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HAHN-BANACH OPERATORS

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Abstract. We consider real spaces only.

Definition. An operator $T : X \to Y$ between Banach spaces X and Y is called a *Hahn-Banach operator* if for every isometric embedding of the space X into a Banach space Z there exists a norm-preserving extension $\tilde{T}$ of T to Z.

A geometric property of Hahn-Banach operators of finite rank acting between finite-dimensional normed spaces is found. This property is used to characterize pairs of finite-dimensional normed spaces (X, Y) such that there exists a Hahn-Banach operator $T : X \to Y$ of rank k. The latter result is a generalization of a recent result due to B. L. Chalmers and B. Shekhtman.

Everywhere in this paper we consider only real linear spaces. Our starting point is the classical Hahn-Banach theorem ([H], [B1]). The form of the Hahn-Banach theorem we are interested in can be stated in the following way.

Hahn-Banach Theorem. *Let X and Y be Banach spaces, $T : X \to Y$ a bounded linear operator of rank 1 and Z a Banach space containing X as a subspace. Then there exists a bounded linear operator $\tilde{T} : Z \to Y$ satisfying*

(a) $||\tilde{T}|| = ||T||$;

(b) $\tilde{T}x = Tx$ *for every* $x \in X$.

Definition 1. An operator $\tilde{T} : Z \to Y$ satisfying (a) and (b) for a bounded linear operator $T : X \to Y$ is called a *norm-preserving extension* of T to Z.

The Hahn-Banach theorem is one of the basic principles of linear analysis. It is quite natural that there exists a vast literature on generalizations of the Hahn-Banach theorem for operators of higher rank. See papers by G. P. Akilov [A], J. M. Borwein [Bor], B. L. Chalmers and B. Shekhtman [CS], G. Elliott and I. Halperin [EH], D. B. Goodner [Go], A. D. Ioffe [I], S. Kakutani [Kak], J. L. Kelley [Kel], J. Lindenstrauss [L1], [L2], L. Nachbin [N1] and M. I. Ostrovskii [O], representing different directions of such generalizations, and references therein. There exist two interesting surveys devoted to the Hahn-Banach theorem and its generalizations; see G. Buskes [Bus] and L. Nachbin [N2].

We shall use the following natural definition.

Definition 2. An operator $T : X \to Y$ between Banach spaces X and Y is called a *Hahn-Banach operator* if for every isometric embedding of the space X into a Banach space Z there exists a norm-preserving extension $\tilde{T}$ of T to Z.

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The existence of non-Hahn-Banach operators was mentioned in the remarks to Chapter IV of Banach's book; see [B2, p. 234]. S. Banach and S. Mazur [BM] proved that the identity operator on l_1 is a non-Hahn-Banach operator (in fact, this operator does not have even continuous extensions for some isometric embeddings). It has been known for a long time that there exist non-Hahn-Banach operators of rank 2 (see F. Bohnenblust [Boh], an important relevant result was proved relatively recently by H. König and N. Tomczak-Jaegermann [KT]). A problem of characterization of Hahn-Banach operators arises in a natural way.

Factorizational characterizations of Hahn-Banach operators are well known. In particular, using a by now standard technique (that goes back to G. P. Akilov [A], D. B. Goodner [Go, pp. 92–93] and R. Phillips [P, p. 538]) it is easy to show that an operator $T : X \to Y$ is a Hahn-Banach operator if and only if for some set Γ there exist operators $T_1 : X \to l_\infty(\Gamma)$ and $T_2 : l_\infty(\Gamma) \to Y$ such that $T_2T_1 = T$ and $||T_2||\,||T_1|| = ||T||$. (See [J] for the undefined terminology from the theory of Banach spaces.)

One of the main purposes of the present paper is to find a geometric property of Hahn-Banach operators of finite rank acting between finite-dimensional normed spaces (see Theorem 1). This property does not imply that the operator is a Hahn-Banach operator (see the remark after Theorem 1), but it can be used to answer the following question: given $k \in \mathbb{N}$, for which pairs of finite-dimensional spaces (X, Y) does there exist a Hahn-Banach operator $T : X \to Y$ of rank k? (See Theorem 2.) This result is a generalization of a recent result due to B. L. Chalmers and B. Shekhtman [CS].

