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A NUMERICAL CONDITION FOR A DEFORMATION OF A GORENSTEIN SURFACE SINGULARITY TO ADMIT A SIMULTANEOUS LOG-CANONICAL MODEL

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ABSTRACT. Let $\pi\colon X \to T$ be a deformation of a normal Gorenstein surface singularity over the complex number field $\mathbb{C}$. We assume that T is a neighborhood of the origin of $\mathbb{C}$. Then we prove that π admits a simultaneous log-canonical model if and only if an invariant $-P_t \cdot P_t$ of each fiber X_t is constant.

1. Introduction

Let $\pi\colon X \to T$ be a deformation of a normal Gorenstein surface singularity over the complex number field $\mathbb{C}$. We assume that T is a sufficiently small neighborhood of the origin of $\mathbb{C}$. In his paper [8], Laufer proved that the deformation admits a simultaneous canonical model if and only if $-K_t \cdot K_t$ is constant. In this paper, we prove the log-version of Laufer's results. Instead of $-K \cdot K$, we adopt an invariant $-P \cdot P$. It is a numerical invariant of a normal surface singularity, and its fundamental properties are stated in [14]. For example, the equality below is proved in [14, Introduction]:

$$-P \cdot P/2 = \limsup_{m\to\infty} \delta_m/m^2. \tag{1.1}$$

In this equality, δ_m denotes the m-th L^2-plurigenus of the singularity. Since $\delta_m(X_t)$ is upper semicontinuous for any m by [3], it follows from the equality (1.1) that $-P \cdot P$ is upper semicontinuous. Our main theorem is the following:

Theorem 1.1. *The following conditions are equivalent*:

(1) π *admits a simultaneous log-canonical model*;
(2) $-P_t \cdot P_t$ *is constant.*

The implication (1) $\Rightarrow$ (2) follows from the invariance of the log-plurigenus $\lambda_m(X_t)$ with $m >> 0$ and an equality similar to (1.1). Let $f\colon Y \to X$ be a log-canonical model of X with the exceptional divisor E. Then $f_t\colon Y_t \to X_t$ is a log-canonical model for $t \in T \setminus \{0\}$ near 0 (Lemma 3.1). We define sheaves $\mathcal{I}_m$ and $\mathcal{A}_m$ by $\mathcal{I}_m = f_*\mathcal{O}_Y(m(K_Y + E))$ and $\mathcal{A}_m = \psi_*\mathcal{O}_M(m(K_M + A))$, where

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$\psi\colon (M, A) \to (X_0, x)$ is a good resolution. Since $Y_0 = \operatorname{Proj} \bigoplus_{m \geq 0} \mathcal{I}_m \otimes \mathbb{C}(0)$, the implication (2) $\Rightarrow$ (1) follows from the claim that if $-P_t \cdot P_t$ is constant, then $\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{A}_m$. To show the claim, we will use Izumi's results on the analytic orders [6] (precisely, Ishii's version [4]) and formulas for the plurigenera, and prove the torsion freeness of the sheaf $\mathcal{O}_X(mK_X)/\mathcal{I}_m$ (Lemma 4.8). We will also prove that if a morphism $f\colon Y \to X$ is a simultaneous log-canonical model, then it is a log-canonical model of X (Lemma 4.2). Our methods will be used in higher dimensions (Remark 4.12).

We denote by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ the set of positive integers, the set of rational numbers and the set of real numbers, respectively.

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2. Preliminaries

2.1. **Log-canonical models.** Let X be a normal variety over $\mathbb{C}$ of dimension $d \geq 2$. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X, where the D_i are distinct prime divisors and $d_i \in \mathbb{Q}$. We put $D_{red} = \sum_{d_i \neq 0} D_i$. Let $f\colon Y \to X$ be a birational morphism of normal varieties and E the maximal reduced exceptional divisor on Y. For a divisor D on X, we denote by $f_*^{-1}(D)$ the strict transform of D under the morphism f. The morphism $f\colon Y \to X$ is called a good resolution of a pair (X, D), if Y is nonsingular and $(f_*^{-1}(D) + E)_{red}$ is a divisor with only simple normal crossings. A $\mathbb{Q}$-divisor B on X is called a boundary if it satisfies $0 \leq B \leq B_{red}$.

