DERIVED SUBGROUPS 
AND CENTERS OF CAPABLE GROUPS

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ABSTRACT. A group $G$ is said to be capable if it is isomorphic to the central factor group $H/Z(H)$ for some group $H$. It is shown in this paper that if $G$ is finite and capable, then the index of the center $Z(G)$ in $G$ is bounded above by some function of the order of the derived subgroup $G'$. If $G'$ is cyclic and its elements of order 4 are central, then, in fact, $|G : Z(G)| \leq |G'|^2$.

1. Introduction

Recall that a group $G$ is said to be capable if there exists some group $H$ such that $H/Z(H)$ is isomorphic to $G$. Of course, there are groups that are not capable (nontrivial cyclic groups, for example), and so the condition that a group is capable imposes certain restrictions on its structure. It is known, for example, that if $G$ is a finite capable $p$-group with $|G'| = p$, then $|G : Z(G)| = p^2$. (I would like to thank A. Mann for informing me of this result, a special case of which appears in [1].)

As extraspecial $p$-groups show, there is no general upper bound on the index of the center of a finite group in terms of the order of its derived subgroup. In this paper, we prove that there is such a bound for all capable groups.

Theorem A. There exists a function $B(n)$ defined on the natural numbers such that if $G$ is finite and capable, then $|G : Z(G)| \leq B(|G'|)$.

As an immediate consequence, we have the following general result. (Recall that the second center of a group $G$ is the preimage in $G$ of $Z(G/Z(G))$.)

Corollary B. Let $G$ be an arbitrary finite group. Then the index of the second center of $G$ is bounded above by some function of $|G'|$.

Proof. The group $G/Z(G)$ is capable, and $|(G/Z(G))'| \leq |G'|$. The result follows by applying Theorem A to $G/Z(G)$.

We shall not attempt to find the optimal function $B(n)$ in Theorem A, but in the special case where $G'$ is cyclic and all elements of order 4 in $G'$ are central, we obtain the best possible bound.
Theorem C. Let $G$ be finite and capable, and suppose that $G'$ is cyclic and that all elements of order 4 in $G'$ are central in $G$. Then $|G : Z(G)| \leq |G'|^2$, and equality holds if $G$ is nilpotent.

Unfortunately, we have been unable to decide whether or not the assumption about elements of order 4 in Theorem C is really necessary.

We mention that we are aware of two other related papers in the literature. In [3], H. Heineken considers capable groups $G$ for which $G'$ is central and elementary abelian of order $p^n$, where $p$ is a prime number. In Proposition 3, he shows that if $n = 2$, then $|G : Z(G)| \leq p^5$. Heineken also asserts that for arbitrary $n$, if $p > 2$, then there exist examples where $|G : Z(G)| = p^m$, where $m = 2n + \binom{n}{2}$. In [4], Heineken and D. Nikolova show that under certain very restrictive additional conditions, the index $|G : Z(G)|$ cannot exceed $p^m$, where $m = 2n + \binom{n}{2}$. (In order to obtain this bound, the authors assume that $G$ has exponent $p$ and that $G' = Z(G)$.)

I would like to thank A. Moreto for informing me of the existence of the papers [2] and [3], and for a number of helpful conversations on the subject of this paper. It was the referee who told me about [4], and I thank him too.

2. The general case

In this section, we work toward a proof of Theorem A. We begin with a couple of preliminary lemmas, which must surely be known. We thank D. S. Passman for helping us to find a proof of the first of these results.

(2.1) Lemma. Let $G$ be a finite capable group. Then there is a finite group $H$ such that $H/Z(H) \cong G$.

Proof. Since $G$ is capable, there is by definition a possibly infinite group $H$ such that $H/Z \cong G$, where $Z = Z(H)$. By choosing one element in each of the finitely many cosets of $Z$ in $H$, we can produce a finitely generated subgroup $K$ of $H$ such that $H = ZK$. Then $Z(K) \subseteq Z/H = Z$, and so $Z(K) = K \cap Z$. It follows that $K/Z(K) = K/(K \cap Z) \cong ZK/Z = H/Z \cong G$, and we can therefore replace $H$ by $K$ and assume that $H$ is finitely generated.