Remark. Let $\alpha \in \mathbb{R},\ \alpha \neq 0$. It is clear that αT is a Hahn-Banach operator if and only if T is a Hahn-Banach operator. Hence, studying Hahn-Banach operators it is enough to consider Hahn-Banach operators of norm 1.

We need the following notation. By $S(X)$ and $B(X)$ we denote the unit sphere and the unit ball of a Banach space X respectively. Let X be a finite-dimensional Banach space. An intersection of $B(X)$ with a supporting hyperplane of $B(X)$ will be called a *support set* of $B(X)$. By the *dimension* of a set in a finite-dimensional space we mean the dimension of its affine hull. (See [S] for the undefined terminology from the theory of convex bodies.) We define $f(X)$ to be the maximal dimension of a support set of $B(X)$. For $x \in S(X)$ we define $d(x)$ to be the dimension of the set $\{x^* \in S(X^*) :\ x^*(x) = 1\}$. It is clear that $d(x) = 0$ if and only if x is a smooth point; in the general case $d(x)$ indicates the number of linearly independent directions of non-smoothness of the norm at x.

Theorem 1. *Let X and Y be finite-dimensional Banach spaces and $T : X \to Y$ be a Hahn-Banach operator of rank k. Assume that $||T|| = 1$ and let $x_0 \in S(X)$ be such that $||Tx_0|| = 1$. Then Tx_0 belongs to a support set of $B(Y)$ of dimension $\geq k - 1 - d(x_0)$.*

Proof. Let $C(S(X^*))$ denote the space of all continuous functions on $S(X^*)$ with the sup norm. We identify X with a subspace of $C(S(X^*))$ in the following way: every vector is identified with its restriction (as a function on X^*) to $S(X^*)$. We introduce the following notation: $C = C(S(X^*))$ and $B_C = B(C(S(X^*)))$.

Since T is a Hahn-Banach operator, there exists $\tilde{T} : C \to Y$ such that $\tilde{T}|_X = T$ and $||\tilde{T}|| = 1$. We shall use $\tilde{T}$ to find a "large" support set of $B(Y)$.

Since $||Tx_0|| = 1$, there exists $h \in S(Y^*)$ such that $h(Tx_0) = 1$. Let $F = \{x^* \in S(X^*) : \; x^*(x_0) = 1\}$. Observe that

$$T^*h \in F. \tag{1}$$

Choose a basis $\{y_1, \dots, y_m\}$ in Y such that $y_1 = Tx_0$ and $y_2, \dots, y_m \in \ker h$. The operator $\tilde{T}$ can be represented in the form

$$\tilde{T} = \sum_{i=1}^{m} \mu_i \otimes y_i, \quad \mu_i \in C^*.$$

By the F. Riesz representation theorem (see e.g. [DS], p. 265), we may identify μ_i with (signed) measures on $S(X^*)$.

Our first purpose is to show that μ_1 is supported on $F \cup (-F)$.

We have $\tilde{T}(B_C) \subset B(Y) \subset \{y : \; |h(y)| \leq 1\}$, $h(y_1) = 1$ and $y_2, y_3, \dots, y_m \in \ker h$. Therefore for every $z \in B_C$ we have

$$\mu_1(z) = h\left(\sum_{i=1}^{m} \mu_i(z) y_i\right) = h(\tilde{T}(z))$$

and $|\mu_1(z)| = |h(\tilde{T}(z))| \leq 1$. Hence,

$$||\mu_1|| \leq 1. \tag{2}$$

Also, since $\tilde{T}x_0 = y_1$, we have

$$\mu_1(x_0) = 1. \tag{3}$$

Conditions (2), (3) and $||x_0|| = 1$ imply that μ_1 is supported on $F \cup (-F)$. (By this we mean that the restriction of μ_1 to $S(X^*) \backslash (F \cup (-F))$ is a zero measure.)

We decompose $\mu_i = \nu_i + \omega_i$, where ν_i is the restriction of μ_i to $F \cup (-F)$. Since μ_1 is supported on $F \cup (-F)$, then $\omega_1 = 0$.

Since T is of rank k, there exists a subspace $L \subset X$ of dimension k such that $T|_L$ is an isomorphism. Let

$$M = \{x \in L : \; \forall x^* \in F, \quad x^*(x) = 0\}.$$

Then $\dim M \geq k - d(x_0) - 1$.

