \quad \|r - r_i\|_{Lip1, K_i} < C\varepsilon \quad (i = 1, 2).
\]
Moreover, for fixed $u \in K_2$, we can choose $r$ such that
\[ (r - r_2)(u) = 0. \]

**Proof.** The proof follows ideas from [8], [9], [11] and [24], and as there, without loss of generality we may assume that $r_2 \equiv 0$, $K_1 \cap K \neq \emptyset$ and $K \cap K_2 \neq \emptyset$. Then we can choose neighborhoods $U_1$ and $U_2$ of $K_1$ and $K_2$ respectively, such that $U_1$ and $U_2$ have $C^1$ boundary, $\bar{U}_1 \cap U_2 = \emptyset$ and $U_2$ contains the exterior of a ball.

Let $M = \mathbb{C} \setminus (U_1 \cup U_2)$ and $H$ be an infinitely differentiable function with compact support such that $0 \leq H \leq 1$, $H|_{U_1} \equiv 1$ and $H|_{U_2} \equiv 0$.

By assumption, there exists a neighborhood $U$ of $K$ with $C^1$ boundary such that $\|r_1\|_{\infty, U} < C\varepsilon$. We set $h = r_1$ on $\bar{U} \cap M$ and extend this function to $M$ verifying $\|h\|_{0, M}^* \leq C\varepsilon$.

Now, we define
\[
 f(z) = \begin{cases} 
 h(z) & \text{if } z \in M, \\
 r_1(z) & \text{if } z \in \mathbb{C} \setminus M,
\end{cases}
\]
and
\[
 F(z) = V_H f(z) = \frac{1}{2\pi} E * H \Delta f = H f + g
\]
where
\[
 g = \sum_{|\gamma| \neq 0} C_{\gamma, \beta} (D^\beta E) * (f H^\gamma)
\]
for certain constants $C_{\gamma, \beta}$. Then, except for finitely many singularities in $U_1$, the function $F$ is $\Delta$-meromorphic in $U_1 \cup U_2 \cup U$. Moreover, since $K_1 \cup K_2 \cup K \neq \mathbb{C}$, there exists a ball $D = D(a, \delta)$ contained in $\mathbb{C} \setminus (K_1 \cup K_2 \cup K)$ and we can choose $\psi \in C^\infty(\mathbb{C})$, $\psi \equiv 1$ outside $D$ and $\psi \equiv 0$ on $\partial D$. Next, take $S = c_0 \psi E(x - a)$, where $c_0 = \frac{1}{2\pi} \int h \Delta H dm$ that verifies $|c_0| \leq C\|h\|_{0, D}$. Thus, if $B$ is a ball containing the support of $H$, it follows that
\[
 ||S||_{\alpha, B} \leq C\|h\|_{0, K} < C\varepsilon
\]
and, since $g - S$ satisfies the hypothesis of [21 Lemma 2] and $\|g\|_{\alpha, B(0, R)} \leq C\|f\|_{\alpha, K}$ (see [20 Lemma 2.4]),
\[
 ||g - S||_{\alpha, \mathbb{C}} < C\varepsilon.
\]

Now, if we define $F_1 = F - S = f H + g - S$, arguing as in [11, Theorem 3] we get that
\[
 ||F_1 - r_i||_{\alpha, K_1 \cup K_2} < C\varepsilon, \quad i = 1, 2,
\]

Although we can use $F_1$ to prove [11] and [2], in order to prove [3] we may consider $F_2 = F_1 - F_1(u)$, for $u$ a fixed point of $K_2$, which is harmonic in $U_1 \cup U_2 \cup U$ except for finitely many singularities in $U_1$. Applying Proposition [11] to $F_2$ we get a $\Delta$-meromorphic function $r$ such that
\[
 ||F_2 - r||_{\alpha, K_1 \cup K_2 < \varepsilon},
\]
\[
 ||F_2 - r||_{\text{Lip}1, K_1 \cup K_2 < \varepsilon}
\]
and $r(u) = F_2(u) = 0$. Thus we obtain
\[
 ||r - r_i||_{\alpha, K_1 \cup K_2} < C\varepsilon \quad (i = 1, 2)
\]
and \( r(u) = r_2(u) \) which prove (1) and (2). Finally, if we are able to show that

\[
\|g - S\|_{\text{lip}, K_1 \cup K_2} < C \varepsilon,
\]

by observing that \( r - r_1 = r - F_1 + g - S \) on \( K_1 \) and \( r - r = F_1 + g - S \) on \( K_2 \), we will conclude that (3) is satisfied and the theorem will be completely proved.

