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ON STABILITY OF C_0-SEMIGROUPS

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ABSTRACT. We prove that if $T(t)$ is a C_0-semigroup on a Hilbert space E, then (a) $1 \in \rho(T(\omega))$ if and only if $\sup\{\| \int_0^t \exp\{(2\pi ik)/\omega\}T(s)x\,ds\| \colon\ t \geq 0, k \in \mathbf{Z}\} < \infty$, for all $x \in E$, and (b) $T(t)$ is exponentially stable if and only if $\sup\{\| \int_0^t \exp\{i\lambda t\}T(s)x\,ds\| \colon\ t \geq 0, \lambda \in \mathbf{R}\} < \infty$, for all $x \in E$. Analogous, but weaker, statements also hold for semigroups on Banach spaces.

1. Introduction and main results

Let $T(t)$ be a strongly continuous one-parameter semigroup (C_0-semigroup) of bounded linear operators on a Banach space E, with generator A. The spectrum, resolvent set, and domain of A will be denoted by $\sigma(A)$, $\rho(A)$ and $D(A)$, respectively. The semigroup is called *exponentially stable* if there exist $M \geq 1$ and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}, \quad \forall t \geq 0,$$

and it is called *asymptotically stable* if $\lim_{t\to\infty} \|T(t)x\| = 0$, $\forall x \in E$.

In this paper, we prove the following theorems.

Theorem 1. *Suppose E is a Hilbert space and $T(t)$, $t \geq 0$, is a C_0-semigroup on E. If, for some $\omega > 0$,*

$$\sup_{t\geq 0, k\in\mathbf{Z}} \left\| \int_0^t e^{\frac{2\pi ik}{\omega}s} T(s)x\,ds \right\| < \infty, \quad \forall x \in E, \tag{1}$$

then $1 \in \rho(T(\omega))$. Conversely, if $T(t)$ is uniformly bounded and $1 \in \rho(T(\omega))$, then (1) holds.

Theorem 2. *Suppose E is a Hilbert space and $T(t)$, $t \geq 0$, is a C_0-semigroup on E. Then $T(t)$ is exponentially stable if and only if*

$$\sup_{\lambda\in\mathbf{R}, t\geq 0} \left\| \int_0^t e^{i\lambda s} T(s)x\,ds \right\| < \infty, \quad \forall x \in E. \tag{2}$$

Theorem 3. *Suppose E is a Banach space and $T(t)$, $t \geq 0$, is a C_0-semigroup on E, with generator A. If*

$$\sup_{t\geq 0} \left\| \int_0^t e^{i\lambda s} T(s)x\,ds \right\| < \infty\ \forall x \in E, \quad \lambda \in \mathbf{R}, \tag{3}$$

then

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(i) $\|T(t)x\| \to 0$ *as* $t \to \infty$ *for all* $x \in D(A^2)$.
(ii) *If, in addition to* (3), $T(t)$ *is uniformly bounded, then* $T(t)$ *is asymptotically stable.*
(iii) *If, in addition to* (3), $T(t)$ *is analytic, then* $T(t)$ *is exponentially stable.*

The proof of Theorems 1–3 is based on Lemma 1, which states that if

$$\left\|\int_0^t T(s)x\,ds\right\| \le M\|x\|, \quad \forall x \in E,$$

then $0 \in \rho(A)$, and moreover, $\|A^{-1}\| \le M$. This fact, and the Principle of Uniform Boundedness, imply that if the condition (2) holds, then the resolvent $R(\lambda) = (\lambda - A)^{-1}$ is uniformly bounded in $\operatorname{Re}\lambda > 0$. Theorem 2 will follow immediately from this and a well-known theorem that for semigroups on Hilbert space the uniformly boundedness of the resolvent in $\{\lambda \in \mathbf{C}\colon \operatorname{Re}\lambda > 0\}$ implies the exponential stability.[1] Theorems 1 and 3 also follow from Lemma 1 in a similar way.

Despite that the results and their proofs as given in this paper are elementary and very natural, to our knowledge they are new and may have some interest in the asymptotic behavior for C_0-semigroups. In fact, after the first version of this paper was complete and circulated, van Neerven has kindly pointed out to the author in a private communication that the fact that the condition (2) implies the uniform boundedness of the resolvent in $\operatorname{Re}\lambda > 0$ was obtained in [7, Corollary 5],[2] earlier and independently. The method of [7] is completely different (and more complicated) and, in our opinion, does not allow to obtain Theorems 1 and 2. Discrete analogs of our results also hold and are presented in Section 3.

2. Proofs

The following lemma plays the key role in our argument.