Let $x \in B(M)$. The definitions of M and ν_i imply that

$$\nu_i(x) = 0, \; i \in \{1, \dots, m\}. \tag{4}$$

(Recall that we identify vectors in M with the corresponding functions in C.)

Now we construct a "mixture" of x and x_0.

It is clear that for each $\delta > 0$ there exists a function $g_\delta \in B_C$ such that

$$g_\delta|_{F \cup (-F)} = x_0|_{F \cup (-F)}$$

and the restrictions of g_δ and x to the complement of the $\delta-$neighbourhood of $F \cup (-F)$ coincide.

We have

$$\lim_{\delta \downarrow 0} \tilde{T} g_\delta = \lim_{\delta \downarrow 0} \sum_{i=1}^{m} \mu_i(g_\delta) y_i = \lim_{\delta \downarrow 0} \sum_{i=1}^{m} (\nu_i(g_\delta) + \omega_i(g_\delta)) y_i.$$

We have $\nu_i(g_\delta) = \nu_i(x_0)$ for every $\delta > 0$ and $i \in \{1, \dots, m\}$.

It is clear that $\omega_i(F \cup (-F)) = 0$. By the definition of g_δ it follows that $\lim_{\delta\downarrow 0} g_\delta(x^*) = x(x^*)$ for $x^* \in S(X^*)\backslash(F \cup (-F))$ and that the functions g_δ are uniformly bounded. By the Lebesgue dominated convergence theorem we get

$$\lim_{\delta\downarrow 0} \omega_i(g_\delta) = \omega_i(x).$$

Therefore

$$\lim_{\delta\downarrow 0} \tilde{T} g_\delta = \sum_{i=1}^{m} (\nu_i(x_0) + \omega_i(x)) y_i.$$

Equation (4) implies that $\nu_i(x_0) = \nu_i(x_0 + x)$. Using this and the fact that $\omega_1 = 0$, we get

$$\lim_{\delta\downarrow 0} \tilde{T} g_\delta = \sum_{i=1}^{m} \mu_i(x_0 + x) y_i - \sum_{i=2}^{m} \omega_i(x_0) y_i = T(x_0 + x) - \sum_{i=2}^{m} \omega_i(x_0) y_i.$$

Since $g_\delta \in B_C$, $||\tilde{T}|| = 1$ and $B(Y)$ is closed, then

$$T(x_0 + x) - \sum_{i=2}^{m} \omega_i(x_0) y_i \in B(Y)$$

for every $x \in B(M)$.

By (1) we have $h(Tx) = 0$ for every $x \in M$. Recall, also, that $y_2, \ldots, y_m \in \ker h$. Therefore

$$h\left(T(x_0 + x) - \sum_{i=2}^{m} \omega_i(x_0) y_i\right) = hTx_0 = 1$$

for every $x \in M$. Since $T|_M$ is an isomorphism and the vector $\sum_{i=2}^{m} \omega_i(x_0) y_i$ does not depend on x, the intersection of $B(Y)$ with the supporting hyperplane $\{y : \ h(y) = 1\}$ has dimension $\geq \dim M \geq k - d(x_0) - 1$. $\square$

Corollary. *The existence of a Hahn-Banach operator $T : X \to Y$ of rank k implies $f(X^*) + f(Y) \geq k - 1$.*

Proof. Observe that $f(X^*) \geq d(x_0)$ and that $f(Y)$ is greater or equal to the dimension of any support set of $B(Y)$ containing Tx_0. $\square$

Remark. There exist operators satisfying the condition of Theorem 1 that are not Hahn-Banach. In fact, let T be the identity mapping of $X = l_1^n$ onto the space Y whose unit ball is the intersection of $(1+\varepsilon)B(l_1^n)$, $(\varepsilon > 0)$ and $B(l_\infty^n)$. It is easy to see that

(1) the norm of this operator is 1;

(2) the only points where the operator attains its norm are $\pm e_1, \pm e_2, \ldots, \pm e_n$, where $\{e_1, \ldots, e_n\}$ is the unit vector basis;

(3) the points $\pm e_1, \pm e_2, \ldots, \pm e_n$ are contained in $(n-1)$-dimensional support sets of $B(Y)$.

