

AN ANALOGUE OF HAJÓS' THEOREM FOR THE CIRCULAR CHROMATIC NUMBER

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ABSTRACT. This paper designs a set of graph operations and proves that starting from G_d^k , by repeatedly applying these operations, one can construct all graphs G with $\chi_c(G) \geq k/d$ (for $k/d \geq 3$). This can be viewed as an analogue of Hajós' Theorem for the circular chromatic number.

1. INTRODUCTION

The circular chromatic number (also known as the star chromatic number) of a graph is a natural generalization of the chromatic number of a graph. Given two integers k, d such that $k \geq d$, a (k, d) -coloring of a graph G is a coloring ϕ of the vertices of G with colors $0, 1, 2, \dots, k-1$ such that for any two adjacent vertices x and y of G , we have $d \leq |\phi(x) - \phi(y)| \leq k-d$. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of the ratio k/d for which there exists a (k, d) -coloring of G . An equivalent definition of $\chi_c(G)$ is as follows: Suppose $r \geq 1$ is a real number. Let C^r be a circle in the Euclidean plane of circumference r . For a graph $G = (V, E)$, an r -circular coloring f of G is a mapping which maps each vertex x of G to a unit length arc $f(x)$ of C^r such that for every edge xy of G , $f(x) \cap f(y) = \emptyset$. The circular chromatic number $\chi_c(G)$ of G is equal to the infimum of those r for which G has an r -circular coloring. (See [12, 17] for the proof of the equivalence.) We shall use this definition of $\chi_c(G)$ in this paper. It was shown in [11] (also see [2, 12] for combinatorial proofs) that for finite graphs, the infimum in the definition is always attained, and hence it can be replaced by minimum.

Note that a $(k, 1)$ -coloring of a graph G is just an ordinary k -coloring of G . This implies that $\chi_c(G) \leq \chi(G)$ for any graph G . On the other hand, it was shown in [2, 11] that $\chi_c(G) > \chi(G) - 1$. Therefore $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. So $\chi(G) = \lceil \chi_c(G) \rceil$. In this sense, $\chi_c(G)$ is a refinement of $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

Many questions that have been studied for the chromatic number can be studied in the framework of circular chromatic number. Quite a few classical results concerning the chromatic number of graphs have been generalized to results concerning the circular chromatic number. For example, it was shown in [13] that for

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any $r \geq 2$ there are graphs G of arbitrary large girth and with $\chi_c(G) = r$ (see [17] for a constructive proof). As a complement to the Four Colour Theorem, it was shown in [7, 15, 16] that a rational number r is the circular chromatic number of a non-trivial planar graph if and only if $2 \leq r \leq 4$.

In this paper, we present an analogue of Hajós' theorem for the circular chromatic number. To be precise, we seek a recursive construction method that constructs all graphs G with $\chi_c(G) \geq r$ by following some simple rules. We present such a construction method for any rational $r \geq 3$. The problem of finding an analogue of Hajós' theorem for the circular chromatic number is posed in [14, 17]. The first attempt at this problem is given in [14], where interesting examples of graph with given circular chromatic number was constructed (although an analogue of Hajós' theorem was not found). In this paper, we use three operations to replace Hajós' sum. Among these operations, Operation 1 has already been used in [14].

2. OPERATIONS THAT DO NOT DECREASE CIRCULAR CHROMATIC NUMBER

Suppose G, G' are graphs, and $e = xy, e' = x'y'$ are edges of G, G' respectively. The Hajós' sum of G and G' (with respect to e and e') is the graph obtained from the disjoint union of G and G' by deleting e and e' , identifying x and x' , and adding an edge that connects y and y' .

Let \mathcal{G}_n denote the class of graphs with chromatic number at least n . Hajós [5] proved the following theorem:

Theorem 1. *The class \mathcal{G}_n is the minimal class of graphs which contains K_n and is closed under the following operations:*

1. *adding vertices and edges;*
2. *identifying non-adjacent vertices;*
3. *taking Hajós' sum of two graphs.*

Given a rational number $r = k/d \geq 2$ (with $\gcd(k, d) = 1$). The graph G_k^d has vertices $0, 1, \dots, k-1$, and ij is an edge if and only if $d \leq |i-j| \leq k-d$. From the point of view of graph homomorphisms, the graphs G_k^d play the same role in the investigation of circular chromatic number as those played by the complete graphs in the investigation of chromatic number: A graph G has circular chromatic number at most k/d if and only if G admits a homomorphism to G_k^d (i.e., an edge-preserving mapping from the vertex set of G to the vertex set of G_k^d). Thus an analogue of Hajós' theorem for the circular chromatic number would construct all graphs of circular chromatic number at least k/d from the graph G_k^d by a set of simple rules. It is easy to see that the operations of adding vertices and edges, and identifying nonadjacent vertices do not decrease the circular chromatic number of a graph. However, the operation of taking Hajós' sum may decrease the circular chromatic number. For example, the Hajós' sum of two triangles produces a pentagon, while the triangle has circular chromatic number 3 and the pentagon has circular chromatic number $5/2$.