Definition 2.1. Let B be a boundary on X. A divisor $K_X + B$ is said to be log-canonical if the following conditions are satisfied:

(1) $K_X + B$ is a $\mathbb{Q}$-Cartier divisor.
(2) There exists a good resolution $f\colon Y \to X$ of (X, B) such that
$$K_Y + f_*^{-1}(B) = f^*(K_X + B) + \sum a_i E_i$$
for $a_i \in \mathbb{Q}$ with the condition that $a_i \geq -1$, where the E_i vary all the exceptional prime divisors on Y.

If $K_X + B$ is log-canonical, then the equality in condition (2) above is satisfied with $a_i \geq -1$ for any resolution of X (see [7, Lemma 0-2-12]). Hence we have the following:

Lemma 2.2. *Let $f\colon Y \to X$ be any good resolution of a pair (X, B) with the exceptional divisor E, and let $B_Y = f_*^{-1}(B) + E$. If $K_X + B$ is log-canonical, then we have $f_*\mathcal{O}_Y(m(K_Y + B_Y)) = \mathcal{O}_X(m(K_X + B))$ for any $m \in \mathbb{N}$.*

Definition 2.3. Let $f\colon Y \to X$ be a birational morphism with the exceptional divisor E. Then the morphism $f\colon Y \to X$ is called a log-canonical model of X, if the divisor $K_Y + E$ is log-canonical and f-ample.

Theorem 2.4 (see [1, 6.16])**.** *Let X be a normal variety of dimension $d \leq 3$. Then there exists a unique log-canonical model $f\colon Y \to X$ of X. In fact, the following morphism is the log-canonical model*:
$$\operatorname{Proj} \bigoplus_{n \geq 0} h_*\mathcal{O}_Z(n(K_Z + F)) \to X,$$
where $h\colon Z \to X$ is any proper birational morphism with the exceptional divisor F such that the divisor $K_Z + F$ is log-canonical.

2.2. **Plurigenera.** Let (X,x) be a normal isolated singularity and $f\colon (M,A) \to (X,x)$ a good resolution of the singularity (X,x), where A is the exceptional divisor.

Definition 2.5 (cf. [9], [15])**.** We define the log-plurigenera $\{\lambda_m(X,x)\}_{m\in\mathbb{N}}$ and the L^2-plurigenera $\{\delta_m(X,x)\}_{m\in\mathbb{N}}$ by

$$\lambda_m(X,x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_M(m(K_M+A)),$$
$$\delta_m(X,x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_*\mathcal{O}_M(m(K_M+A)-A).$$

These definitions are independent of the choice of a good resolution.

Proposition 2.6. *Let $g\colon Y \to X$ be a partial resolution with the exceptional divisor E such that K_Y+E is log-canonical. Then we have*

$$\lambda_m(X,x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/g_*\mathcal{O}_Y(m(K_Y+E)).$$

Proof. It follows from Lemma 2.2. □

Let (X,x) be a normal surface singularity and $f\colon (M,A) \to (X,x)$ the minimal good resolution. Let K be a canonical divisor on M. Let $A = \bigcup_{i=1}^k A_i$ be the decomposition of A into irreducible components. By [12, Theorem A.1], $K+A$ admits a unique Zariski decomposition $P+N$, where P and N are elements of $\sum_{i=1}^k \mathbb{Q}A_i$ satisfying the following:

(1) $(K+A)\cdot A_i = (P+N)\cdot A_i$ for all i;
(2) P is f-nef and N is effective;
(3) $P\cdot N = 0$.

Theorem 2.7 (see [14], [11])**.** *For every $m \in \mathbb{N}$, we have*

$$\lambda_m(X,x) = -(P\cdot P)m^2/2 + (P\cdot K)m/2 + b_1(m),$$
$$\delta_m(X,x) = -(P\cdot P)(m-1)^2/2 - (P\cdot K)(m-1)/2 + b_2(m),$$

where $b_1(m)$ and $b_2(m)$ are bounded functions of m.

