ROUGH SINGULAR INTEGRALS ASSOCIATED TO SURFACES OF REVOLUTION

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(Communicated by Christopher D. Sogge)

Abstract. Let $1 < p < \infty$ and $n \geq 2$. The authors establish the $L^p(\mathbb{R}^{n+1})$-boundedness for a class of singular integral operators associated to surfaces of revolution, $\{(t, \phi(|t|)) : t \in \mathbb{R}^n\}$, with rough kernels, provided that the corresponding maximal function along the plane curve $\{(t, \phi(|t|)) : t \in \mathbb{R}\}$ is bounded on $L^p(\mathbb{R}^2)$.

1. Introduction

Let $n \geq 2$ and $y \in \mathbb{R}^n$. For the Calderón-Zygmund type kernel

$$K(y) = \frac{\Omega(y)}{|y|^n b(|y|)}$$

and a suitable function $\phi$ on $[0, \infty)$, we define the singular integral operator $T$ along the surface

$$\Gamma = \{(y, \phi(|y|)) : y \in \mathbb{R}^n\}$$

by

$$Tf(x, s) = \text{p. v.} \int_{\mathbb{R}^n} f(x - y, s - \phi(|y|)) K(y) \, dy.$$  

(1)

Here and in what follows, we always assume that $b$ is a measurable function on $[0, \infty)$, $\Omega$ is homogeneous of order zero on $\mathbb{R}^n$, integrable on $S^{n-1}$ and satisfies

$$\int_{S^{n-1}} \Omega(y) \, d\sigma(y) = 0.$$  

(2)

The kernel $K(y)$, which has radial roughness introduced by the factor $b(|y|)$, was first studied by R. Fefferman in the context of singular integrals on $\mathbb{R}^n$ (17).

In 10, Kim, Wainger, Wright and Ziesler proved the following theorem.
Theorem A (11). Let \( \phi \in C^2([0, \infty)) \) be convex, increasing and \( \phi(0) = 0 \). Let \( \Omega \in C^\infty(S^{n-1}) \) satisfy (2) and \( b \equiv 1 \). Then \( T \) in (1) is bounded on \( L^p(\mathbb{R}^{n+1}) \) for \( 1 < p < \infty \).

In [3], Chen and Fan generalized the above result by requiring that \( \Omega \) belongs to a Block space introduced in [11] and \( b \in L^\infty([0, \infty)) \).

Theorem B (3). Suppose \( \Omega \in B^\beta_r(S^{n-1}) \) for some \( \beta > 0 \) and \( r > 1 \). If the maximal operator \( \nu_\phi \) on \( \mathbb{R} \) given by

\[
(\nu_\phi g)(x) = \sup_{k \in \mathbb{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |g(x - \phi(t))| \, dt
\]

is a bounded operator on \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \), then \( T \) is bounded on \( L^p(\mathbb{R}^{n+1}) \) for \( 1 < p < \infty \).

The main purpose of this paper is to consider the \( L^p \) boundedness of \( T \) when \( \Omega \in H^1(S^{n-1}) \), the Hardy space on the sphere; see [5] and [4] for the definition. The method that we use in this paper comes from the work of Duoandikoetxea and Rubio de Francia (6) and its extension obtained in Fan-Pan (8).

To state our main result, we need to introduce the maximal function \( \mathcal{M}_\phi \), associated to the plane curve \( \{(x, \phi(|x|)) : x \in \mathbb{R}\} \). For any measurable function \( f \) on \( \mathbb{R}^2 \), \( \mathcal{M}_\phi f \) is defined by

\[
(\mathcal{M}_\phi f)(x_1, x_2) = \sup_{k \in \mathbb{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(x_1 - t, x_2 - \phi(|t|))| \, dt.
\]

Here is our main theorem.

Theorem 1. Let \( \phi : [0, \infty) \to \mathbb{R} \) be continuously differentiable on \( (0, \infty) \) and satisfy

\[
|\phi(t) - \phi(0)| \leq Ct^\alpha
\]

for some \( \alpha > 0 \) and small \( t \), where \( C \) is a constant independent of \( t \). Let \( \Omega \in H^1(S^{n-1}) \), \( b \in L^\infty([0, \infty)) \) and \( T \) be given by (1). Then \( T \) is bounded on \( L^p(\mathbb{R}^{n+1}) \) for \( 1 < p < \infty \), provided that \( \mathcal{M}_\phi \) in (3) is bounded on \( L^p(\mathbb{R}^2) \).