Now $|H : Z| < \infty$, and hence by Schreier’s theorem, the abelian group $Z$ is finitely generated, and thus we can write $Z = T \times F$, where $T$ is finite and $F$ is torsion free. Certainly $Z/F \subseteq Z(H/F)$, and we claim that equality holds here. To see this, let $h \in H$ be central modulo $F$, so that $[H, h] \subseteq F$, and in particular $[H, h]$ is central in $H$. It follows that the map $x \mapsto [x, h]$ defines a homomorphism from $H$ into $F$. Since $Z$ is in the kernel of this homomorphism and $H/Z$ is finite, we see that $[H, h]$ is a finite subgroup of $F$. But $F$ is torsion free, and thus $[H, h] = 1$ and $h \in Z$.

Now $H/F$ is a finite group, and we have $(H/F)/Z(H/F) = (H/F)/(Z/F) \cong H/Z \cong G$. This completes the proof. \hfill \Box

(2.2) Lemma. Let $A \subseteq G$, where $A$ is abelian, and suppose that $|G : A| = m < \infty$ and that $|G'| = n < \infty$. Then

$$|G : Z(G)| \leq m^{1+\log(n)},$$

where the logarithm is to the base 2.
Proof. First, we argue that we can choose a subset $X \subseteq G$ such that $G = \langle X, A \rangle$ and $|X| \leq \log(m)$. To prove this, write $A_0 = A$ and recursively construct subgroups $A_i$ such that $A_i = \langle A_{i-1}, x_i \rangle$, where $x_i$ is chosen arbitrarily in $G - A_{i-1}$ as long as $A_{i-1} \triangleleft G$. We thus have $A = A_0 < A_1 < \cdots < A_r = G$ for some integer $r \leq \log(|G : A|) = \log(m)$. The set $X = \{x_i \mid 1 \leq i \leq r\}$ has the desired properties.

Each conjugacy class of $G$ is contained in some coset of $G'$ in $G$, and thus each of the classes of $G$ has cardinality no larger than $|G'|$. It follows that $|G : C_G(x)| \leq |G'|$ for each element $x \in G$, and thus $|G : C_G(X)| \leq |G'|^{|X|}$. Since $A$ is abelian and $G = \langle A, X \rangle$, we see that $A \cap C_G(X) \subseteq Z(G)$, and thus

$$|G : Z(G)| \leq |G : A||A : A \cap C_G(X)| \leq |G : A||C_G(X)| \leq |G : A||G'|^{|X|} \leq mn^{\log(m)} = mn^{\log(n)},$$

and the result follows. \qed

Of course, if we know the smallest prime divisor $p$ of $m = |G : A|$ in the situation of Lemma 2.2, we can work with logarithms to the base $p$, and thereby obtain a better bound.

Every group with trivial center is clearly capable, and our next result establishes Theorem A for such groups.

(2.3) Theorem. There is a function $F(n)$ defined on the natural numbers such that if $Z(G) = 1$ and $|G'| = n < \infty$, then $|G| \leq F(n)$.

Proof. Let $C = C_G(G')$, and write $m = |G : C|$, so that $m$ is bounded above by some function of $n$. (For example, since $G/C'$ is isomorphically embedded in $\text{Aut}(G')$, it follows that $m \leq n!$.) We have $[G, C] \subseteq G'$, and thus $[G, C, C] = 1$. We conclude by the three subgroups lemma that $C'$ centralizes $G$, and since we are assuming that $Z(G) = 1$, we see that $C' = 1$ and $C$ is abelian.

We can now apply Lemma 2.2 to the abelian subgroup $C \subseteq G$ of index $m$, and we conclude that $|G| = |G : Z(G)| \leq m^{1+\log(n)}$. The result follows since $m$ is bounded in terms of $n$. \qed

We mention that a bound significantly better than the inequality $m \leq n!$ is available in the situation of Theorem 2.3. If we argue as in the proof of Lemma 2.2, we can find a generating set $X$ for $G'$ with $|X| \leq \log(n)$, where $n = |G'|$. Then $C = C_G(G') = C_G(X)$, and it follows that $m = |G : C| \leq |G'|^{|X|} \leq n^{\log(n)}$. (A version of this easy argument goes back at least as far as 1939, where it appears in the proof of statement (35) of the paper [3], by H. Wielandt.)

Before we begin the proof of Theorem A, we recursively define the relevant function $B(n)$. We start by setting $B(1) = 1$, and we assume that we have fixed some particular bounding function $F(n)$ as in Theorem 2.3. (We assume, as we may, that $F(1) = 1$ and that $F$ is monotonically increasing.) If $n > 1$, we let $M$ be the maximum of the quantities $B(n/q)$, where $q$ runs over prime divisors of $n$, and we set

$$B(n) = \max\{F(n), (nM)^{1+\log(n)}\}.$$ 

It is easy to see that with this definition, we have $B(m) \leq B(n)$ whenever $m$ divides $n$, and of course, $F(n) \leq B(n)$ for all $n$. 