We now proceed to verify (4). Since \( E \) is an infinitely differentiable function except at the origin, if \( z_1 \) and \( z_2 \) belong to \( K_1 \cup K_2 \), then

\[
\left| \frac{g(z_1) - g(z_2)}{z_1 - z_2} \right| \leq \sum_{|\alpha| \neq 0} |C_{\alpha\beta}| \|f \|_{\infty, \mathbb{C}} \int_{\text{supp} DH} \left| \frac{D^{\beta} E(y - z_1) - D^{\beta} E(y - z_2)}{z_1 - z_2} \right| dm(y).
\]

Since the support of \( DH \) is at positive distance from \( K_1 \cup K_2 \), there exists \( \delta > 0 \) such that \( |y - z_i| > \delta \), for \( i = 1, 2 \) and \( y \in \text{supp} DH \). Owing to the fact that \( D^{\beta} E \in \text{Lip}(1, \mathbb{C} \setminus B(0, \delta)) \) it is readily seen that

\[
\|g\|_{\text{lip}, K_1 \cup K_2} < C \varepsilon.
\]

To finish, since \( S \) is defined as the product of \( E \) and a \( C^\infty \) function that is identically equal to 1 outside a disc, it follows that \( \|S\|_{\text{lip}, K_1 \cup K_2} < C \varepsilon \) which together with (4) gives (3).

We are now in a position to state the main result of this section which establishes the equivalence of the \( LE \)-approximation and the \( \text{Lip} \alpha \) approximation.

**Theorem 1.** Let \( F \) be a relatively closed subset of a domain \( G \) in \( \mathbb{C} \) and \( 0 \leq \alpha < 1 \). Then the following statements are equivalent:

(a) \( f \) can be approximated in \( \text{Lip} \alpha \)-norm on \( F \) by functions in \( m_F(G) \).
(b) If \( K \) is a compact subset of \( F \), then \( f|_K \in [m_K(\mathbb{C})]^{*}\alpha,K \).
(c) \( f \) is \( LE \)-approximable on \( F \) by functions in \( m_F(G) \).

Proof. (a) \( \Rightarrow \) (b) and (c) \( \Rightarrow \) (a) are trivial.

(b) \( \Rightarrow \) (c) Analogously to the proof of [2] Theorem 3.3, we may suppose that \( F \) is bounded. Thus, without loss of generality we can assume that \( \partial F \cap \partial G \) is bounded and \( \overline{B}(0, \rho) \cap \overline{F} = \emptyset \) for some \( \rho > 0 \). For \( n \geq 1 \), let \( \Omega_n = B(0, n) \setminus \overline{B}(0, \frac{1}{n}) \). Note that \( \overline{\Omega}_n \subset G_{n+1} \) and \( \bigcup \Omega_n = \mathbb{C} \setminus \{0\} \). Let \( G_n \) be a sequence of bounded domains such that \( \overline{G}_n \subset G_{n+1} \), \( \bigcup G_n = G \) and \( \text{dist}(\partial G_n, \partial F \cap \partial G) = \frac{1}{n} \), where \( \partial F \cap \partial G \) is a compact set. If we denote \( V_n = G_n \cap \Omega_n \), then \( \overline{V}_n \subset V_{n+1} \) and \( \bigcup V_n = G \setminus \{0\} \) and \( \overline{V}_n \subset G_n \).

For each \( n = 1, 2, \ldots \), we now apply the fusion lemma (Proposition 2) with \( K_1 = \overline{V}_n \), \( K_2 = \mathbb{C} \setminus B(0, n+1) \) and \( K = F_n = F \cap \overline{V}_{n+1} \). Then, there exist constants \( a_n \) that correspond to the constant \( C \) in Proposition 2 and we may assume that the \( a_n \) are increasing and \( a_n > 1 \).