In this section, we shall present 3 operations that do not decrease the circular chromatic number. These operations will be used to replace the Hajós' sum to construct all graphs of circular chromatic number at least k/d from copies of G_k^d .

Operation 1. Given $2k+1$ ($k \geq 1$) graphs $G_0, G_1, G_2, \dots, G_{2k}$. For $i = 0, 1, 2, \dots, 2k$, let $e_i = x_i y_i$ be an edge of G_i . Let $C_{2k+1} = (c_0, c_1, \dots, c_{2k})$ be a cycle of length $2k+1$. Construct a new graph from the disjoint union of $G_0, G_1, G_2, \dots, G_{2k}, C_{2k+1}$

as follows:

- delete the edges e_i , for $i = 0, 1, 2, \dots, 2k$,
- identify all the x_i 's into a single vertex and name it x ,
- identify y_i and c_i for $i = 0, 1, 2, \dots, 2k$.

We shall denote the resulting graph by $S_1(G_0, e_0, G_1, e_1, G_2, e_2, \dots, G_{2k}, e_{2k})$, or simply by S_1 if the graphs G_i and edges e_i are clear from the context.

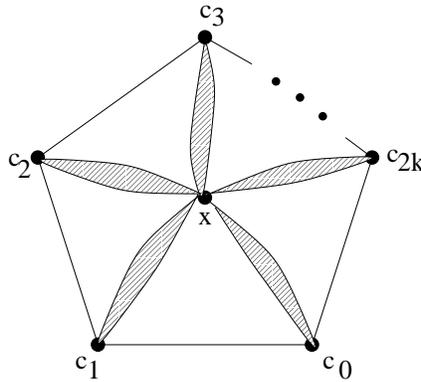


FIGURE 1. Operation 1 (each shaded area represents a $G_i - e_i$)

Lemma 1. *If $k \geq 1$ and $\chi_c(G_i) \geq r \geq 2$ for $i = 0, 1, \dots, 2k$, then $\chi_c(S_1) \geq r$.*

Proof. We refer to Figure 1 for the names of the vertices. Assume to the contrary that for some $r' < r$, there is an r' -circular coloring of S_1 . Since $\chi_c(G_i) \geq r$ for each i , we conclude that $f(x) \cap f(c_i) \neq \emptyset$. Assume that the two end points of $f(x)$ are p_1 and p_2 . Since $f(c_1) \cap f(c_0) = \emptyset$, we conclude that one of $f(c_0), f(c_1)$ contains p_1 and the other contains p_2 . Without loss of generality, we assume that $p_1 \in f(c_1)$ and $p_2 \in f(c_0)$. Since $f(c_1) \cap f(c_2) = \emptyset$, we conclude that $p_2 \in f(c_2)$. By induction, we conclude that $p_2 \in f(c_{2j})$ and $p_1 \in f(c_{2j-1})$ for any $j \geq 1$. In particular, $p_2 \in f(c_{2k})$. So $f(c_0) \cap f(c_{2k}) \neq \emptyset$, contrary to the assumption that f is an r' -coloring of S_1 . □

Operation 2. Given n graphs $G_0, G_1, G_2, \dots, G_{n-1}$. For $i = 0, 1, 2, \dots, n - 1$, let $e_i = x_i y_i$ be an edge of G_i . Construct a new graph from the disjoint union of $G_0, G_1, G_2, \dots, G_{n-1}$ as follows:

- delete the edges e_i , for $i = 0, 1, 2, \dots, n - 1$,
- identify y_i with x_{i+1} , for $i = 0, 1, 2, \dots, n - 1$ (here and below the sum in the indices are modulo n),
- add edges to connect x_i and x_{i+2} for $i = 0, 1, 2, \dots, n - 1$,
- add a vertex u and connect u to x_i for $i = 0, 1, 2, \dots, n - 1$.

We shall denote the resulting graph by $S_2(G_0, e_0, G_1, e_1, G_2, e_2, \dots, G_{n-1}, e_{n-1})$, or simply by S_2 if the graphs G_i and edges e_i are clear from the context.