The condition imposed on \( \phi(t) \) for \( t \sim 0 \) ensures that the integral in (1) exists in principle-value sense when, say, \( f \in S(\mathbb{R}^{n+1}) \).

The \( L^p(\mathbb{R}^2) \) boundedness of \( \mathcal{M}_\phi \) is known for many \( \phi \)'s. Below we shall mention a few prominent cases:

(i) If \( \phi \) is a real-valued polynomial, then \( \mathcal{M}_\phi \) is bounded on \( L^p(\mathbb{R}^2) \) for \( p > 1 \); see [12].

(ii) Let \( h(t) = t\phi'(t) - \phi(t) \) for \( t > 0 \). If \( \phi : \mathbb{R} \to \mathbb{R} \) is of class \( C^2(0, \infty) \), convex on \( [0, \infty) \) and \( \phi(0) = \phi'(0) = 0 \) and there exists an \( \varepsilon > 0 \) so that for each \( t > 0 \), \( h'(t) > \varepsilon h(t)/t \), then \( \mathcal{M}_\phi \) in (3) is bounded on \( L^p(\mathbb{R}^2) \) for \( p > 1 \); see Theorem 1.5 in [2]. Moreover, if \( \phi \) is either even or odd, convex on \( [0, \infty) \), and there exists a \( 0 < C < \infty \) so that for each \( t > 0 \), \( \phi'(Ct) \geq 2\phi'(t) \), then \( \mathcal{M}_\phi \) in (3) is bounded on \( L^p(\mathbb{R}^2) \) for \( p > 1 \). For details, see [11] or [2].

(iii) For \( \phi(t) = t^\alpha \) with \( \alpha \in (0, 1] \), \( \mathcal{M}_\phi \) is bounded on \( L^p(\mathbb{R}^2) \) for \( p > 1 \); see [12].
2. Proof of Theorem 1

We begin with the definition of the space $H^1(S^{n-1})$. For $f \in L^1(S^{n-1})$ and $x \in S^{n-1}$, we define

$$P^+ f(x) = \sup_{0 < t < 1} \left| \int_{S^{n-1}} P_{tx}(y) f(y) \, d\sigma(y) \right|,$$

where

$$P_{tx}(y) = \frac{1 - t^2}{|y - tx|^n}$$

for $y \in S^{n-1}$.

**Definition 1.** An integrable function $f$ on $S^{n-1}$ is in the space $H^1(S^{n-1})$ if and only if

$$\|P^+ f\|_{L^1(S^{n-1})} = \int_{S^{n-1}} |P^+ f(x)| \, d\sigma(x) < \infty$$

and we define

$$\|f\|_{H^1(S^{n-1})} = \|P^+ f\|_{L^1(S^{n-1})}.$$

A very useful characterization of the space $H^1(S^{n-1})$ is its atomic decomposition. Let us first recall the definition of atoms.

**Definition 2.** A function $a(\cdot)$ on $S^{n-1}$ is a regular atom if there exist $\xi \in S^{n-1}$ and $\rho \in (0, 2]$ such that

(i) $\text{supp } a \subset S^{n-1} \cap B(\xi, \rho)$, where $B(\xi, \rho) = \{y \in \mathbb{R}^n : |y - \xi| < \rho\}$;

(ii) $\|a\|_{L^\infty(S^{n-1})} \leq \rho^{-n+1}$;

(iii) $\int_{S^{n-1}} a(y) \, d\sigma(y) = 0$.

A function $a(\cdot)$ on $S^{n-1}$ is an exceptional atom if $a(\cdot) \in L^\infty(S^{n-1})$ and

$$\|a\|_{L^\infty(S^{n-1})} \leq 1.$$

The following can be found in [5] and [4].

**Lemma 1.** For any $f \in H^1(S^{n-1})$ there are complex numbers $\lambda_j$ and atoms (regular or exceptional) $a_j$ such that

$$f = \sum_j \lambda_j a_j$$

and

$$\|f\|_{H^1(S^{n-1})} \sim \sum_j |\lambda_j|.$$

The following lemma is a simple corollary of Theorem B.