Let \( \varepsilon > 0 \). By hypothesis \( f|_{F_n} \in [m_{F_n}(\mathbb{C})]^{*}\alpha,F_n \); hence we can find a \( \Delta \)-meromorphic function \( q_n \) without poles on \( F_n \) such that

\[
\|f - q_n\|^{*}_{\alpha,F_n} < \frac{\varepsilon}{2^{n+1}a_n(n+1)}
\]

for \( n = 1, 2, \ldots \). Thus, since \( F_n \subset F_{n+1} \), \( n = 1, 2, \ldots \), we have

\[
\|q_{n+1} - q_n\|^{*}_{\alpha,F_n} < \frac{\varepsilon}{2^{n}a_n(n+1)}.
\]
Now, for each $n$, there exists a $\Delta$-meromorphic function $r_n$ which satisfies Proposition 3 and, by following ideas of [7], this implies the Lip $\alpha$ convergence on $C^\infty \setminus G$ of

$$g_n(z) = \sum_{k=1}^{n-1} (r_k(z) - q_{k+1}(z))$$

as $n \to \infty$. Besides on $\partial F \setminus \partial G$, since $\text{lip}(\alpha, \partial F \setminus \partial G)$ is a closed subalgebra, $g_n$ converges to a function $\phi \in \text{lip}(\alpha, \partial F \setminus \partial G)$.

Finally, we define

$$m(z) = \sum_{k=1}^{n-1} (r_k - q_{k+1}) + q_n + \sum_{k=n}^\infty (r_k - q_k)$$

and

$$e(z) = \begin{cases} f(z) - m(z) & \text{if } z \in F, \\ \phi(t) & \text{if } z \in \overline{F} \setminus F. \end{cases}$$

Then, in a similar way as [11] Theorem 3.3] if $\alpha = 0$ and as [11] Theorem 3.3] if $0 < \alpha < 1$, it follows that $m(z) \in m_F(G \setminus \{0\})$, $e \in \text{lip}(\alpha, \overline{F})$ and

$$\|f - m\|_{\alpha, F} < \varepsilon.$$

If $0 \notin G$, the proof is finished. Otherwise, let $h$ be a harmonic function on $\mathbb{C}\setminus\{0\}$ such that $m - h$ has a removable singularity at $z = 0$. Since $h$ is harmonic on the compact subset $\overline{F}$ of $\mathbb{C}$, there exists $s \in m_{\overline{F}}(\mathbb{C})$ such that

$$\|h - s\|_{\alpha, \overline{F}} < \frac{\varepsilon}{2}.$$

Let $p = m - h - s$. Then $p$ is $\Delta$-meromorphic on $G$ and $f - p$ extends $\text{lip} \alpha$ to $\partial F \setminus \partial G$. This completes the proof. \hfill $\square$

The following corollaries are a direct consequence of the above theorem and the characterization of the Lip-$\alpha$-approximation of $a_\alpha(F)$ by $m_F(G)$ given for $\alpha = 0$ in [2] and for $0 < \alpha < 1$ in [20], [21] and [10]:

**Corollary 1.** All functions of $a_0(F)$ can be LE-approximated by functions in $m_F(G)$ if and only if $G \setminus F$ and $G \setminus F^\circ$ are thin at the same points.

**Corollary 2.** Let $0 < \alpha < 1$. All functions of $a_\alpha(F)$ can be LE-approximated by functions in $m_F(G)$ if and only if there exists a constant $C > 0$ such that

$$M^\alpha_*(D \setminus F^\circ) \leq CM^\alpha(D \setminus F)$$

for every disc $D \subset G$.

Now our goal is to study when the LE-approximation by functions in $h(G)$ is possible. In other words, we look for conditions on the domain $G$ and the closed set $F$ in order to apply a pushing poles method in Lip $\alpha$ norm for harmonic functions. The conditions that appear are the same as in Arakeljan’s Theorem for the uniform approximation ([1]). We collect them in the following theorem.

**Theorem 2.** If $G^\infty \setminus F$ is connected and locally connected at $\{\infty\}$, $0 \leq \alpha < 1$ and $m$ is a function in $m_F(G)$, the restriction $m|_F$ is LE-approximable on $F$ by functions in $h(G)$. 
Suppose that $f \in \text{Lip}(\alpha, \partial K)$ and let $F \in \text{Lip}(\alpha, \hat{K})$ and 

$$
\|F\|_{\alpha, \hat{K}} \leq A(\alpha, K, \Omega) \|f\|_{\alpha, \partial K}.
$$

Proof. Without loss of generality we can suppose that $\hat{K}$ is connected and $\hat{K} = K^o \cup \partial K^o$. Let $\eta = \text{dist}(\hat{K}, \partial \Omega)$ and $\beta < \frac{\eta}{2}$. We need only to show that if $\|f\|_{\alpha, \partial K} \leq 1$, then for any $x \neq a \in \hat{K}$ we have 

$$
|F(x) - F(a)| \leq A(\alpha)|x - a|^\alpha.
$$

Fix $x$ and $a$ in $\hat{K}$, put $b = x - a$, and $G(y) = F(y + b) - F(y)$. Then $G(y)$ is harmonic on the set $\hat{K}_1 = \{y \in \hat{K}^o : y + b \in \hat{K}^o\}$ and so attains its modulus maximum on $\partial \hat{K}_1$, that is, at some point $y$ with $y \in \partial \hat{K}$ or $y + b \in \partial \hat{K}$. So it is enough to consider the case $a \in \partial \hat{K}$ and $x \in \hat{K}^o$.