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Proof of Theorem A. We are given a finite capable group $G = H/Z$, where $Z = \mathbf{Z}(H)$. Write $U = H'Z$ and let $n = |G'| = |U/Z|$. By Lemma 2.1, we can assume that $H$ is finite, and we show that $|G : \mathbf{Z}(G)| \leq B(n)$ by induction on $|H|$. (Note that if $H = 1$, then $G = 1$ and the inequality is trivially true.)

If $Z = 1$, then $H = G$, and by Lemma 2.3, we have $|G : \mathbf{Z}(G)| = |G| \leq F(n) \leq B(n)$. There is nothing further to prove in this case, and so we assume that $Z > 1$, and we choose a subgroup $T \subseteq Z$ of prime order $p$. Let $Y/T = \mathbf{Z}(H/T)$, and note that $Y \supseteq Z$ and $H/Y \cong (H/T)/\mathbf{Z}(H/T)$ is capable.

Suppose first that $Y \cap U = Z$. Let $A/Y = \mathbf{Z}(H/Y)$ and note that $UY/Y = (H/Y)'$, and this subgroup has order $n$. Since $|H/T| < |H|$, we can apply the inductive hypothesis to deduce that $|H : A| \leq B(n)$. But $[H, A] \subseteq Y \cap H' \subseteq Y \cap U = Z$, and thus $A/Z \subseteq \mathbf{Z}(H/Z)$. This shows that $|G : \mathbf{Z}(G)| \leq |H : A| \leq B(n)$, as desired.

We can now assume that $Y \cap U > Z$. Let $y$ be an element of $Y \cap U$ that does not lie in $Z$ and set $C = C_H(y) < H$. We have $[H, Y] \subseteq T \subseteq Z$, and thus the map $h \mapsto [h, y]$ defines a homomorphism from $H$ into $T$ with kernel $C$, and we see that $C < H$ and $[H : C]$ divides $|T| = p$. It follows that $|H : C| = p$ and also, since $H/C$ is abelian, we have $U \subseteq C$. Because $[H, Y] \subseteq Z$, we also have $1 = [h, y]^p = [y, y]^p$ for all elements $h \in H$. We deduce that $y^p \in Z$, and thus $y$ has order $p$ modulo $Z$.

Let $X = \mathbf{Z}(C)$ and observe that $y \in X \cap U$, and hence $|X \cap U : Z|$ is divisible by $p$ and $[U : X \cap U]$ is a divisor of $|U : Z|/p = n/p$. Now $(H/X)' = U/X \cong U/(X \cap U)$, and thus $|(H/X)'|$ divides $n/p$. It follows that $C/X$ is a capable group whose derived subgroup has order dividing $|(H/X)'|$, which in turn divides $n/p$.

Write $V/X = \mathbf{Z}(C/X)$. Since $C < H$, the inductive hypothesis applies, and we conclude that $|C : V| \leq B(|(C/X)'|) \leq B(n/p)$. Now let $h \in H - C$ and write $S/X = C_{V/X}(h)$. Since $H/C$ has prime order, we see that $h$ generates $H$ modulo $C$, and thus $S/X \subseteq \mathbf{Z}(H/X)$. But $|(H/X)'| \leq n/p$, and thus $|(H/X) : C_{H/X}(h)| \leq n/p$, and we have $|V : S| \leq n/p$. Now $|H : C| = p$ and $|C : V| \leq B(n/p)$, and so we see that $[H : S] \leq nB(n/p)$. In particular, $|H : S| \leq nM$, where $M$ is the maximum value of $B(n/q)$, as $q$ runs over all prime divisors of $n$.

Since $S/X \subseteq \mathbf{Z}(H/X)$, we have $[H, S] \subseteq X$. But $S \subseteq C$ and $X = \mathbf{Z}(C)$, and so $[H, S, S] \subseteq [X, C] = 1$, and thus we see by the three subgroups lemma that $S'$ centralizes $H$. We conclude that $S' \subseteq Z$, and thus $S/Z$ is abelian. Since $|H : S| \leq nM$ and $|(H/Z)'| = n$, Lemma 2.2 yields that $|G : \mathbf{Z}(G)| = |(H/Z) : \mathbf{Z}(H/Z)| \leq (nM)^{1 + \log(n)}$. In particular, $|G : \mathbf{Z}(G)| \leq B(n)$, as required. \qed

3. Cyclic derived subgroups

We begin work toward a proof of Theorem C by studying groups that are not necessarily capable, but which have a cyclic derived subgroup.