Lemma 1. *Assume that* E *is a Banach space and* $T(t)$, $t \ge 0$, *is a* C_0*-semigroup on* E*, with generator* A*. If*

$$\sup_{t\ge 0}\left\|\int_0^t T(s)x\,ds\right\| < \infty, \quad \forall x \in E, \tag{4}$$

then $0 \in \rho(A)$*. Moreover, if*

$$\left\|\int_0^t T(s)\,ds\right\| \le M, \quad \forall t \ge 0, \tag{5}$$

then $\|A^{-1}\| \le M$.

Proof. From (4) and the Principle of Uniform Boundedness it follows that there exists $M > 0$ such that (5) holds. Let us define, for every $\lambda > 0$, the following operator R_λ:

$$R_\lambda x \equiv \int_0^\infty e^{-\lambda t}T(t)x\,dt \equiv \lim_{r\to\infty}\int_0^\tau e^{-\lambda t}T(t)x\,dt, \quad \forall x \in E.$$

[1]This theorem is usually attributed to Gearhart who proved it for contraction semigroups [2]. For general C_0-semigroups it was obtained independently by Herbst [3], Howland [4] and Prüss [11]. See also [6].

[2]and hence the exponential stability, though this latter fact was not stated explicitly in [7], nor in [8].

From (4) and

$$\int_0^\infty e^{-\lambda t}T(t)x\,dt = \int_0^\infty e^{-\lambda t} d\int_0^t T(s)x\,ds$$
$$= e^{-\lambda t}\int_0^t T(s)x\,ds\Big|_0^\infty + \lambda\int_0^\infty \left(\int_0^t T(s)x\,ds\right)e^{-\lambda t}dt,$$

it follows that R_λ is well defined (for $\lambda > 0$) and is a bounded linear operator on E, and

$$R_\lambda x = \lambda\int_0^\infty \left(\int_0^t T(s)x\,ds\right)e^{-\lambda t}dt. \tag{6}$$

We show that $R_\lambda = (\lambda - A)^{-1}$. From the well-known formula

$$A\int_0^t T(s)x\,ds = T(t)x - x, \qquad x \in E,$$

it follows that

$$\begin{aligned} AR_\lambda x &= \lambda\int_0^\infty e^{-\lambda t}A\left(\int_0^t T(s)x\,ds\right)dt \\ &= \lambda\int_0^\infty e^{-\lambda t}(T(t)x - x)dt \\ &= \lambda R_\lambda x - x, \end{aligned} \tag{7}$$

and hence $(\lambda - A)R_\lambda x = x$. Since $T(t)Ax = AT(t)x$ for all x in $D(A)$, we have

$$R_\lambda(\lambda - A)x = (\lambda - A)R_\lambda x = x.$$

This implies, together with (7), that $R_\lambda = (\lambda - A)^{-1}$.

Now from (6) we have

$$\|(A-\lambda)^{-1}\| = \|R_\lambda\| \le \lambda M\int_0^\infty e^{-\lambda t}dt = M, \quad \forall\lambda > 0. \tag{8}$$

This implies that $0 \in \rho(A)$,

$$A^{-1}x = \lim_{\lambda\to 0^+}\int_0^\infty e^{-\lambda t}T(t)x\,dt,$$

and

$$\|A^{-1}\| \le M. \qquad \square \tag{9}$$

The following lemma is in a sense a converse to Lemma 1. It must be a well-known fact but we include a proof because we could not find a convenient reference.

Lemma 2. *If* $\|T(t)\| \le M$, $\forall t \ge 0$, *and* $0 \in \rho(A)$, *then*

$$\sup_{t\ge 0}\left\|\int_0^t T(s)ds\right\| \le (M+1)\|A^{-1}\|.$$

Proof. We have

$$\int_0^t T(s)x\,ds = \int_0^t T(s)AA^{-1}x\,ds$$
$$= \int_0^t d(T(s)A^{-1}x) = T(t)A^{-1}x - A^{-1}x.$$

Therefore,

$$\left\| \int_0^t T(s)x\,ds \right\| \le \|T(t)-I\|\,\|A^{-1}\|\,\|x\| \le (M+1)\|A^{-1}\|\,\|x\|. \qquad \square$$

Lemmas 1 and 2 imply the following statement.

Lemma 3. *Assume that E is a Banach space and $T(t)$, $t \ge 0$, is a C_0-semigroup on E, with generator A. Consider the following statements:*

(i) *For every $x \in E$*

$$\sup_{\lambda\in\mathbf{R},t\ge 0} \left\| \int_0^t e^{i\lambda s} T(s)x\,ds \right\| < \infty;$$

(ii) *$i\mathbf{R} \subset \rho(A)$ and*

$$M_1 \equiv \sup_{\lambda\in\mathbf{R}} \|(A-i\lambda)^{-1}\| < \infty. \tag{10}$$

Then (i) *implies* (ii). *If, in addition, $T(t)$ is uniformly bounded, then* (ii) *implies* (i).