Denote by $K' = \{x \in \hat{K} : \text{dist}(x, \partial \hat{K}) \geq \beta\}$ and consider a first case where $a \in \partial \hat{K}^o$ and $x \in K'$. In this case, 

$$
|F(x) - F(a)| \leq 2 \frac{\|f\|_{\alpha, \partial \hat{K}}}{\beta^\alpha} |x - a|^\alpha.
$$

If $x \notin K'$, using the triangle inequality, we can reduce the situation to the case that $a$ is the nearest point to $x$ on $\partial \hat{K}$. Write $\delta = |x - a| = \text{dist}(x, \partial \hat{K})$. For $j = 1, 2, \ldots$ consider the sets 

$$
\Gamma_j = \{y \in \partial \hat{K} : (j - 1)\delta \leq |y - a| < j\delta\}. $$

Note that \( \Gamma_j \neq \emptyset \) only for \( j \leq \frac{d}{\delta} + 1 \), where \( d = \text{diam} \partial \mathring{K} \).

If \( \Omega \) is a domain in \( \mathbb{C} \) and \( E \subset \partial \Omega \) is a Borel set, write \( \omega(y, \Omega, E) \) for the harmonic measure of \( E \) at the point \( y \) relative to \( \Omega \). Put \( \omega_j = \omega(x, \mathring{K}^\circ, \Gamma_j) \). Let \( k_0 \) be such that \( k_0 \delta < \eta \leq (k_0 + 1)\delta \). Then for \( k = 1, 2, ..., k_0 \) we claim that

\[
(6) \quad \sum_{j > k} \omega_j \leq \frac{A}{\sqrt{k}}
\]

with \( A \geq 1 \) an absolute constant.

For \( k = 1 \), (6) is immediate because \( \sum_{j > 1} \omega_j \leq 1 \). For \( 2 \leq k \leq k_0 \) consider the connected component \( D_k \) of \( \{ y \in \mathring{K}^\circ : |y - a| \leq k\delta \} \) which contains \( x \). Note that by construction \( D_k \) is simply connected. From the properties of harmonic measure it follows that

\[
(7) \quad \sum_{j > k} \omega_j \leq \omega(x, D_k, S_k)
\]

where \( S_k = \{ y \in \partial D_k : |y - a| = k\delta \} \).

Consider a linear mapping from the disc \( \{ y \in \mathbb{R}^2 : |y - a| \leq k\delta \} \) onto \( \{ t \in \mathbb{R}^2 : |t| \leq 1 \} \). Let \( D'_k \) be the image of \( D_k \) under this mapping, \( S'_k \) the image of \( S_k \), and \( t \) the image of \( x \). Then \( |t| = \frac{1}{k} \) and according to the theorem of Milloux-Carleman ([18, pp. 347-350]), we have

\[
(8) \quad \omega(x, D_k, S_k) = \omega(t, D'_k, S'_k) \leq \frac{2}{\pi} \left( \frac{\pi}{2} - \arcsin \left( \frac{1 - \frac{1}{k}}{1 + \frac{1}{k}} \right) \right).
\]

Hence, if \( k \leq k_0 \), then \( \omega(x, D_k, S_k) \leq \frac{A}{\sqrt{k}} \) and thus (6) is proved.

Now from the definition of harmonic measure and the maximum principle we can get

\[
(9) \quad |F(x) - F(a)| \leq \sum_{j=1}^J (j\delta)^\alpha \omega_j
\]

where \( J \) is the integer part of \( \frac{d}{\delta} + 1 \). Indeed, consider the function \( G(y) = F(y) - F(a) \) which belongs to \( a(\mathring{K}) \) and satisfies

\[
|G(y)| = |f(y) - f(a)| \leq (j\delta)^\alpha
\]

for \( y \in \Gamma_j \). Then for all \( y \in \mathring{K}^\circ \)

\[
|G(y)| \leq \sum_{j=1}^J (j\delta)^\alpha \omega(y, \mathring{K}^\circ, \Gamma_j),
\]