Lemma 2. *If $n \geq 4$ and $\chi_c(G_i) \geq r \geq 2$ for $i = 0, 1, \dots, n - 1$, then $\chi_c(S_2) \geq r$.*

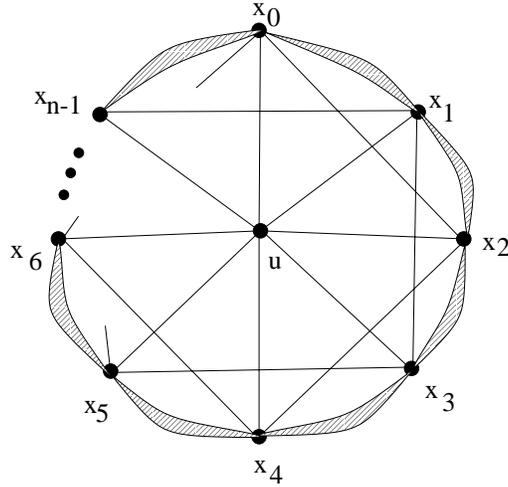


FIGURE 2. Operation 2 (each of the shaded area represents a $G_i - e_i$)

Proof. Again we refer to Figure 2 for the names of the vertices. Assume to the contrary that there is an $r' < r$ and an r' -circular coloring f of S_2 . Since $\chi_c(G_i) \geq r > r'$, it follows that for $i = 0, 1, \dots, n - 1$, $f(x_i) \cap f(x_{i+1}) \neq \emptyset$. On the other hand $f(x_i) \cap f(x_{i+2}) = \emptyset$ (for $i = 0, 1, 2, \dots, n - 1$). It follows that the relative positions of the arcs $f(x_i)$ must be as depicted in Figure 3.



FIGURE 3. Relative positions of $f(x_i)$ on $C^{r'}$

However, $f(x_{n-1})$ must intersect $f(x_0)$. The only possibility is that $f(x_{n-1})$ goes around the circle $C^{r'}$ to “meet” $f(x_0)$ at the left end point. Therefore the union

$$f(x_0) \cup f(x_1) \cup f(x_2) \cup \dots \cup f(x_{n-1})$$

covers all the circle $C^{r'}$. Now $f(u)$ is disjoint from each of the arcs $f(x_i)$. This is an obvious contradiction. \square

Operation 3. Given seven graphs G_1, G_2, \dots, G_7 . Let P_4 be the path with five vertices p_1, p_2, \dots, p_5 . For $i = 1, 2, \dots, 7$, let $e_i = x_i y_i$ be an edge of G_i . Construct a new graph from the disjoint union of G_1, G_2, \dots, G_7 and P_4 as follows:

- delete the edges e_i , for $i = 1, 2, \dots, 7$,
- identify y_1, y_2, y_3, y_4, y_5 into a single vertex, and name it y , and identify x_i with p_i for $i = 1, 2, \dots, 5$, and name it x_i ,
- identify x_6 with x_2 , y_6 with x_5 , x_7 with x_1 , y_7 with x_4 .

We shall denote the resulting graph by $S_3(G_1, e_1, G_2, e_2, \dots, G_7, e_7)$, or simply by S_3 if the graphs G_i and edges e_i are clear from the context.

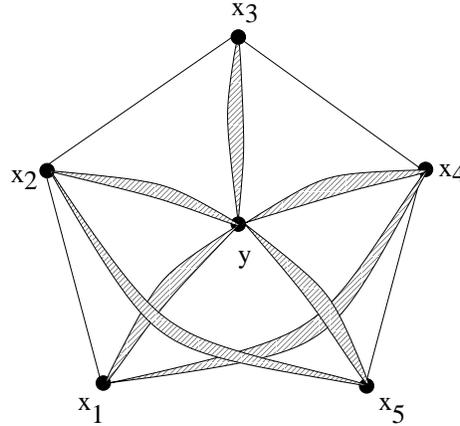


FIGURE 4. Operation 3 (each shaded area represents a $G_i - e_i$)

Lemma 3. *If $r \geq 3$ and G_1, G_2, \dots, G_7 are graphs with circular chromatic number at least r , then $\chi_c(S_3) \geq r$.*

Proof. Assume to the contrary that there is an $r' < r$ and an r' -circular coloring f of S_3 .

Since $\chi_c(G_i) \geq r$ for each i , $f(x_i) \cap f(y) \neq \emptyset$ for $i = 1, 2, \dots, 5$. Assume that $f(x_1) \cap f(y)$ is an arc of length δ , $f(x_2) \cap f(y)$ is an arc of length δ' .