Proof. Assume that (i) holds. By the Principle of Uniform Boundedness, there exists a constant M such that

$$\sup_{\lambda\in\mathbf{R},t\ge 0} \left\| \int_0^t e^{i\lambda s} T(s)\,ds \right\| \le M.$$

Applying Lemma 1 (and estimate (11)) to the semigroup $e^{-i\lambda t}T(t)$, we have $i\lambda \in \rho(A)$ and $\|(A-i\lambda)^{-1}\| \le M$, $\forall \lambda \in \mathbf{R}$.

Conversely, suppose that (ii) holds and $\sup_{t\ge 0}\|T(t)\| \equiv M < \infty$. Let

$$M_1 \equiv \sup_{\lambda\in\mathbf{R}} \|(A-i\lambda)^{-1}\|.$$

Applying Lemma 2 to the semigroup $e^{i\lambda t}T(t)$, we have

$$\sup_{t\ge 0,\lambda\in\mathbf{R}} \left\| \int_0^t e^{i\lambda t} T(s)\,ds \right\| \le (M+1)\|(A+i\lambda)^{-1}\| \le M_1(M+1).$$

$\square$

Remark. If (10) holds, then it follows from the well-known equality

$$\operatorname{dist}(i\lambda, \sigma(A)) \ge \frac{1}{\|(A-i\lambda)^{-1}\|} \tag{11}$$

that the *spectral bound* $s(A)$ of A is negative, where

$$s(A) \equiv \sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma(A)\}.$$

In fact, (10) and (11) imply

$$s(A) \le -\frac{1}{M_1}.$$

As mentioned in the Introduction, the implication (i) $\Rightarrow$ (ii) in Lemma 3 was obtained recently by van Neerven [7, Theorem 4] (see also [8, Proposition 4.5.3]), by an entirely different method (even though he does not formulate it in that way).[3]

The following lemma can be proved completely analogously to Lemma 3.

Lemma 4. *Assume that E is a Banach space and $T(t)$, $t \geq 0$, is a C_0-semigroup on E, with generator A. Consider the following statements:*

(i) *For every $x \in E$*

$$\sup_{k\in\mathbf{Z},t\geq 0}\left\|\int_0^t e^{\frac{2\pi ik}{\omega}s}T(s)x\,ds\right\|<\infty;$$

(ii) $\frac{2\pi ik}{\omega}\in\rho(A)$, $\forall k\in\mathbf{Z}$ *and*

$$M_1\equiv\sup_{k\in\mathbf{Z}}\left\|\left(A-\frac{2\pi ik}{\omega}\right)^{-1}\right\|<\infty. \tag{12}$$

Then (i) *implies* (ii)*. If, in addition, $T(t)$ is uniformly bounded, then* (ii) *implies* (i)*.*

Now we are able to provide proofs of Theorems 1–3.

Proof of Theorem 1. Assume that (i) holds. By Lemma 4, we have $\frac{2\pi ik}{\omega}\in\rho(A)$, $\forall k\in\mathbf{Z}$ and $\sup_{k\in\mathbf{Z}}\|(A-\frac{2\pi ik}{\omega})^{-1}\|<\infty$. By [11, Theorem 2], we have $1\in\rho(T(\omega))$. The converse statement follows easily from Lemma 2. □

Remark. It is easy to see that the converse statement in Theorem 1 does not hold without the uniform boundedness assumption.

Proof of Theorem 2. The "only if" part is obvious, so we need only to prove the "if" part. By Lemma 3, $i\mathbf{R}\subset\rho(A)$ and

$$\sup_{\lambda\in\mathbf{R}}\|(A-i\lambda)^{-1}\|<\infty.$$

By [11, Corollary 5], the semigroup $T(t)$ admits exponential dichotomy. From

$$T(t)x-x=\int_0^t T(s)Ax\,dx,\qquad x\in D(A),$$

it follows that $\{T(t)x\colon t\geq 0\}$ is bounded for every $x\in D(A)$. Exponential dichotomy and the latter property imply that $T(t)$ is exponentially stable. □

Remark. From the proof of Lemma 1 (namely, formula (8) it also follows that, if condition (2) holds, then the resolvent $(\lambda-A)^{-1}$ is uniformly bounded not only on $i\mathbf{R}$, but also in $\{\lambda\in\mathbf{C}\colon \operatorname{Re}\lambda>0\}$. Therefore, the semigroup is exponentially stable by the mentioned theorem of Gearhart.