and (10) follows from the last inequality by replacing \( y \) with \( x \).
Let us maximize the sum in (9) as a function of \( \omega = (\omega_1, \omega_2, ..., \omega_J) \) under the restrictions

\[
\begin{align*}
\omega_1 &\leq 1, \\
\omega_2 + \omega_3 + ... + \omega_J &\leq \frac{A}{\sqrt{1}} \\
\omega_3 + \omega_4 + ... + \omega_J &\leq \frac{A}{\sqrt{2}} \\
&\vdots \\
\omega_{k_0} + \omega_{k_0+1} + ... + \omega_J &\leq \frac{A}{\sqrt{k_0 - 1}} \\
\omega_{k_0+1} + \omega_{k_0+2} + ... + \omega_J &\leq \frac{A}{\sqrt{k_0}},
\end{align*}
\]

and \( \omega_j \geq 0 (j = 1, 2, ..., J) \). It is not difficult to see that the maximum is attained at \( \omega_1 = 1, \omega_j = \frac{\lambda_j}{\lambda_j - 1} = \frac{A}{\sqrt{j}} \) for \( j = 2, 3, k_0 \) and \( \omega_j \) for \( j = k_0, ..., J \) verifies the last restriction with the equality. Finally, from (11) we get

\[
|F(x) - F(a)| \leq \sum_{j=1}^{J} (j\delta)\omega_j \leq \sum_{j=1}^{k_0} (j\delta)\omega_j + \sum_{j=k_0+1}^{J} (j\delta)\omega_j \\
\leq \sum_{j=1}^{k_0} (j\delta)\omega_j + \frac{A(\text{diam}K)^\alpha}{\sqrt{k_0}} \leq A(\delta)^\alpha + \frac{A(\text{diam}K)^\alpha}{\sqrt{k_0}} \\
\leq \left(A(\alpha) + \frac{A(\text{diam}K)^\alpha}{\sqrt{\delta}}\right) \delta^{\frac{\alpha - 1}{2}} = A(\alpha, K, \Omega) \delta^\alpha.
\]

\[\square\]

**Theorem 4.** Suppose that under the conditions of the above theorem we also have \( f \in \text{lip}(\alpha, \partial K) \). Then \( F \) belongs to \( \text{lip}(\alpha, K) \).

**Proof.** Without loss of generality we can suppose that there exists a function \( \varepsilon(\delta) \) non-decreasing, such that \( \varepsilon(\delta) \to 0 \) as \( \delta \to 0 \) and

\[
|f(x) - f(y)| \leq \varepsilon(|x - y|)|x - y|^\alpha
\]

for any \( x, y \in \partial K \). Now repeating the proof of the above theorem we get

\[
|F(x) - F(a)| \leq \sum_{j=1}^{J} \varepsilon(j\delta)(j\delta)\omega_j \leq \sum_{j=1}^{k_0} \varepsilon(j\delta)(j\delta)\omega_j + \sum_{j=k_0+1}^{J} \varepsilon(j\delta)(j\delta)\omega_j \\
\leq \sum_{j=1}^{k_0} \varepsilon(j\delta)(j\delta)\omega_j + \frac{A\varepsilon(\text{diam}K)(\text{diam}K)^\alpha}{\sqrt{k_0}} \\
\leq \sum_{j=1}^{k_0} A\delta^\alpha \varepsilon(j\delta)j^{\alpha - \frac{1}{2}} + A\varepsilon(\text{diam}K)\delta^{\frac{\alpha - 1}{2}} \delta^\alpha = \delta^\alpha \varepsilon_1(\delta)
\]

where \( \varepsilon_1(\delta) \to 0 \) as \( \delta \to 0 \). \[\square\]
Our main result is the following:

**Theorem 5.** Let $F$ be a relatively closed subset of a domain $G \subset \mathbb{C}$ and $0 \leq \alpha < \frac{1}{2}$. Then the following statements are equivalent:

a) All functions of $a_\alpha(F)$ can be LE-approximated by functions in $h(G)$.

b) All functions of $a_\alpha(F)$ can be approximated in $\text{Lip}_\alpha$-norm on $F$ by functions in $h(G)$.

c) i) $\partial F = \partial \hat{F}$. 

ii) The holes of $F$ satisfy the long islands condition.

iii) $G^\infty \setminus \hat{F}$ is locally connected at $\{\infty\}$.