(3.1) Lemma. Let $G$ be finite and assume that $G'$ is a cyclic $p$-group for some prime $p$. If $G' \cap \mathbf{Z}(G)$ is nontrivial, then $G$ has a normal $p$-complement.

Proof. Let $P \in \text{Syl}_p(G)$. Then $G' \subseteq P$, and so $P \leq G$ and $G$ has a $p$-complement $H$. By Fitting’s lemma, we can write $G' = C_{G'}(H) \times [G', H]$, and we observe that the first factor is nontrivial since we are assuming that $G' \cap \mathbf{Z}(G) > 1$. Since the cyclic $p$-group $G'$ is indecomposable, we conclude that $[G', H] = 1$, and thus $[P, H] = [P, H, H] = 1$, where the first equality follows because $|H|, |P| = 1$. We conclude that $P$ normalizes $H$, and thus $H < G$, as required. \qed
**Theorem.** Let $G$ be finite and assume that $G'$ is cyclic. Let $\pi$ be the set of prime divisors of $|G' \cap Z(G)|$ and let $b$ be the $\pi'$-part of $|G'|$. Then:

(a) $G$ has a normal $\pi$-complement $M$ and $G/M$ is nilpotent.
(b) $|M : M \cap Z(G)|$ divides $b\varphi(b)$, where $\varphi$ is Euler’s function.
(c) $|G : Z(G)|$ divides $b\varphi(b)|G : V|$, where $V/M = Z(G/M)$.

**Proof.** Let $p \in \pi$ and note that since $G'$ is cyclic, the derived subgroup of $G/O_p(G)$ is a cyclic $p$-group. Also, since $G' \cap Z(G) \not\subseteq O_p(G)$ by the definition of the set $\pi$, we conclude that the group $G/O_p(G)$ satisfies the hypotheses of Lemma 3.1, and hence it has a normal $p$-complement. It follows that $G$ has a normal $p$-complement, and since this is true for every prime $p \in \pi$, we see that $G$ has a normal $\pi$-complement $M$ and that $G/M$ is nilpotent. This proves (a).

Now let $B = M \cap G'$ and write $C = C_M(B)$. Since $B$ is cyclic of order $b$ and $B \lhd G$, we see that $|M : C|$ divides $b\varphi(b)$, and thus to prove (b), it suffices to show that $|C : C \cap Z(G)|$ divides $b\varphi(b)\varphi(q)$. Since $C \lhd G$, we have $[G, C] \subseteq C \cap G' \subseteq M \cap G' = B$, and thus $[G, C, C] \subseteq [B, C] = 1$. By the three subgroups lemma, it follows that $[C', G] = 1$, and thus $C' \subseteq G' \cap Z(G)$. But $C$ is a $\pi'$-group, and by the definition of $\pi'$ it follows that no prime divisor of $|C|$ divides the order of $G' \cap Z(G)$. We conclude that $C' = 1$ and $C$ is abelian.

Now let $Q \in \text{Syl}_q(C)$, where $q$ is an arbitrary prime in $\pi'$, and note that $Q \lhd G$ since $C$ is abelian and $C \lhd G$. Since $q \in \pi'$ is arbitrary, we see that to establish that $|C : C \cap Z(G)|$ divides $b\varphi(b)$, as claimed, it suffices to show that the $q$-part of this index divides $b\varphi(b)$, or equivalently, that $|Q : Q \cap Z(G)|$ divides $b\varphi(b)$.

Now, $[G, M] \subseteq B \subseteq C$, and thus $M/C \subseteq Z(G/C)$. Since $G/M$ is nilpotent, it follows that $G/C$ is nilpotent, and we let $K/C$ be the normal $q'$-complement of $G/C$. Now $K$ acts on $Q$, and we prove next that $C_Q(K) \subseteq Z(G)$. To this end, we let $L = [C_Q(K), G]$ and we note that $L \subseteq G'$ and also $L \subseteq C_Q(K)$ since $C_Q(K) \lhd G$. Since $K$ centralizes $L$ and $L \lhd G$, we see that the $q$-group $G/K$ acts on the $q'$-group $L$, and thus if $L > 1$, we have $1 < C_L(G) \subseteq G' \cap Z(G)$. This is a contradiction since $q \notin \pi$, and it follows that $L = 1$, and thus $C_Q(K) \subseteq Z(G)$, as claimed, and thus, in fact, $C_Q(K) = Q \cap Z(G)$.