3. Deformations and some invariants

Let $\pi\colon X \to T$ be a deformation of a normal Gorenstein surface singularity $(X_0,x) = \pi^{-1}(0)$, where T is a neighborhood of the origin of $\mathbb{C}$. Then X is a Gorenstein variety. Therefore, for any $t \in T$, we have $\mathcal{O}_{X_t}(mK_X) \cong \mathcal{O}_{X_t}(mK_{X_t})$. For any morphism $h\colon W \to X$, we denote by W_t the fiber $(\pi\circ h)^{-1}(t)$ and set $h_t = h|_{W_t}$. Let $f\colon Y \to X$ be the log-canonical model of X with the maximal reduced exceptional divisor E on Y. Then $\mathcal{O}_Y(m(K_Y+E))$ is a Cohen-Macaulay $\mathcal{O}_Y$-module for $m \in \mathbb{N}$. We define sheaves $\mathcal{I}_m$ and $\mathcal{Q}_m$ by $\mathcal{I}_m = f_*\mathcal{O}_Y(m(K_Y+E))$ and $\mathcal{Q}_m = \mathcal{O}_X(mK_X)/\mathcal{I}_m$ for every $m \in \mathbb{N}$. Put $T^* = T \setminus \{0\}$. We assume that T is small enough to satisfy the following conditions:

(1) there exists a good resolution $g\colon M \to Y$ of (Y,E) which induces a good resolution $M_t \to X_t$ for any $t \in T^*$;
(2) Y_t is a normal surface with the exceptional divisor $E_t = E \cap Y_t$ for any $t \in T^*$.

Lemma 3.1. *For any $t \in T^*$, the restriction $f_t\colon Y_t \to X_t$ is the log-canonical model of X_t.*

Proof. Let t be a point of T^*. Since the normal sheaf of Y_t in Y is trivial and $\mathcal{O}_Y(m(K_Y+E))$ is Cohen-Macaulay, we obtain that

$$\mathcal{O}_{Y_t}(m(K_Y+E)) \cong \mathcal{O}_{Y_t}(m(K_{Y_t}+E_t))$$

for every $m \in \mathbb{N}$. Hence the f_t-ampleness of the divisor $K_{Y_t}+E_t$ follows from the f-ampleness of K_Y+E. Let $g\colon M \to Y$ be a resolution as in (1) above. Let F be the exceptional divisor of g. Then we obtain that

$$K_M + g_*^{-1}(E) + F = g^*(K_Y+E) + \Delta,$$

where Δ is an effective divisor supported in F. Restricting those divisors to M_t, we see that $K_{Y_t}+E_t$ is log-canonical. □

Let W be a normal variety whose singular locus W_{sing} is a finite set. Then we put

$$\lambda_m(W) = \sum_{w \in W_{sing}} \lambda_m(W,w) \quad \text{and} \quad \delta_m(W) = \sum_{w \in W_{sing}} \delta_m(W,w).$$

Let $\psi(t)\colon M_t \to X_t$ be the minimal good resolution of the singularities and K_t the canonical divisor on M_t. Let $A_{t,p}$ be a connected component of the exceptional set A_t on M_t which blows down to $p \in (X_t)_{sing}$. Let $P_{t,p}+N_{t,p}$ be the Zariski decomposition of $K_t + A_{t,p}$, where $P_{t,p}$ and $N_{t,p}$ are $\mathbb{Q}$-divisors supported in $A_{t,p}$. We define a $\mathbb{Q}$-divisor P_t on M_t by $P_t = \sum_{p\in(X_t)_{sing}} P_{t,p}$, and regard $-P_t \cdot P_t$ as a function of t.

Proposition 3.2. *For any $m \in \mathbb{N}$, we obtain that*

$$\lambda_m(X_t) = -(P_t\cdot P_t)m^2/2 + (P_t \cdot K_t)m/2 + b_t(m),$$
$$\delta_m(X_t) = -(P_t\cdot P_t)(m-1)^2/2 - (P_t \cdot K_t)(m-1)/2 + b'_t(m),$$

where b_t and b'_t are bounded functions of m. Furthermore, $-P_t\cdot P_t$ is upper semi-continuous.