We may assume that $r - r' < \min\{\delta, \delta'\}$. Otherwise, we choose an $\epsilon > 0$ so that $0 < r - r' - \epsilon < \min\{\delta, \delta'\}$, choose a point $p \notin f(y)$ on the circle $C^{r'}$, then insert a segment of length ϵ at the point p , so we obtain a circle of length $r' + \epsilon$. Now each arc $f(v)$ in the new circle has length 1 if $f(v)$ does not contain p , and has length $1 + \epsilon > 1$ if $f(v)$ does contain p . For each v such that $f(v)$ contains p , we may fix one end point of $f(v)$ and move the other end point of $f(v)$ (in the new circle of circumference $r' + \epsilon$) so that $f(v)$ has length 1 in the new circle. It is obvious that this modification gives an $(r' + \epsilon)$ -circular coloring of S_3 . (Indeed, no new intersections between the arcs are introduced.) By properly choosing the end point of $f(v)$ to be fixed, we can assure that for the new $(r' + \epsilon)$ -circular coloring, the intersections $f(x_1) \cap f(y)$ and $f(x_2) \cap f(y)$ still have length δ and δ' .

Let the two end points of $f(y)$ be p, q . Since $f(x_i) \cap f(x_{i+1}) = \emptyset$ for $i = 1, 2, 3, 4$, we may assume that $p \in f(x_1), f(x_3), f(x_5)$ and $q \in f(x_2), f(x_4)$. Referring to Figure 4, we see that $f(x_1) \cap f(x_4) \neq \emptyset$. The intersection $f(x_1) \cap f(x_4)$ does not contain p or q . So either $f(x_1) \cap f(x_4)$ is contained in $f(y)$ or $f(x_1) \cap f(x_4)$ is disjoint from $f(y)$.

If $f(x_1) \cap f(x_4)$ is disjoint from $f(y)$, then the union $f(x_1) \cup f(x_4) \cup f(y)$ covers the whole circle which has length r' . However, since $f(x_1) \cap f(y)$ has length $\delta > r - r'$, the union $f(x_1) \cup f(x_4) \cup f(y)$ has length at most $3 - \delta < 3 - (r - r') < r'$ (as $r \geq 3$), which is a contradiction. Therefore $f(x_1) \cap f(x_4)$ is contained in $f(y)$.

Similarly, we have $f(x_2) \cap f(x_5) \neq \emptyset$ and this implies that the intersection $f(x_2) \cap f(x_5)$ is contained in $f(y)$. But this is impossible, because $f(x_2)$ is disjoint from $f(x_1)$ and $f(x_5)$ is disjoint from $f(x_4)$. \square

3. THE MAIN RESULT

Now we shall prove that Operations 1, 2, and 3, together with the operation of adding vertices and edges and identifying nonadjacent vertices, enable us to construct all graphs of circular chromatic number greater than or equal to k/d from copies of G_d^k (for $k/d \geq 3$).

Theorem 2. *Suppose $k/d \geq 3$. Let \mathcal{G}_k^d be the set of graphs that contains G_k^d and is closed under the following operation:*

- adding vertices and edges;
- identifying non-adjacent vertices;
- applying Operations 1, 2, 3.

Then \mathcal{G}_k^d consists of all graphs G with $\chi_c(G) \geq k/d$.

It follows from Lemmas 1, 2, and 3 that all the graphs in \mathcal{G}_k^d have circular chromatic number at least k/d . It remains to show that every graph G with $\chi_c(G) \geq r$ is contained in \mathcal{G}_k^d .

For this purpose, we need the following theorem, which is proved in [18].

Theorem 3. *If G is a graph such that for every vertex x of G , $N_G[x]$ induces a perfect graph, and $V(G) - N_G[x]$ induces a bipartite graph with no induced P_4 , then $\chi_c(G) = \omega_c(G)$, where $\omega_c(G)$ is the minimum k/d for which G_d^k is a subgraph of G .*

Proof of Theorem 2. Assume to the contrary that there is a graph G with $\chi_c(G) \geq k/d$ and $G \notin \mathcal{G}_d^k$. We may choose the graph G so that by adding any edge to G , the resulting graph G' is no longer a counterexample to Theorem 2. Since $\chi_c(G') \geq r$, so $G' \in \mathcal{G}_d^k$. In other words, $G \in \mathcal{G}_d^k$, but adding any edge results in a graph $G' \in \mathcal{G}_d^k$.