Since $Q \subseteq C$ and $C$ is abelian, the $q'$-group $K/C$ acts on $Q$, and hence we have $Q = [Q, K] \times C_Q(K)$ by Fitting’s lemma. We have established that $C_Q(K) = Q \cap Z(G)$, and therefore $|Q : Q \cap Z(G)| = |[Q, K]|$, and this divides $b$ since $[Q, K] \subseteq B$. This shows that $|C : C \cap Z(G)|$ divides $b\varphi(b)$, as claimed, and the proof of (b) is complete.

Finally, to prove (c), we let $V/M = Z(G/M)$ and $W = C_V(B)$. (Note that $Z(G) \subseteq W$ and that $W \cap M = C$.) Since $B$ is cyclic of order $b$, we see that $|V : W|$ divides $\varphi(b)$, and thus $|G : W|$ divides $|G : V|\varphi(b)$. It suffices, therefore, to show that $|W : Z(G)|$ divides $b\varphi(b)$.

First, we prove that a Hall $\pi$-subgroup $H$ of $W$ is central in $G$ by showing separately that $H$ centralizes the Hall $\pi'$-subgroup $M$ of $G$ and that it centralizes a Hall $\pi$-subgroup $S$ of $G$, where $S$ is chosen to contain $H$. We have $[M, H] \subseteq M \cap G' = B$, and thus $[M, H] = [M, H, H] \subseteq [B, H] = 1$, as desired, where the last equality follows since $H \subseteq W = C_V(B)$. Also, $H, G \subseteq M$ since $H \subseteq V$, and thus since $H \subseteq S$, we have $[H, S] \subseteq M \cap S = 1$. This shows that $H \subseteq Z(G)$, as claimed, and thus $|W : Z(G)|$ is a $\pi'$-number. We now have $W = (W \cap M)Z(G) = C(Z(G),$
and thus $|W : \mathbf{Z}(G)| = |C : C \cap \mathbf{Z}(G)|$, which, as we have seen, is a divisor of $b$. This completes the proof. □

Next, we quote a known result.

(3.3) Lemma. Let $\sigma \in \text{Aut}(G)$ and assume that $\sigma$ fixes all elements of prime order and of order 4 in $[G, \sigma]$. Then $[G, \sigma]$ has exponent dividing the order $o(\sigma)$.

Proof. This is Theorem A(c) of [5]. □

Somewhat surprisingly, we need the following result, which establishes a lower bound on the index of the center of our group. We remark that if $G$ is a $p$-group and $G'$ is cyclic, then a generator of $G'$ must actually be a commutator in $G$. The same conclusion therefore holds for finite nilpotent groups with cyclic derived subgroups. (One can work with one Sylow subgroup at a time.)

(3.4) Lemma. Let $G$ be nilpotent and assume that $G'$ is cyclic and that all elements of order 4 in $G'$ are central in $G$. Then $|G : \mathbf{Z}(G)| \geq |G'|^2$.

Proof. As we remarked, some generator of $G'$ must be a commutator, and thus we can choose elements $a$ and $b$ in $G$ such that $\langle [a, b] \rangle = G'$, and we see that $X' = G'$, where $X = \langle a, b \rangle$. Since $|X : \mathbf{Z}(X)| \leq |G : \mathbf{Z}(G)|$, it is no loss to assume that $G = X$, and thus we have $G = \langle a, b \rangle$. Write $Z = \mathbf{Z}(G)$ and let $A = \langle Z, a \rangle$ and $B = \langle Z, b \rangle$. Then both $A$ and $B$ are abelian, and since $G = \langle A, B \rangle$, we see that $A \cap B = Z$.

Consider the inner automorphism $\sigma$ of $G$ induced by $a$, and note that the order of $\sigma$ is exactly $|\langle a \rangle : \langle a \rangle \cap Z| = |A : Z|$. Since $[G, \sigma] = G'$ is cyclic, every subgroup of prime order in $[G, \sigma]$ is normal, and hence is central in $G$ since $G$ is nilpotent. Thus $\sigma$ fixes all elements of prime order in $[G, \sigma]$ and also, by hypothesis, $\sigma$ fixes all elements of order 4 in $[G, \sigma]$. Thus Lemma 3.3 applies, and so $|A : Z| = o(\sigma)$ is a multiple of the exponent of $[G, \sigma]$. Since $[G, \sigma] = G'$ is cyclic, its exponent is equal to its order, and we have $|A : Z| \geq |G'|$. Similarly, $|B : Z| \geq |G'|$, and thus $|G : Z| \geq |AB|/|Z| = |A : Z||B : Z| \geq |G'|^2$, as desired. □

The following result is very closely related to Theorem 1 of [2].