**Lemma 2.** Let $\phi$ be the same as in Theorem 1. Let $\Omega \in L^r(S^{n-1})$ for some $1 < r \leq \infty$, $n \geq 2$ and $T$ be given by (1). Then $T$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, provided that $M_\phi$ in (3) is bounded on $L^p(\mathbb{R}^2)$. 

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boundedness of the maximal operator $\nu^{N}_\phi$. 

For $N \in \mathbb{N}$, let 

$$(\nu^{N}_\phi g)(x) = \sup_{-\infty < k \leq N} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |g(x - \phi(t))| \, dt.$$  

Then for $f(x, y) = \chi_{[0,2N+2]}(x)g(y)$, 

$$\chi_{[0,2N+1]}(x)(\nu^{N}_\phi g)(y) \leq (\mathcal{M}_\phi f)(x, y).$$  

Thus 

$$2^{(N+1)/p}\|\nu^{N}_\phi g\|_{L^p(\mathbb{R})} \leq \|\mathcal{M}_\phi f\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})} = C_p 2^{(N+1)/p}\|g\|_{L^p(\mathbb{R})}.$$  

By letting $\mathbb{N} \to \infty$ (after dividing both sides by $2^{(N+1)/p}$), we obtain 

$$\|\nu^{\phi} g\|_{L^p(\mathbb{R})} \leq C_p 2^{1/p}\|g\|_{L^p(\mathbb{R})}.$$  

This finishes the proof of Lemma 4.  

The following lemma in [7] is one of our main tools.

**Lemma 3.** Let $l, m \in \mathbb{N}$ and $\{\sigma_{s,k} : 0 \leq s \leq l \text{ and } k \in \mathbb{Z}\}$ be a family of measures on $\mathbb{R}^m$ with $\sigma_{0,k} = 0$ for every $k \in \mathbb{Z}$. Let $\{\alpha_{k} : 1 \leq s \leq l \text{ and } 1 \leq j \leq 2\} \subset (0, \infty), \{\eta_{s} : 1 \leq s \leq l\} \subset (0, \infty) \setminus \{1\}, \{M_{s} : 1 \leq s \leq l\} \subset \mathbb{N}$, and $L_{s} : \mathbb{R}^m \to \mathbb{R}^{M_{s}}$ be linear transformations for $1 \leq s \leq l$. Suppose

(i) $\|\sigma_{s,k}\| \leq 1$ for $k \in \mathbb{Z}$ and $1 \leq s \leq l$;

(ii) $|\sigma_{s,k}(\xi)| \leq C(\eta^{k}_{s}|L_{s}\xi|)^{-\alpha_{s,k}}$ for $\xi \in \mathbb{R}^m, k \in \mathbb{Z}$ and $1 \leq s \leq l$;

(iii) $|\sigma_{s,k} - \sigma_{s-1,k}(\xi)| \leq C(\eta^{k}_{s}|L_{s}\xi|)^{-\alpha_{s,k}}$ for $\xi \in \mathbb{R}^m, k \in \mathbb{Z}$ and $1 \leq s \leq l$;

(iv) For some $q > 1$, there exists $A_{q} > 0$ such that 

$$\left\| \sup_{k \in \mathbb{Z}} |\sigma_{s,k} \ast f| \right\|_{L^{q}(\mathbb{R}^{m})} \leq A_{q} \|f\|_{L^{q}(\mathbb{R}^{m})}$$

for all $f \in L^{l}(\mathbb{R}^{m})$ and $1 \leq s \leq l$.

Then for every $p \in (\frac{2q}{q+1}, \frac{2q}{q-1})$, there exists a positive constant $C_{p}$ such that

(a) 

$$\left\| \sum_{k \in \mathbb{Z}} \sigma_{l,k} \ast f \right\|_{L^{p}(\mathbb{R}^{m})} \leq C_{p} \|f\|_{L^{p}(\mathbb{R}^{m})}$$

and

(b) 

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{l,k} \ast f|^2 \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{m})} \leq C_{p} \|f\|_{L^{p}(\mathbb{R}^{m})}$$

hold for all $f \in L^{p}(\mathbb{R}^{m})$. The constant $C_{p}$ is independent of the linear transformations $\{L_{s}\}_{s=1}^{l}$.