**Proof.** a) $\Rightarrow$ b) is trivial.

b) $\Rightarrow$ c) For $\alpha = 0$, see [12, Theorem 6] and [15, Corollary 1] (the conditions ii) and iii) together are equivalent to $(G, F)$ satisfies the $(K, L)$-condition defined in [18]). We consider now the case $0 < \alpha < \frac{1}{2}$. If $a_\alpha(F) = [h(G)]_{\alpha,F}^*$, it is readily seen that $h(F) \subset [h(G)]_{\alpha,F}^*$ and therefore $h(F) \subset [h(G)]_{0,F}^*$. Thus $(G, F)$ is a uniform Runge pair for harmonic functions, i.e., all functions of $h(F)$ can be uniformly approximated by functions from $h(G)$. Now by [12] and [4] the conditions i), ii) and iii) hold.

c) $\Rightarrow$ a) Suppose now that i), ii) and iii) hold. Since $G^\infty \setminus \hat{F}$ is connected, for each point $z \in \hat{F}$ we choose a disk $U_z \subset G$ such that the complement of $U_z \cap \hat{F}$ is connected. Take an exhausting sequence $\{G_n\}$ of $G$, where each $G_n$ is a bounded domain and $\partial G_n$ consists of finitely many Jordan curves. Now suppose $z \in \hat{F}_n = F \cap G_n$ and choose a disk $V_z \subset U_z$ such that the complement of $(V_z \cap \hat{F}_n)$ is connected and hence $f|_{V_z \cap \hat{F}_n}$ is approximable in $\text{Lip}_\alpha$-norm by harmonic functions in a neighborhood of $V_z \cap \hat{F}_n$ ([20]). Besides, by taking into account the localization theorem given by P. Paramonov and J. Verdera [21], we have that $f$ can be approximated in $\text{Lip}_\alpha$-norm by harmonic functions in a neighborhood of $\hat{F}_n$. Now, from Proposition [14] and Theorem [14] $f$ can be approximated in $\text{Lip}_\alpha$-norm on $\hat{F}$ by functions in $m_{\hat{F}}(G)$. Thus, since $G^\infty \setminus \hat{F}$ is connected and locally connected at $\{\infty\}$, by Theorem [2] one has that every function in $a_\alpha(\hat{F})$ can be LE-approximated by functions in $h(G)$.

The proof will be complete if it is proved that under these conditions every function in $a_\alpha(F)$ can be extended to a function in $a_\alpha(\hat{F})$. But the conditions i), ii) and iii) imply that $a_0(F) = [h(G)]_{0,F}^*$ and consequently all functions in $a_0(F)$ can be extended to a function in $a_0(\hat{F})$ (see [12, 4]). Hence all functions $f$ in $a_\alpha(F)$ can be extended to a function in $a_0(\hat{F}) \cap a_\alpha(F)$. On the other hand, the condition ii) guarantees that there exists a sequence $\{K_n\}$ of compact sets verifying:

a) $\hat{F} = \bigcup K_n$, $K_n \subset K_{n+1}$,

b) $\partial K_n \cap (\hat{F} \setminus F) = \emptyset$.

Hence $f \in a_0(K_n) \cap a_\alpha(\partial K_n)$, and if $0 < \alpha < \frac{1}{2}$ from Theorem [4] we prove that $f \in a_\alpha(\hat{F})$.

Remark that if $\frac{1}{2} < \alpha < 1$, then Theorem [5] is not true. It is still an open question if it holds for $\alpha = \frac{1}{2}$. Also observe that for $\mathbb{R}^n$, with $n \geq 3$ and $\alpha \in [0, 1]$, there is no purely geometric criterion in order for $a_\alpha(F)$ to coincide with the closure of harmonic functions on $G$ in the norm $\|\|_{\alpha,F}$ (see [22] for details).
Finally, an immediate consequence of the proof of Theorem 5 is the next result of decomposition for the class \([h(G)]_{a,F}^\alpha\).

**Corollary 3.** Let \(F\) be a relatively closed subset of \(G\) such that \(G^\infty \setminus F\) is connected and locally connected at \(\{\infty\}\) and \(0 \leq \alpha < 1\). Then \([h(G)]_{a,F}^\alpha = a_{\alpha u}(F) + h(G)\), i.e. if \(\varepsilon > 0\) and \(g \in [h(G)]_{a,F}^\alpha\), there exist \(g_1 \in a_{\alpha u}(F)\) and \(g_2 \in h(G)\) such that \(\|g_1\|_{a,F}^\alpha < \varepsilon\) and \(g = g_1 + g_2\) on \(F\).

**Acknowledgment**

The authors are grateful to the referee for several suggestions that improved the presentation of the paper.

**References**


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