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A REMARK ON A PAPER OF E. B. DAVIES

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ABSTRACT. We explain the existence of open sets of complex quasi-modes in terms of Hörmander's commutator condition.

In a recent paper [1], Davies proves the following interesting result: if $P(h) = (hD_x)^2 + V(x)$, $D_x = (1/i)d/dx$, $V \in \mathcal{C}^\infty(\mathbb{R})$, then

$$\begin{gathered} \operatorname{Im} V'(x_0) \neq 0\,, \quad E = \xi_0^2 + V(x_0)\,,\ \xi_0 \in \mathbb{R} \setminus \{0\} \implies \\ \exists\, u(h) \in L^2(\mathbb{R})\,,\ \|u(h)\|_{L^2(\mathbb{R})} = 1\,,\ (P(h) - E)u(h) = \mathcal{O}(h^\infty)\,, \end{gathered} \tag{1}$$

where $\mathcal{O}(h^\infty)$ is a bound in any norm, for instance $\mathcal{C}^k$ for any k.

The purpose of this note is to point out that a more general statement than (1) follows immediately from the now standard results in microlocal analysis described in Chapter 26 of [5].

Let X be a manifold and $S^{m,k}(T^*X)$ be the space of semi-classical symbols on T^*X: $a \in \mathcal{C}^\infty(T^*X \times (0,1])$,

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) \leq C_{\alpha,\beta} h^{-m} (1 + |\xi|)^{k-|\beta|}\,, \quad (x,\xi) \in T^*X\,,\ h \in (0,1]\,. \tag{2}$$

The local quantization formula

$$a(x, hD_x; h)u = \frac{1}{(2\pi h)^n} \int\int a(x,\xi;h) e^{i\langle x-y,\xi\rangle} u(y) dy d\xi$$

defines a class of semi-classical pseudo-differential operators on X, $\Psi_h^{m,k}(X)$, with a symbol map, σ, and a short exact sequence,

$$0 \longrightarrow \Psi_h^{m-1,k-1}(X) \longrightarrow \Psi_h^{m,k}(X) \xrightarrow{\ \sigma\ } S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X) \longrightarrow 0\,.$$

If we strengthen the condition (2) to

$$(hD_h)^l \partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) \leq C_{l,\alpha,\beta} h^{-m} (1 + |\xi|)^{k-|\beta|}\,, \quad (x,\xi) \in T^*X\,,\ h \in (0,1], \tag{3}$$

we can then use, without any modifications, the $\mathcal{C}^\infty$ theory of pseudo-differential operators when working in compact subsets of T^*X. We can simply introduce a new variable $t \in \mathbb{R}$ and consider $h = 1/\tau$, where τ is the dual variable to t. Working microlocally in the region $|\xi/\tau| \leq C$ corresponds to working semi-classically in compact subsets of T^*X. In particular we have a correspondence between the

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frequency set and the wave front set:

$$(x,\xi) \notin WF_h(u(h)) \iff \begin{cases} \exists\, Q \in \Psi^0(\mathbb{R}\times X) \;\text{ elliptic at }\; (t,x,\tau,\tau\xi) \;\text{ for all } (t,\tau)\in T^*\mathbb{R}\setminus 0 \\ \text{such that } Q\mathcal{F}^{-1}_{\tau\mapsto t}(u(1/\bullet)) \in \mathcal{C}^\infty(\mathbb{R}\times X), \end{cases}$$

where $\mathcal{F}^{-1}_{\tau\mapsto t}$ is the inverse Fourier transform (see [3] and [6, Proposition 9] for the definition and the relation to other wave front sets). The characterization given here follows from the one in [6].

This allows us to adapt directly the results of Hörmander and of Duistermaat and Sjöstrand (see [2] and [5, Sect.26.3]) to obtain the following statement: if $P(x,hD_x;h)$ is a semi-classical pseudo-differential operator, in the class satisfying (3), and with the principal symbol p, then

$$\begin{gathered} p(m)=0\,, \quad \{\operatorname{Re} p, \operatorname{Im} p\}(m) < 0\,, \quad m \in T^*X \implies \\ \exists\, u(h) \in \mathcal{C}^\infty(X) \;\text{ such that } WF_h(u(h)) = \{m\} \text{ and } WF_h(Pu)=\emptyset. \end{gathered} \tag{4}$$

If we take $P(x,hD_x;h) = (hD_x)^2 + V(x) - E$ we immediately recover Davies's result: if $\chi \in \mathcal{C}^\infty_{\rm c}(X)$ is equal to 1 near x_0, $m=(x_0,\xi_0)$, then, because of the WF_h statements, $P(\chi u) = \mathcal{O}(h^\infty)$ and $\chi u \in \mathcal{C}^\infty_{\rm c}(X)$. The condition on the Poisson bracket generalizes the one dimensional condition $\operatorname{Im} V'(x_0)\neq 0$. For Schrödinger operators in higher dimensions, we need $E=\xi_0^2+V(x_0)\,,\ \operatorname{Im}\xi_0\cdot\nabla V(x_0)\neq 0$.

We conclude with a few remarks. The microlocal statement (4) is a generalization of the celebrated commutator condition of Hörmander [4]. The crucial part of his argument was in fact a geometric optics "quasi-mode" construction which, in a very special case, is repeated in [1]. As was pointed out to the author by Victor Ivrii, the microlocal commutator condition (4) has the following well known global analogue: let Q be an unbounded operator on a Hilbert space $\mathcal{H}$, with the domain $\mathcal{D}$. Let us also assume that Q^* has the same domain as Q and that $Q-z\,,\ Q^*-\bar z \;:\; \mathcal{D}\to\mathcal{H}$, are Fredholm operators for all z. Then

$$\pm[Q,Q^*] \geq C > 0 \implies \operatorname{spec} Q = \mathbb{C}\,.$$

In the case of the $+$ sign, $Q-z$ fails to be injective for all z, and in the case of the $-$ sign, surjective. In fact, we conclude that for all $z\in\mathbb{C}$ and $u\in\mathcal{D}$,

$$\mp\|(Q-z)u\|^2 \pm \|(Q^*-\bar z)u\|^2 \geq C\|u\|^2\,.$$

If $Q-z_0$ were invertible at some z_0, then index $(Q-z) =$ index $(Q-z_0)=0$. In the $-$ case $Q-z$ would then be invertible everywhere, and hence the spectrum is either empty or equal to $\mathbb{C}$. In the $+$ case we draw the same conclusion for the adjoint.

We finally remark that, as pointed out by Davies, the existence of complex quasi-modes implies lower bounds on the resolvent, and that has computation consequences.

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