The following result is just Lemma 5.1 in [7], which follows immediately from Lemma 6.2 in [8] and is an extension of an earlier theorem due to Duoandikoetxea and Rubio de Francia in [9].

**Lemma 4.** Let $s, m \in \mathbb{N}, \eta \in (0, \infty) \setminus \{1\}, \delta_{1}, \delta_{2} > 0$, and $L : \mathbb{R}^s \to \mathbb{R}^m$ be a linear transformation. Suppose that $\{\sigma_{k}\}_{k \in \mathbb{Z}}$ is a sequence of measures on $\mathbb{R}^m$.
such that

$$2S$$

for all $$f \in L^q(\mathbb{R}^m)$$. Then for $$p \in (\frac{2q}{q+1}, \frac{2q}{q-1})$$, there exists a positive constant $$C_p = C(p, s, m, \eta, \delta_1, \delta_2)$$ such that

$$(a) \quad \left\| \sum_{k \in \mathbb{Z}} \sigma_k * f \right\|_{L^p(\mathbb{R}^m)} \leq C_p \| f \|_{L^p(\mathbb{R}^m)}$$

and

$$(b) \quad \left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_k * f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \leq C_p \| f \|_{L^p(\mathbb{R}^m)}$$

hold for all $$f \in L^p(\mathbb{R}^m)$$. The constant $$C_p$$ is independent of the linear transformation $$L$$.

In order to handle truncation in the phase space, we need the following useful lemma, which is Lemma 6.4 in [8].

**Lemma 5.** For $$s \leq d$$, let $$H : \mathbb{R}^s \to \mathbb{R}^s$$ and $$G : \mathbb{R}^d \to \mathbb{R}^d$$ be two nonsingular linear transformations and $$\varphi \in S(\mathbb{R}^s)$$. Define $$J$$ and $$X_r = X_r(\varphi, G, H)$$ by

$$(Jf)(x) = f(G^t(\varphi^t \otimes \text{id}_{\mathbb{R}^d}))((x)$$

and

$$X_r f(x) = J^{-1}([|\Phi_r| \otimes \delta_{\mathbb{R}^{d-s}}] * Jf)(x),$$

where $$x \in \mathbb{R}^d$$, $$r > 0$$, $$G^t$$ and $$H^t$$ are respectively the transposes of $$G$$ and $$H$$, $$\text{id}_{\mathbb{R}^{d-s}}$$ is the identity operator on $$\mathbb{R}^{d-s}$$, $$\delta_{\mathbb{R}^{d-s}}$$ is the Dirac delta operator on $$\mathbb{R}^{d-s}$$, and $$\Phi \in S(\mathbb{R}^s)$$ satisfies $$\hat{\Phi} = \varphi$$. Let $$X = X(\varphi, G, H)$$ be given by

$$X f(x) = \sup_{r > 0} |X_r f(x)|.$$ 

Then for $$1 < p \leq \infty$$, there exists a positive constant $$C_p = C(p, \varphi, s, d)$$ such that

$$\|X f\|_{L^p(\mathbb{R}^d)} \leq C_p \| f\|_{L^p(\mathbb{R}^d)}$$

for all $$f \in L^p(\mathbb{R}^d)$$. The constant $$C_p$$ is independent of the linear transformations $$G$$ and $$H$$.

Now let $$A_1(0, \infty)$$ denote the set of functions $$b$$ on $$(0, \infty)$$ satisfying

$$\sup_{R > 0} \frac{1}{R} \int_0^R |b(t)|^o \; dt < \infty.$$ 

For $$y = (y_1, \cdots, y_n) \in \mathbb{R}^n$$, let $$\tilde{y} = (y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1}$$. We denote the north pole $$(0, \cdots, 0, 1)$$ on $$S^{n-1}$$ by $$\rho_1$$. Let $$F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$ be of the form

$$(4) \quad F(t, y) = t^o q(\tilde{y}) + W_1(t, y) + W_2(t),$$
where \( q : \mathbb{R}^{n-1} \to \mathbb{R} \) is a polynomial, \( W_1 \) satisfies
\[
\frac{\partial^j W_1}{\partial t^j}(t, y) = 0,
\]
and \( W_2(\cdot) \) is an arbitrary function.