Proof. It follows from Theorem 2.7 and Introduction. □

Let $\mathbb{C}(t)$ be the residue field of $t \in T$, i.e., $\mathbb{C}(t) = \mathcal{O}_{T,t}/\mathcal{M}_t$, where $\mathcal{M}_t$ is the maximal ideal. We write $\otimes\mathbb{C}(t)$ instead of $\otimes_{\mathcal{O}_T}\mathbb{C}(t)$. By Nakayama's lemma, we obtain the inequality

$$\dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) \le \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0). \tag{3.1}$$

Let $\mathcal{I}_{m,0}$ be the image of the natural homomorphism $\mathcal{I}_m \otimes \mathbb{C}(0) \to \mathcal{O}_{X_0}(mK_{X_0})$. Then $\mathcal{Q}_m \otimes \mathbb{C}(0) \cong \mathcal{O}_{X_0}(mK_{X_0})/\mathcal{I}_{m,0}$.

Lemma 3.3. *The following conditions are equivalent*:

(1) $\dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0)$ *for $t \in T^*$;*
(2) $\mathcal{Q}_m$ *is a torsion free $\mathcal{O}_T$-module*;
(3) $\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0}$.

Lemma 3.4. *There exists a closed analytic subset $S \subset T$ such that $\lambda_m(X_t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t)$ for any $t \in T \setminus S$ and any $m \in \mathbb{N}$.*

Proof. There exists an integer m_0 such that $R^1 f_* \mathcal{O}_Y(m(K_Y+E)) = 0$ for $m > m_0$. Let S be a closed analytic subset of T such that the $\mathcal{O}_T$-coherent sheaf

$R^1 f_* \mathcal{O}_Y(m(K_Y+E))$ is torsion free outside S for $m \le m_0$. Then, for any $t \in T \setminus S$, we have the exact sequence

$$0 \to f_*\mathcal{O}_Y(m(K_Y+E)) \xrightarrow{\times\tau} f_*\mathcal{O}_Y(m(K_Y+E)) \to f_*\mathcal{O}_{Y_t}(m(K_{Y_t}+E_t))$$
$$\to R^1 f_*\mathcal{O}_Y(m(K_Y+E)) \xrightarrow{\times\tau} R^1 f_*\mathcal{O}_Y(m(K_Y+E)),$$

where τ is a local parameter at t. Since the last arrow is injective, it follows that $f_*\mathcal{O}_{Y_t}(m(K_{Y_t}+E_t))$ is the image of $f_*\mathcal{O}_Y(m(K_Y+E))$. Hence we obtain that

$$\mathcal{Q}_m \otimes \mathbb{C}(t) = \mathcal{O}_{X_t}(mK_{X_t})/f_{t*}\mathcal{O}_{Y_t}(m(K_{Y_t}+E_t)).$$

Now the assertion follows from Proposition 2.6 and Lemma 3.1. □

Let $\psi\colon (M,A) \to (X_0,x)$ be a good resolution. For each $m \in \mathbb{N}$, we put $\mathcal{A}_m = \psi_*\mathcal{O}_M(m(K_M+A))$ and define invariants ϵ_m and θ_m by

$$\epsilon_m = \dim_{\mathbb{C}} \mathcal{A}_m/(\mathcal{I}_{m,0} \cap \mathcal{A}_m), \quad \text{and}$$
$$\theta_m = \dim_{\mathbb{C}} \mathcal{I}_{m,0}/\left(\mathcal{A}_m \cap \mathcal{I}_{m,0}\right), \quad \text{respectively.}$$

Using Proposition 3.4 and the equality (3.1), we obtain the inequality

$$\lambda_m(X_t) \le \lambda_m(X_0) + \epsilon_m - \theta_m. \tag{3.2}$$

Lemma 3.5. *There exist $a, b \in \mathbb{Q}$ such that $\epsilon_m \le am + b$.*