Note that Theorems 1 and 2 do not hold, in general, for semigroups on Banach spaces, because the uniform boundedness of the resolvent $(\lambda-A)^{-1}$ for all $\lambda=\frac{2\pi ik}{\omega}$ (for all $\lambda\in i\mathbf{R}$) does not imply $1\in\rho(T(\omega))$ (resp., does not imply the exponential dichotomy). Standard examples of such semigroups can be found in [6,

[3] The author thanks van Neerven who brought to the author's attention the corresponding results in [7, 8]. Note that Theorem 2 is not formulated in the paper [7], nor in the recent monograph [8], which contains a detailed discussion of conditions of exponential stability and, in particular, a weaker result for semigroups (satisfying condition (2)) on Banach spaces (cf. [7, Corollary 5] and [8, p. 94 and p. 136]).

10]. However, Theorem 3 shows that the asymptotic stability holds for semigroups on Banach spaces even under a weaker assumption than (2).

Proof of Theorem 3. (i) Applying Lemma 1 to the semigroups $e^{i\lambda t}T(t)$, we obtain that $i\mathbf{R} \subset \rho(A)$. As shown in the proof of Theorem 2, if $x \in D(A^2)$ then $T(t)x$, $T(t)Ax$ are bounded and

$$T(t)x = x + \int_0^t T(s)Ax\,ds,$$

which implies that $T(t)x$ is uniformly continuous. By a theorem in [1], $\lim_{t\to\infty} \|T(t)x\| = 0$.

It is obvious that (i) implies (ii).

(iii) If $T(t)$ is an analytic semigroup, then $\sigma(A) \cap i\mathbf{R}$ is compact. Since $\{\lambda \in \mathbf{C}\colon \operatorname{Re}\lambda \geq 0\} \subset \rho(A)$, it follows that $s(A) < 0$, which implies that $T(t)$ is exponentially stable. □

Note that condition (3) is not sufficient to guarantee exponential stability even on Hilbert space, as the following example shows.

Example. Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}$, $\alpha_n = -\frac{1}{n} + in$, and let A be a diagonal operator on H defined by $Ae_n = \alpha_n e_n$. It is clear that $i\mathbf{R} \subset \rho(A)$ and $T(t) \equiv e^{tA}$ is a contraction semigroup. Moreover, from $\|T(t)e_n\| = e^{-\frac{1}{n}t}$ it follows that $\|T(t)\| = 1$, hence $T(t)$ is not exponentially stable. Applying Lemma 2 to semigroups $e^{-i\lambda t}T(t)$, we have

$$\sup_{t\geq 0}\left\|\int_0^t e^{-i\lambda s}T(s)ds\right\| \leq \|(i\lambda - A)^{-1}\|, \qquad \lambda \in \mathbf{R},$$

i.e. condition (3) is satisfied. This example shows that there is an essential difference between (2) and (3).

Remarks. 1. There are some known results on exponential stability of C_0-semigroups that are similar in the spirit to Theorems 1–3, namely:

(i) A well-known result of Datko (see [10, Theorem 4.1]) that $T(t)$ is exponentially stable if and only if for some $1 \leq p < \infty$

$$\int_0^\infty \|T(t)x\|^p dx < \infty, \quad \forall x \in E.$$

(ii) A theorem of Weiss [14] that if $T(t)$ is acting on Hilbert space then $T(t)$ is exponentially stable if and only if for some $1 \leq p < \infty$

$$\int_0^\infty |\langle T(t)x, x^*\rangle|^p dx < \infty, \quad \forall x \in E,\ x^* \in E^*.$$

(iii) A recent result of van Neerven [9] that $T(t)$ is exponentially stable if and only if for every almost periodic function $f\colon \mathbf{R} \to E$, the function

$$\int_0^t T(s)f(s)ds$$

is bounded (a new and shorter proof of this result is contained in [12]).

Theorem 2 implies a strengthening of a particular case of Datko's theorem (for $p = 1$ and Hilbert space). It also implies van Neerven's result (on Hilbert space), but neither does it imply any of the above results in the general case, nor vice versa.