The following estimate on the oscillatory integrals is Proposition 5.3 in [8] and is of considerable importance to us.

**Lemma 6.** Let \( \rho \in (0, 1/4), \ l \in \mathbb{N}, \ m \geq 0, \ q(\vec{y}) = \sum_{j=0}^{m} q_j(\vec{y}), \) where \( q_j(\cdot) \) is a homogeneous polynomial of degree \( j \) on \( \mathbb{R}^{n-1} \) for \( 0 \leq j \leq m. \) Let \( F(t, y) \) be given by \((4)\) and \((5)\). Suppose that \( b(\cdot) \in \Delta_\gamma \) for some \( \gamma > 1 \) and \( \Omega(\cdot) \) is a function satisfying
\[
\text{(a) supp}(\Omega) \subset B(\rho_1, \rho); \quad \text{(b) } \left\| \Omega \right\|_{L^\infty(S^{n-1})} \leq \rho^{-n+1}.
\]

If we assume \( q_m(\vec{y}) = \sum_{|\beta|=m} \alpha_\beta \vec{y}^\beta \) and \( \left\| q_m \right\| = \sum_{|\beta|=m} |\alpha_\beta|, \) then there exists a positive constant \( C \) such that
\[
\int_{2^{k+1}}^{2^k} \left| \int_{S^{n-1}} e^{iF(t, y)} \Omega(y) \, d\sigma(y) \right| \frac{\left| b(t) \right|}{t} \, dt \leq C \left( 2^k \rho^m \left\| q_m \right\| \right)^{-\frac{1}{n-1}}.
\]

The constant \( C \) may depend on \( l, \ m, \ n, \) and \( b(\cdot), \) but it is independent of \( k, \ \rho, \ \Omega(\cdot), \) and the coefficients of \( q(\cdot). \)

**Proof of Theorem 1.** Since \( \Omega \in H^1(S^{n-1}) \) and \( \int_{S^{n-1}} \Omega(y) \, d\sigma(y) = 0, \) there are regular atoms \( a_j(\cdot) \) and \( \{ C_j \} \subset \mathbb{C} \) such that
\[
\Omega(y) = \sum_j C_j a_j(y)
\]
by Lemma 1.

Therefore, we only need to be concerned with the case where \( \Omega(y) \) is a regular atom on \( S^{n-1}. \) By Lemma 2 and using a rotation if necessary, we may assume that there is a \( \rho \in (0, 1/4) \) such that
\[
\text{supp}(\Omega) \subset B(\rho_1, \rho), \text{ where } \rho_1 = (0, \cdots, 0, 1);
\]
\[
\left\| \Omega \right\|_{L^\infty(S^{n-1})} \leq \rho^{-(n-1)}; \quad \int_{S^{n-1}} \Omega(y) \, d\sigma(y) = 0.
\]

For any integrable function \( a(\cdot) \) on \( S^{n-1} \) and a suitable mapping \( \Gamma : \mathbb{R}^n \to \mathbb{R}^{n+1}, \) we define the sequence of measures \( \{ \sigma_{a, \Gamma, k} \}_{k \in \mathbb{Z}} \) by
\[
\int_{\mathbb{R}^{n+1}} F \, d\sigma_{a, \Gamma, k} = \int_{\{ y \in \mathbb{R}^{n+1} : 2^k \leq |y| < 2^{k+1} \}} F(\Gamma(y)) \frac{a(\vec{y})}{|y|^n} b(|y|) \, dy.
\]