Proof. Let ω be a section of $\psi_*\mathcal{O}_M(m(K_M+A)-A)$. By [3, Theorem 2.1], there exists a section ω' of $f_*\mathcal{O}_Y(m(K_Y+E)-E)$ such that the image of ω' in $\mathcal{O}_{X_0}(mK_{X_0})$ is ω. Since $f_*\mathcal{O}_Y(m(K_Y+E)-E) \subset \mathcal{I}_m$, we see that ω belongs to $\mathcal{I}_{m,0}$. Hence we obtain that $\psi_*\mathcal{O}_M(mK_M+(m-1)A) \subset \mathcal{I}_{m,0} \cap \mathcal{A}_m$. This implies that

$$\epsilon_m \le \dim_{\mathbb{C}} \mathcal{A}_m/\psi_*\mathcal{O}_M(mK_M+(m-1)A) = \delta_m(X_0,x) - \lambda_m(X_0,x).$$

From Theorem 2.7, we obtain the assertion. □

4. The main theorem

In this section, we prove the main theorem. Let $\pi\colon X \to T$ be as in the preceding section.

Definition 4.1. Let $f\colon Y \to X$ be a birational morphism with the maximal reduced exceptional divisor E. Suppose that $K_Y + E$ is $\mathbb{Q}$-Cartier. We call f a simultaneous log-canonical model if for any $t \in T$, $f_t\colon Y_t \to X_t$ is the log-canonical model of X_t and E_t is a reduced divisor.

A simultaneous log-canonical model of $\pi\colon X \to T$ is unique if it exists. In fact, we have the following:

Lemma 4.2. *Let $f\colon Y \to X$ be a simultaneous log-canonical model with the exceptional divisor E. Then f is the log-canonical model of X and E_t is the exceptional set on Y_t for any $t \in T$.*

Proof. Let B be the union of the curves C on Y such that $C \not\subseteq E$ and $f(C) = \{x\}$, the singular point of X_0. Then $E \cup B$ is the exceptional set of f, and hence $E_0 \cup B$ is the exceptional set of f_0. Since X is Gorenstein, we may assume that $(K_{Y_0})_{red} \subset E_0$. By assumption, $K_{Y_0} + E_0 + B$ is f_0-ample. Thus there exists a positive divisor D supported on $E_0 \cup B$ such that $K_{Y_0} + E_0 + B = -D$. Hence B is void, and E_0 is the exceptional set of f_0. Since $K_{Y_t} + E_t$ is f_t-ample for any $t \in T$, we see that $K_Y + E$ is f-ample. By inversion of adjunction [1, Chap. 17], $K_Y + E$ is log-canonical, since $K_{Y_t} + E_t$ is log-canonical for any $t \in T$. □

Remark 4.3. In the situation above, we see that $h^0(\mathcal{O}_{E_t}) = 1$ for any t. This implies that E_t is connected. Hence X_t has at most one non-log-canonical singularity.

In the following, the notation is the same as in the preceding section, unless otherwise specified; so $f\colon Y \to X$ denotes the canonical model of X. We assume that T is sufficiently small in each case. For each $m \in \mathbb{N}$, we regard $\lambda_m(X_t)$ as a function of t.

Proposition 4.4. *Suppose that $f\colon Y \to X$ is the simultaneous log-canonical model of the deformation $\pi\colon X \to T$. Then $\lambda_m(X_t)$ is constant for $m >> 0$.*

Proof. Consider the exact sequence

$$0 \to f_*\mathcal{O}_Y(m(K_Y + E)) \xrightarrow{\times\tau} f_*\mathcal{O}_Y(m(K_Y + E)) \to f_*\mathcal{O}_{Y_0}(m(K_{Y_0} + E_0)) \to R^1 f_*\mathcal{O}_Y(m(K_Y + E)),$$

where τ is a local parameter at 0. Since $K_Y + E$ is f-ample, $R^1 f_*\mathcal{O}_Y(m(K_Y+E)) = 0$ for $m >> 0$. We fix such an integer m. Then $f_*\mathcal{O}_{Y_0}(m(K_{Y_0}+E_0)) = \mathcal{I}_m \otimes \mathbb{C}(0)$. It is clear that $\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0}$. Thus $\dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0) = \lambda_m(X_0)$ by Proposition 2.6. Then Lemma 3.3 and 3.4 imply that $\lambda_m(X_t) = \lambda_m(X_0)$ for $t \in T^*$. $\square$

In [13], Tomari and Watanabe proved a result on the order of growth of the L^2-plurigenera by applying Izumi's results on the analytic orders [6]. We will use their argument. The following lemma is Ishii's version [4, Lemma 1.5].