(3.5) Lemma. Let $G$ be nilpotent and assume that $G'$ is cyclic and that all elements of order 4 in $G'$ are central in $G$. Then there exist subgroups $X$ and $Y$ of $G$ such that $XY = G$, $X' = G'$, $[X, Y] = 1$ and $|G : Y| = |G'|^2$.

Proof. Since $G$ is nilpotent, we can find (as in the previous proof) a two-generator subgroup $X = \langle a, b \rangle$ of $G$ such that $X' = G'$. Let $Y = \mathbf{C}_G(X)$. By Lemma 3.4, we see that $|G : Y| \geq |X : |X \cap Y| = |X : \mathbf{Z}(X)| \geq |X'|^2 = |G'|^2$. But $Y = \mathbf{C}_G(a) \cap \mathbf{C}_G(b)$, and so $|G : Y| \leq |G : \mathbf{C}_G(a)||G : \mathbf{C}_G(b)| = |\text{cl}(a)||\text{cl}(b)| \leq |G'|^2$. It follows that we have equality throughout, and thus in particular, we see that $|G'|^2 = |G : Y| = |X : X \cap Y|$, and it follows that $XY = G$, as required. □

Proof of Theorem C. It suffices to prove the inequality in the statement of the theorem; the fact that equality must hold if $G$ is nilpotent will then follow via Lemma 3.4.

Since $G$ is capible, we can assume that $G = H/Z$, where $H$ is some finite group and $Z = \mathbf{Z}(H)$. We wish to apply Theorem 3.2 to $G$, and so as in that theorem, we let $\pi$ be the set of prime divisors of $|G' \cap \mathbf{Z}(G)|$. Then $H/Z$ has a normal $\pi$-complement $M/Z$, and we let $V/M = \mathbf{Z}(H/M)$. By Theorem 3.2(c), we know that
Prove that which is cyclic, and thus \( Y \) is abelian, then since \( K/Y \), as desired. This completes the proof.

Recall that our goal is to show that \( Y' = Z \) when \( K = Z \). By Theorem 3.2(a), we know that \( K/Y \) is nilpotent, and thus by Lemma 3.5, there exist subgroups \( X \) and \( Y \) of \( K \), each of them containing \( Z \), and such that \( XY = K \) and \( [X, Y] \subseteq Z \). Also \( (X/Z)' = (K/Z)' \) and \( |K : Y| = |(K/Z)'| = 1 \), and thus by the three subgroups lemma, \( X' \) centralizes \( Y \), and similarly \( Y' \) centralizes \( X \). Since \( (X/Z)' = (K/Z)' \), we see that \( X'Z = K'Z \), and it follows that \( K' \) centralizes \( Y \). But \( Y' \subseteq K' \), and thus \( Y' \) centralizes both \( Y \) and \( X \), and thus \( Y' \subseteq Z(K) \) since \( K = XY \).

Recall that our goal is to show that \( Y' \subseteq Z \). Since we now know that \( Y' \) centralizes \( K \), it suffices to show that \( Y' \) also centralizes \( M \). Now \( Y'Z/Z \subseteq (H/Z)' \), which is cyclic, and thus \( Y'Z \leq H \) and of course, \( Y'Z/Z \) is a \( \pi \)-subgroup of \( H/Z \). Thus \( Y'Z/Z \) and \( M/Z \) are normal subgroups of coprime orders in \( H/Z \), and it follows that \( [M, Y'Z] \subseteq Z \). If \( m \in M \), it follows that the map \( y \mapsto [m, y] \) is a homomorphism from \( Y'Z \) into \( Z \), and \( Z \) is contained in the kernel of this map. Therefore \( [M, Y'Z] \) divides \( Y'Z/Z \), and it follows that \( [M, Y'Z] \) is a \( \pi \)-group. Also, we can interchange the roles of \( M \) and \( Y'Z \) in this argument, and we deduce that \( [M, Y'Z] \) is a \( \pi' \)-group. It follows that \( [M, Y'] = 1 \), and thus \( Y' \subseteq Z(MK) = Z(H) = Z \), as desired. This completes the proof.

References


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