For \( y \in \mathbb{R}^n \setminus \{0\}, \) let \( \vec{y} = (y_1/|y|, \cdots, y_{n-1}/|y|). \) Let \( N = [\frac{3(n-1)}{2}] + 2 \) (this \( N \) is chosen so that we can have both \((9)\) and \((10)\) for \( j = N). \) For \( j = 1, \cdots, N - 2, \) let \( b_j = (-1)^j \frac{1}{2} \left( \frac{1}{2} - j \right) \cdots \left( \frac{1}{2} - j + 1 \right)/j! . \) Then
\[
|(1 - t)^{1/2} - 1 - \sum_{l=1}^{j-1} b_l t^l| \leq C_j t^j
\]
for \( t \in [0, 1/4]. \)
We now define the mappings $\Gamma_0$, $\Gamma_1$, \ldots, $\Gamma_N$ by
\[
\Gamma_N(y) = (y, \phi(|y|)),
\]
\[
\Gamma_j(y) = (|y|y, |y|(1 + b_1|y|^2 + \cdots + b_{j-1}|y|^{2(j-1)}), \phi(|y|)), \quad j = 2, \ldots, N - 1,
\]
\[
\Gamma_1(y) = (|y|y, |y|, \phi(|y|)),
\]
and
\[
\Gamma_0(y) = (0, |y|, \phi(|y|)).
\]

For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$, we shall establish estimates (ii) and (iii) in Lemma 3 for $\{\hat{\sigma}_{\Omega, \Gamma, k}(\xi, \eta) : 1 \leq j \leq N$ and $k \in \mathbb{Z}\}$. By an inequality on page 551 of [6], we have
\[
|\hat{\sigma}_{\Omega, \Gamma, k}(\xi, \eta)| \leq \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} e^{-i[(\xi, \eta) \cdot (y + \xi + \sum_{s=1}^{n-1} b_s|y|^{2s})]} \Omega(y) d\sigma(y) \right| b(t) \frac{dt}{t}
\]
\[
\leq C \left[2^k |\xi||^{-1/6} \right|^{\Omega}_{L^2(S^{n-1})} \]
\[
\leq C \left[2^k |\rho|^{n-1} \xi_n \right|^{-1/6}.
\]

One observes that the variable $\eta$ does not appear in the previous inequality. The same is true for the Fourier estimates obtained from here on.

Now, for $2 \leq j \leq N - 1$, we have
\[
|\hat{\sigma}_{\Omega, \Gamma, j,k}(\xi, \eta)| \leq \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} e^{-i[(\xi, \eta) \cdot (y + \xi + \sum_{s=1}^{n-1} b_s|y|^{2s})]} \Omega(y) d\sigma(y) \right| b(t) \frac{dt}{t}.
\]
By applying Lemma 6 with $q(y) = -[(\xi_1, \cdots, \xi_{n-1}) \cdot y + \xi_n \sum_{s=1}^{n-1} b_s|y|^{2s}]$, $m = 2(j-1)$, $\gamma = 2$ and $l = 1$, we obtain
\[
|\hat{\sigma}_{\Omega, \Gamma, j,k}(\xi, \eta)| \leq C \left[2^k |\rho^{n-1} \xi_n \right|^{-1/6}.
\]
Finally, by Lemma 6 with $m = 1$, $\gamma = 2$ and $l = 1$, we have
\[
|\hat{\sigma}_{\Omega, \Gamma, 1,k}(\xi, \eta)| \leq \int_{2^k}^{2^{k+1}} \left| \int_{S^{n-1}} e^{-i[(\xi, \eta) \cdot (y + \xi + \sum_{s=1}^{n-1} b_s|y|^{2s})]} \Omega(y) d\sigma(y) \right| b(t) \frac{dt}{t}
\]
\[
\leq C \left[2^k |\rho|^{n-1} \xi_n \right|^{-1/6}.
\]
Let
\[
L_1(\xi, \eta) = \rho(\xi_1, \cdots, \xi_{n-1}), \quad \theta_1 = \frac{1}{8};
\]
\[
L_j(\xi, \eta) = \rho^{2(j-1)} \xi_n, \quad \theta_j = \frac{1}{16(j-1)}, \quad 2 \leq j \leq N - 1;
\]
\[
L_N(\xi, \eta) = \rho^{n-1} \xi_n, \quad \theta_N = \frac{1}{6}.
\]
Then by (6)–(8), we have
\[
|\hat{\sigma}_{\Omega, \Gamma, j,k}(\xi, \eta)| \leq C \left[2^k |L_j(\xi, \eta)| \right]^{-\theta_j}
\]
for $1 \leq j \leq N$, $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$. Next we shall verify that for $(\xi, \eta) \in \mathbb{R}^{n+1}$, $k \in \mathbb{Z}$ and $1 \leq j \leq N$,
\[
|\hat{\sigma}_{\Omega, \Gamma, j,k}(\xi, \eta) - \hat{\sigma}_{\Omega, \Gamma, j-1,k}(\xi, \eta)| \leq C 2^k |L_j(\xi, \eta)|.
\]
Let us begin with $j = N$. In this case, we have
\[
|\tilde{\sigma}_{\Omega, r_{N}, k}(\xi, \eta) - \tilde{\sigma}_{\Omega, r_{N-1}, k}(\xi, \eta)|
\leq \int_{2^{k}}^{2^{k+1}} \left| \int_{S^{n-1}} \left( e^{-i \xi_{n} b_{j-1} |\tilde{y}|^{2(j-1)} - 1 \right) \Omega(y) \, d\sigma(y) \right| |b(t)| \frac{dt}{t}
\leq C \int_{2^{k}}^{2^{k+1}} t|\xi_{n}| \rho^{2(n-1)} \|\Omega\|_{L^{1}(S^{n-1})} \frac{dt}{t}
\leq C 2^{k} \rho^{2(n-1)} \xi_{n} = C 2^{k} L_{N}(\xi, \eta).
\]