Lemma 4.5. *Let (W, w) be a d-dimensional normal isolated singularity, and let $h\colon W_1 \to W$ be a resolution of the singularity which factors through the blowing up by the maximal ideal of the singular point. Let $F = \bigcup_{i=1}^k F_i$ be the exceptional divisor on W_1, where the F_i are irreducible components. Then there exist positive numbers $\beta \in \mathbb{R}$ and $b \in \mathbb{N}$ such that*:

For an $\mathcal{O}_W$-ideal $J = h_*\mathcal{O}_{W_1}(-\sum_{i=1}^k a_i F_i)$ with $a_i > b$ for some i, the inequalities $\dim_{\mathbb{C}} \mathcal{O}_W/J \geq \beta(a_i)^d$ $(i = 1, \dots, k)$ hold.

Lemma 4.6. *Assume that $-P_t \cdot P_t$ is constant. Then $\mathcal{I}_{m,0} \subset \mathcal{A}_m$ for all $m \in \mathbb{N}$.*

Proof. It suffices to show that $\theta_m = 0$ for all $m \in \mathbb{N}$. Note that Proposition 3.2, the inequality (3.2) and Lemma 3.5 imply that θ_m is bounded by a linear function. Assume that $\theta_r \neq 0$. Let ω be a section of $\mathcal{I}_{r,0}$ which does not belong to $\mathcal{A}_r$. We define a homomorphism $\varphi_m\colon \mathcal{O}_{X_0} \to \mathcal{I}_{mr,0}$ by $\varphi_m(s) = s\omega^m$ for every $m \in \mathbb{N}$. We denote by J_m the inverse image $\varphi_m^{-1}(\mathcal{A}_{mr} \cap \mathcal{I}_{mr,0})$. Then we have the injection

$$\mathcal{O}_{X_0}/J_m \to \mathcal{I}_{mr,0}/(\mathcal{A}_{mr} \cap \mathcal{I}_{mr,0}).$$

We put $a_i = \min\{\, v_i(\omega) + r, 0 \,\}$, where v_i is the valuation at an irreducible component A_i of A. Then $J_m = \psi_*\mathcal{O}_M(\sum m a_i A_i)$. By the choice of ω, there exists a component A_i such that $a_i < 0$. We may assume that the good resolution $\psi\colon (M, A) \to (X_0, x)$ factors through the blowing up by the maximal ideal of the point x. By Lemma 4.5, there exists a positive number $c \in \mathbb{R}$ such that $\theta_{mr} \geq cm^2$ for any $m \in \mathbb{N}$. It contradicts the fact noted above. $\square$

The technique above is used in the proof of [5, Theorem 1]. In [5], a claim similar to "$\mathcal{A}_m \subset \mathcal{I}_{m,0}$" is proved by using [10, Lemma 1]. The upper semi-continuity of the plurigenus γ_m follows from the claim. Unfortunately we have no log-version of [10, Lemma 1]. However, we can show the claim "$\mathcal{A}_m \subset \mathcal{I}_{m,0}$" under the condition that $-P_t \cdot P_t$ is constant.

Lemma 4.7. *Let F be any prime divisor on Y such that $f(F) = \{x\}$. Then there exists $m > 0$ such that $f_*\mathcal{O}_Y(m(K_Y + E) + F)/\mathcal{I}_m \neq 0$.*

Proof. We note that F is a projective surface. Let $\mathcal{I}_F$ be an $\mathcal{O}_Y$-ideal of the subvariety F, and let $L_m = m(K_Y + E)$. Since L_1 is f-ample, there exists an integer $n \in \mathbb{N}$ such that $\mathcal{O}_F(L_n)$ is a very ample invertible sheaf and the following sequence is exact for any $m \in \mathbb{N}$:

$$0 \to f_*(\mathcal{I}_F\mathcal{O}_Y(L_{mn} + F)) \to f_*\mathcal{O}_Y(L_{mn} + F) \to H^0(\mathcal{O}_F(L_{mn} + F)) \to 0.$$