2. Since the function $t \mapsto \int_0^t T(t-s)f(s)ds$ is the mild solution of the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), \ t \geq 0, \\ u(0) = 0, \end{cases} \tag{13}$$

conditions (1)–(3) in Theorems 1–3 have equivalent reformulation in terms of boundedness of solutions of (13) with $f(t) = e^{i\lambda t}x$, $x \in E$. Thus, for instance, for C_0-semigroups on Hilbert space the exponential stability is equivalent to the boundedness, for all $x \in E$, of the solutions to

$$\begin{cases} u'(t) = Au(t) + e^{i\lambda t}x, \ t \geq 0, \\ u(0) = 0, \end{cases}$$

uniformly in $\lambda \in \mathbf{R}$. Analogously, if E is a Hilbert space, then $1 \in \rho(T(\omega))$ if for every $x \in E$, the solutions to

$$\begin{cases} u'(t) = Au(t) + e^{\frac{2\pi k}{\omega}t}x, \ t \geq 0, \\ u(0) = 0, \end{cases}$$

are uniformly bounded in $k \in \mathbf{Z}$ (Theorem 1). For semigroups on Banach spaces the following weaker version of Theorem 1 holds:

If for every continuous ω-periodic function $f\colon \mathbf{R} \to E$, the mild solution to the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), \ t \geq 0, \\ u(0) = 0, \end{cases}$$

is bounded, then $1 \in \rho(T(\omega))$; see [13].

3. Discrete semigroups

The following lemma is a discrete analog of Lemma 1.

Lemma 5. *Suppose that, for every $x \in E$, the series*

$$\sum_{k=0}^{n} T^k x$$

is bounded. Then $1 \in \rho(T)$.

Proof. By the Uniform Boundedness Principle, $\sup_{n\geq 0} \|T^n\| \equiv M < \infty$, so that $\lambda \in \rho(T)$ for all $\lambda > 1$. Let $S_k = \sum_{i=0}^{k-1} T^i$, $k \geq 1$, $S_0 \equiv 0$. From the following equality which is valid for any sequence a_k, b_k

$$\sum_{k=0}^{n} a_k(b_{k+1} - b_k) = a_n b_{n+1} - a_0 b_0 - \sum_{k=1}^{n} b_k(a_k - a_{k-1})$$

it follows that

$$\sum_{k=0}^{n} \lambda^{-(k+1)} T^k = \sum_{k=0}^{n} \lambda^{-(k+1)} (S_{k+1} - S_k)$$
$$= \lambda^{-(n+1)} S_{n+1} - \sum_{k=1}^{n} (\lambda^{-(k+1)} - \lambda^{-k}) S_k$$
$$= \lambda^{-(n+1)} S_{n+1} - (1-\lambda) \sum_{k=1}^{n} \lambda^{-(k+1)} S_k.$$

Since S_k are uniformly bounded, the series converges (in the uniform operator topology) to some operator R. It follows that $R = (\lambda - T)^{-1}$ and

$$|R\| = \|(\lambda - T)^{-1}\| \leq (\lambda - 1) M \sum_{1}^{\infty} \lambda^{-(k+1)} < M.$$

Therefore,

$$\operatorname{dist}(\lambda, \sigma(T)) \geq \frac{1}{\|(\lambda - T)^{-1}\|} > \frac{1}{M}, \quad \forall \lambda > 1,$$

which implies that $1 \in \rho(T)$ and

$$\operatorname{dist}(1, \sigma(T)) \geq \frac{1}{M}. \qquad \square \tag{14}$$

From Lemma 5 we obtain the following theorem which is a discrete analog of Theorem 2.

Theorem 4. *Suppose that T is a bounded linear operator on an arbitrary Banach space E. Then $\|T^n\| \to 0$ if and only if*

$$\sup_{n \geq 0} \left\| \sum_{k=0}^{n} \lambda^k T^k x \right\| < \infty$$

for all $x \in E$ and λ, $|\lambda| = 1$. Moreover, if

$$\left\| \sum_{k=0}^{n} \lambda^k T^k \right\| < M, \quad \forall n, \lambda, |\lambda| = 1,$$

then,

$$r(T) \leq \frac{M-1}{M}. \tag{15}$$

For the proof of Theorem 4 we apply Lemma 5 to operator λT to obtain that $\lambda^{-1} \in \rho(T)$ for all λ, $|\lambda| = 1$, hence the spectra radius of T is < 1. Moreover, from (14) it follows that

$$\operatorname{dist}(\lambda, \sigma(T)) \geq \frac{1}{M}, \quad \forall \lambda, \ |\lambda| = 1,$$

which implies (15).

Remark. The discrete analog of the results of Datko and Weiss are also true (see [15]) and their relationship with Theorem 4 is the same as the relationship between the corresponding results in the continuous parameter case. A particular case, obtained earlier by McCabe [5], that $r(T) < 1$ if and only if $\sum_{n=0}^{\infty} \|T^n x\| < \infty$, $\forall x \in E$, is also implied by Theorem 4.

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