Assume that $\chi_c(G) = k'/d' \geq k/d$. First we assume that $\omega_c(G) = k'/d'$. Then G contains a copy of $G_{d'}^{k'}$. It is well known (cf. [17]) that $G_{d'}^{k'}$ admits a homomorphism to G_d^k . Hence G_d^k admits a homomorphism to G . Let f be a homomorphism from G_d^k to G . By identifying those vertices of G_d^k which are mapped to the same vertex of G , and by adding some vertices and edges (if necessary), we obtain a copy of G .

Now we assume that $\omega_c(G) < k'/d'$. Then by Theorem 3, we know that there is a vertex x of G such that one of the following is true:

1. $V(G) - N_G[x]$ is not bipartite;
2. $V(G) - N_G[x]$ contains an induced P_4 ;
3. $N_G[x]$ is not a perfect graph.

We shall derive a contradiction in each of these three cases.

First assume that $V(G) - N_G[x]$ contains an odd cycle $C = (x_0, x_1, \dots, x_{2k})$. For $i = 0, 1, \dots, 2k$, let G_i be obtained from G by adding edge $e_i = xx_i$. Then $G_i \in \mathcal{G}_d^k$. Let $G' = S_1(G_0, e_0, G_1, e_1, \dots, G_{2k}, e_{2k})$. Then G' is as depicted in Figure 1, where each shaded area is a copy of G (as each $G_i - e_i$ is a copy of G). Moreover, for each copy of G , the vertex x in G' is also the vertex x in G , and the vertex c_i in G' is the vertex x_i in G .

For each vertex v of G , let v_i be the copy of v in G_i . Now we identify all the v_i 's into a single vertex. Because the vertices x_0, x_1, \dots, x_{2k} already induce a cycle in G , the resulting graph is G . Therefore $G \in \mathcal{G}_d^k$, contrary to our assumption.

Assume next that $V(G) - N_G(x)$ contains an induced path $(x_1, x_2, x_3, x_4, x_5)$ of length 4. For $i = 1, 2, 3, 4, 5$, let G_i be obtained from G by adding the edge

$e_i = xx_i$. Let G_6 be the graph obtained from G by adding the edge $e_6 = x_1x_4$ and G_7 be the graph obtained from G by adding the edge $e_7 = x_2x_5$. Then $G_i \in \mathcal{G}_d^k$ for $i = 1, 2, \dots, 7$.

Let $G' = S_3(G_1, e_1, G_2, e_2, \dots, G_7, e_7)$. Then G' is as depicted in Figure 4, where each shaded area is a copy of G (as $G_i - e_i = G$). Similarly as above, for each $v \in G$, we identify the different copies of v in G' into a single vertex. We obtain the graph G . So $G \in \mathcal{G}_d^k$, contrary to our assumption.

Now we assume that $N_G[x]$ induces a non-perfect graph. Then the complement of $N_G[x]$ contains an induced cycle of length $n \geq 4$ (recall that triangulated graphs are perfect, and complements of perfect graphs are perfect).

Assume $(x_0, x_1, \dots, x_{n-1})$ is an induced cycle of the complement of $N_G[x]$ of length $n \geq 4$. Let G_i be obtained from G by adding the edge $e_i = x_ix_{i+1}$ for $i = 0, 1, 2, \dots, n-1$ (summation in indices modulo n). Let

$$G' = S_2(G_0, e_0, G_1, e_1, \dots, G_{n-1}, e_{n-1}).$$

Then G' is as depicted in Figure 2, where each shaded area represents a copy of G . Identifying the different copies of the same vertex of G into a single vertex, and u (as in Figure 2) with the vertex x of G , we obtain a copy of G . So $G \in \mathcal{G}_d^k$, contrary to our assumption. This completes the proof of Theorem 2. \square

Two remarks. 1. The method presented here constructs all graphs with $\chi_c(G) \geq k/d$ from \mathcal{G}_d^k with a few rules, provided that $k/d \geq 3$. The method does not work for $2 < k/d < 3$. After the submission of this paper, the author found another set of graph operations which works for $2 < k/d < 3$. By that method, we need four more graph operations for the case $2 < k/d < 3$ [19].

2. A different generalization of Hajós' theorem to H -coloring was given by Nešetřil [8]. Given any graph H , a method is presented in [8] that constructs all graphs which do not admit a homomorphism to H , starting from a finite set of graphs. These operations are adding vertices and edges, identifying nonadjacent vertices and a "generalized" Hajós' sum. In particular, for any $k/d \geq 2$, that method construct all graphs G with $\chi_c(G) > k/d$ (note that here it is necessary to have the strict inequality), starting with a finite set of graphs. The finite set of graphs is not known explicitly and is usually a big set.

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