By [2, III, Ex. 5.2], there exists a polynomial q' of degree 2 such that

$$\dim_{\mathbb{C}} f_*\mathcal{O}_Y(L_{mn} + F)/f_*(\mathcal{I}_F\mathcal{O}_Y(L_{mn} + F)) = q'(m)$$

for $m >> 0$. Since $\mathcal{I}_F\mathcal{O}_Y(L_{mn} + F)$ is isomorphic to $\mathcal{O}_Y(L_{mn})$ outside a one-dimensional subvariety of F, there exists a polynomial q of degree 2 such that $\dim_{\mathbb{C}} f_*\mathcal{O}_Y(L_{mn} + F)/\mathcal{I}_{mn} \geq q(m)$ for $m >> 0$. □

Lemma 4.8. *Assume that $-P_t \cdot P_t$ is constant. Then $\mathcal{Q}_m$ is a torsion free $\mathcal{O}_T$-module for any $m \in \mathbb{N}$.*

Proof. Assume that $\mathcal{Q}_m$ has a torsion element. Then there exists the maximal reduced divisor $D \neq 0$ on Y such that $f(D) = \{x\}$. Let $h\colon Z \to Y$ be the blowing up by an ideal sheaf $\mathcal{O}_Y(-D)$. Then h is an isomorphism in codimension one and $h_*^{-1}(D)$ is a Cartier divisor. Thus there exists a prime divisor $F \subset h_*^{-1}(D)$ such that the intersection $F \cap (f \circ h)_*^{-1}(X_0)$ is one-dimensional. By Lemma 4.7, there exist $r > 0$ and $\omega \in \mathcal{O}_X(rK_X)$ with the condition that the image of ω in $\mathcal{Q}_r$ is a torsion element and $v_F(\omega) < -r$, where v_F denotes the valuation at F. Suppose that $v_F(\omega) \leq v_F(\omega')$ for all $\omega' \in \mathcal{O}_X(rK_X)$ with the condition above. We denote by η_1 (resp. η_2) the image of $\eta \in \mathcal{O}_X(mK_X)$ in $\mathcal{Q}_m \otimes \mathbb{C}(0)$ (resp. $\mathcal{O}_{X_0}(mK_{X_0})/\mathcal{A}_m$). We may assume that the good resolution $\psi\colon (M, A) \to (X_0, x)$ factors through $(f \circ h)_*^{-1}(X_0) \to X_0$. We denote by $\mathcal{T}_m$ the torsion submodule of $\mathcal{Q}_m$ for $m \in \mathbb{N}$. By the choice of ω, $\omega_1 \in \mathcal{T}_r \otimes \mathbb{C}(0)$, and ω_1 and ω_2 are nonzero. For each $m \in \mathbb{N}$, define a homomorphism $\varphi_m\colon \mathcal{O}_{X_0} \to \mathcal{O}_{X_0}(mrK_{X_0})/\mathcal{A}_{mr}$ by $\varphi_m(s) = (s\omega^m)_2$. By an argument similar to that in the proof of Lemma 4.6, there exists a positive number c such that $\dim_{\mathbb{C}} \varphi_m(\mathcal{O}_{X_0}) \geq cm^2$. By Lemma 4.6, we have the natural surjection $\mathcal{Q}_{mr} \otimes \mathbb{C}(0) \to \mathcal{O}_{X_0}(mrK_{X_0})/\mathcal{A}_{mr}$. Since the image of $\mathcal{T}_{mr} \otimes \mathbb{C}(0)$ in $\mathcal{O}_{X_0}(mrK_{X_0})/\mathcal{A}_{mr}$ contains $\varphi_m(\mathcal{O}_{X_0})$, we have $\dim_{\mathbb{C}} \mathcal{T}_{mr} \otimes \mathbb{C}(0) \geq cm^2$. Then, for $t \in T^*$, we have the inequality

$$\dim_{\mathbb{C}} \mathcal{Q}_{mr} \otimes \mathbb{C}(t) \leq \dim_{\mathbb{C}} \mathcal{Q}_{mr} \otimes \mathbb{C}(0) - cm^2.$$