For $2 \leq j \leq N - 1$, we have
\[
|\tilde{\sigma}_{\Omega, r_{j}, k}(\xi, \eta) - \tilde{\sigma}_{\Omega, r_{j-1}, k}(\xi, \eta)|
\leq \int_{2^{k}}^{2^{k+1}} \int_{S^{n-1}} \left| e^{-i \xi_{n} b_{j-1} |\tilde{y}|^{2(j-1)} - 1 \right) \Omega(y) \, d\sigma(y) |b(t)| \frac{dt}{t}
\leq C 2^{k} \rho^{2(j-1)} \xi_{n} = C 2^{k} L_{j}(\xi, \eta).
\]

Finally, for $j = 1$, we have
\[
|\tilde{\sigma}_{\Omega, r_{1}, k}(\xi, \eta) - \tilde{\sigma}_{\Omega, r_{0}, k}(\xi, \eta)|
\leq \int_{2^{k}}^{2^{k+1}} \int_{S^{n-1}} \left| e^{-i \xi_{1} b_{0} |\tilde{y}|^{2} - 1 \right) \Omega(y) \, d\sigma(y) |b(t)| \frac{dt}{t}
\leq C 2^{k} \rho^{1} |(\xi_{1}, \ldots, \xi_{n-1})| = C 2^{k} L_{1}(\xi, \eta).
\]

This completes the proof of (10). \qed

We still need to verify condition (iv) in Lemma 3. It suffices to establish the $L^{p}(\mathbb{R}^{n})$ boundedness of the operators $\sigma^{*_j}_{\Omega,1}$ defined by
\[
\sigma^{*_j}_{\Omega,1}(f)(x, s) = \sup_{k \in \mathbb{Z}} |(\sigma_{\Omega, r_{j}, k} \ast f)(x, s)|,
\]
where $j = 1, \ldots, N$, $x \in \mathbb{R}^{n}$, $s \in \mathbb{R}$ and $1 < p < \infty$.

Let us begin with $\sigma^{*_1}_{\Omega,1}$ which is given by
\[
\sigma^{*_1}_{\Omega,1}(f)(x, s) = \sup_{k \in \mathbb{Z}} |(\sigma_{\Omega, r_{1}, k} \ast f)(x, s)|.
\]