This implies the following:

$$\lambda_{mr}(X_t) \leq \lambda_{mr}(X_0) + \varepsilon_{mr} - cm^2.$$

By Proposition 3.2 and Lemma 3.5, we get $-P_t \cdot P_t < -P_0 \cdot P_0$. □

Remark 4.9. Assume that $-P_t \cdot P_t$ is constant. From the proof above, we see that Y_0 is irreducible. Hence any irreducible component of E dominates T. Since Y_0 is a principal divisor, for any irreducible component F of E, the intersection $F \cap Y_0$ is a one-dimensional variety.

Lemma 4.10. *Assume that $-P_t \cdot P_t$ is constant. Then $\mathcal{I}_{m,0} = \mathcal{A}_m$ for any $m \in \mathbb{N}$.*

Proof. By Lemma 4.6, it suffices to show that $\mathcal{A}_m \subset \mathcal{I}_{m,0}$. Let ω be a section of $\mathcal{A}_m$ and ω' a section of $\mathcal{O}_X(mK_X)$ whose image in $\mathcal{O}_{X_0}(mK_{X_0})$ is ω. We may assume that $\psi\colon (M,A) \to (X_0,x)$ factors through $f_0\colon Y_0 \to X_0$. If $v_F(\omega') < -m$ for an irreducible component F of E, then there exists an irreducible component A_i of A lying over the variety $F \cap Y_0$ such that $v_{A_i}(\psi^*\omega) < -m$. It contradicts the choice of ω. Hence ω' belongs to $\mathcal{I}_m$, and ω belongs to $\mathcal{I}_{m,0}$. □

Now we are ready to prove the main theorem which is the log-version of Laufer's result on simultaneous canonical models.

Theorem 4.11. *The following conditions are equivalent*:

(1) *$\pi\colon X \to T$ admits the simultaneous log-canonical model.*
(2) *The function $\lambda_m(X_t)$ of t is constant for any $m \in \mathbb{N}$.*
(3) *The function $-P_t \cdot P_t$ of t is constant.*

Proof. Consider the condition: $(2)'$ $\lambda_m(X_t)$ is constant for $m >> 0$. By Proposition 3.2 and 4.4, we obtain the implication $(1) \Rightarrow (2)' \Rightarrow (3)$. We assume that $-P_t \cdot P_t$ is constant. Then, from Lemma 3.3, 4.8 and 4.10, we obtain the following equalities for any $m \in \mathbb{N}$:

$$\mathcal{I}_m \otimes \mathbb{C}(0) = \mathcal{I}_{m,0} = \mathcal{A}_m, \quad \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(t) = \dim_{\mathbb{C}} \mathcal{Q}_m \otimes \mathbb{C}(0).$$

Now it is clear that (2) holds, and that $Y_0 = \operatorname{Proj}(\bigoplus_{m\geq 0} \mathcal{I}_m \otimes \mathbb{C}(0))$ is the log-canonical model of X_0. Since $\mathcal{A}_m = f_*\mathcal{O}_{Y_0}(m(K_{Y_0} + E_0))$ for $m >> 0$ (cf. the proof of Proposition 4.4), and $K_{Y_0} + E_0$ is f_0-ample, we see that E_0 is reduced. On the other hand, $f_t\colon Y_t \to X_t$ is the log-canonical model for $t \in T^*$ by Lemma 3.1. Hence we obtain the condition (1). □

Remark 4.12. The techniques in this paper can be used in higher dimensions. In fact, we can generalize the main theorem to the higher-dimensional case if the following claims hold:

(1) there exists a log-canonical model $f\colon Y \to X$ of X;
(2) for an isolated singularity (W,w) of dimension n, there exists $p = p(W,w) \in \mathbb{R}$ such that $\lambda_m(W,w) = pm^n + O(m^{n-1})$.

Note that $\delta_m(W,w) = \lambda_m(W,w) + O(m^{n-1})$ by an argument in [14, p. 627].

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