Choose $\theta \in C_{0}^{\infty}(\mathbb{R}^{n-1})$ such that $\theta(t) \equiv 1$ for $|t| \leq 1/2$ and $\theta(t) \equiv 0$ for $|t| \geq 1$. For $k \in \mathbb{Z}$, we define $\nu_{k}$ by
\[
\nu_{k}(\xi, \eta) = \theta(2^{k} \rho(\xi_{1}, \ldots, \xi_{n-1})) \tilde{\sigma}_{\Omega, r_{0}, k}(\xi, \eta)
\]
for $\xi \in \mathbb{R}^{n}$ and $\eta \in \mathbb{R}$. Let $\tau_{k} = \tilde{\sigma}_{\Omega, r_{1}, k} - \nu_{k}$. Then by (10) and $|\tilde{\sigma}_{\Omega, r_{0}, k}(\xi, \eta)| \leq C$, we have
\[
|\tilde{\tau}_{k}(\xi, \eta)| \leq |\tilde{\sigma}_{\Omega, r_{1}, k}(\xi, \eta) - \tilde{\sigma}_{\Omega, r_{0}, k}(\xi, \eta)|
\leq |1 - \theta(2^{k} \rho(\xi_{1}, \ldots, \xi_{n-1}))| |\tilde{\sigma}_{\Omega, r_{0}, k}(\xi, \eta)|
\leq C \left[ 2^{k} L_{1}(\xi, \eta) + |2^{k} \rho(\xi_{1}, \ldots, \xi_{n-1})| \right]
= C 2^{k} L_{1}(\xi, \eta).
\]

If $2^{k} |L_{1}(\xi, \eta)| > 1$, by (9), we have
\[
|\tilde{\tau}_{k}(\xi, \eta)| \leq C (2^{k} |L_{1}(\xi, \eta)|)^{-1/8}.
\]
Let
\[ |\tau_k(\xi, \eta)| \leq C \left[ \min \{2^k|L_1(\xi, \eta)|, (2^k|L_1(\xi, \eta)|)^{-1}\} \right]^{1/8}. \]

By the

\[ L \]

Also, from (11), it is easy to deduce that
\[ \tau^*(f)(x, s) = \sup_{k \in \mathbb{Z}} |(|\tau_k| * f)(x, s)|, \quad \nu^*(f)(x, s) = \sup_{k \in \mathbb{Z}} |(|\nu_k| * f)(x, s)| \]
and
\[ g_\tau(f)(x, s) = \left\{ \sum_{k \in \mathbb{Z}} |(|\tau_k| * f)(x, s)|^2 \right\}^{1/2}. \]

Then
\[ \sigma^*_{[\Omega, 1]}(f)(x, s) \leq g_\tau(f)(x, s) + \nu^*(f)(x, s) \]
and
\[ \tau^*(f)(x, s) \leq \sigma^*_{[\Omega, 1]}(|f|(x, s)) \leq g_\tau(|f|(x, s)) + 2\nu^*|f|(x, s). \]

By the $L^p(\mathbb{R}^2)$ boundedness of $M_\varphi$ and Lemma 5, for $1 < p < \infty$, we have
\[ \|\nu^*(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{n+1})}. \]

Also, from (11), it is easy to deduce that
\[ \|g_\tau(f)\|_{L^2(\mathbb{R}^{n+1})} \leq C \|f\|_{L^2(\mathbb{R}^{n+1})}. \]

Thus, (13) implies that
\[ \|\tau^*(f)\|_{L^2(\mathbb{R}^{n+1})} \leq C \|f\|_{L^2(\mathbb{R}^{n+1})}. \]

By invoking Lemma 4, we obtain
\[ \|g_\tau(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{n+1})} \]
for $4/3 < p < 4$. Thus, by (13) again, we obtain
\[ \|\tau^*(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{n+1})} \]
for $4/3 < p < 4$. By using (14), (13) and repeating the preceding argument, we obtain
\[ \|g_\tau(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{n+1})} \]
for $1 < p < \infty$. Now, from (12), it follows that
\[ \|\sigma^*_{[\Omega, 1]}(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{n+1})} \]
for $1 < p < \infty$.

Similarly, we can show that
\[ \|\sigma^*_{[\tilde{\Omega}, j]}(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{n+1})} \]
for $1 \leq j \leq N$. Now, by (9), (10), (15) and Lemma 3, we have
\[ \left\| \sum_{k \in \mathbb{Z}} \sigma_{[\Omega_1, \Gamma, k] * f} \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{n+1})}. \]
for $1 < p < \infty$. Noting that
\[
\sum_{k \in \mathbb{Z}} \left( \sigma_{\Omega, r, k} \ast f \right)(x, s) = \int_{\mathbb{R}^n} f(x - y, s - \phi(|y|)) \frac{\Omega(y)}{|y|^n} b(|y|) \, dy,
\]
we thus obtain a proof of our theorem.

Acknowledgements

The authors would like to thank the referee for helpful comments.

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