Therefore T satisfies the condition of Theorem 1 with $k = n$. On the other hand, the operator T is not a Hahn-Banach operator if $n \geq 3$ and ε is small enough. In fact, $||T^{-1}|| = 1 + \varepsilon$. Therefore, if T were a Hahn-Banach operator it would imply that for every Banach space Z containing l_1^n as a subspace there exists a projection onto l_1^n with the norm $\leq 1 + \varepsilon$. It remains to apply the well-known result of B. Grünbaum (see [Gr] or [J, p. 81]).

B. L. Chalmers and B. Shekhtman [CS] characterized 2-dimensional spaces having isomorphisms that are Hahn-Banach operators. One of the steps in their approach is an embedding of the considered spaces into L_1. This is why they got the restriction on the dimension (spaces of dimension ≥ 3 may be non-isometric to any subspace of L_1; see J. Lindenstrauss [L2, p. 494]). Our next purpose is to extend their results and to characterize pairs (X, Y) of finite-dimensional normed linear spaces such that there exists a Hahn-Banach operator $T : X \to Y$ of rank k.

Theorem 2. *Let X and Y be finite-dimensional normed linear spaces and let k be a positive integer satisfying $k \leq \min\{\dim X, \dim Y\}$. There exists a Hahn-Banach operator $T : X \to Y$ of rank k if and only if $f(X^*) + f(Y) \geq k - 1$.*

Proof. The necessity has already been proved (see the corollary).

Sufficiency. Suppose that

$$k \leq \min\{\dim X, \dim Y, f(X^*) + f(Y) + 1\}.$$

It is well known (see e.g. [KS]) that in order to show that $T : X \to Y$ is a Hahn-Banach operator it is enough to show that it has a norm-preserving extension to the space $C = C(S(X^*))$ (The space X is embedded into C in the same way as in Theorem 1). Therefore, if an operator $Q : C \to Y$ is such that the restriction of Q to X has rank k and $||Q|| = ||Q|_X|| = 1$, then $T = Q|_X$ is a Hahn-Banach operator of rank k.

Our purpose is to construct such Q. Let $n = f(Y)$ and $m = f(X^*)$. Let $y_0, y_1, \ldots, y_n \in Y$ be linearly independent and such that

$$\{y : \; y = \theta y_0 + \sum_{i=1}^{n} a_i y_i, \text{ where } \theta = \pm 1, \; |a_i| \leq 1\} \subset S(Y).$$

Let $x_0^*, x_1^*, \ldots, x_m^* \in X^*$ be linearly independent and such that

$$\{x^* : \; x^* = \theta x_0^* + \sum_{i=1}^{m} b_i x_i^*, \text{ where } \theta = \pm 1, \; |b_i| \leq 1\} \subset S(X^*).$$

Let $x_0 \in S(X)$ be such that $x_0^*(x_0) = 1$. Let

$$x_0^*, x_1^*, \ldots, x_m^*, x_{m+1}^*, \ldots, x_r^*,$$

where $r = \dim X - 1$ be a basis in X^* satisfying the condition $x_{m+1}^*(x_0) = \cdots = x_r^*(x_0) = 0$. (Observe that the condition $x_1^*(x_0) = \cdots = x_m^*(x_0) = 0$ follows from our choice of the vectors.) Let $x_0, x_1, \ldots, x_r$ be its biorthogonal vectors.

Let $y_0, y_1, \ldots, y_s$, where $s = \dim Y - 1$, be a basis in Y.

We suppose that $k > m + 1$. (It will be clear from our argument which changes should be made if it is not the case.)

We define an operator $Q_1 : C \to Y$ as follows. Let $\mu_0, \mu_{m+1}, \ldots, \mu_k$ be norm-preserving extensions of $x_0^*, x_{m+1}^*, \ldots, x_k^*$ to C. Let

$$Q_1(f) = \mu_0(f) y_0 + \sum_{i=1}^{k-m-1} \frac{\mu_{m+i}(f)}{||\mu_{m+i}||} y_i.$$

It is clear that for $x \in X$

$$Q_1(x) = x_0^*(x) y_0 + \sum_{i=1}^{k-m-1} \frac{1}{||\mu_{m+i}||} x_{m+i}^*(x) y_i. \tag{5}$$

We have supposed that $k-m-1\le n$. This and the choice of $y_0,\dots,y_n$ implies that $||Q_1||\le 1$.

Our next step is to show that there exist signed measures $\nu_0,\nu_1,\dots,\nu_m$ of norm 1 on $S(X^*)$ satisfying the conditions

$$\nu_i(x_j)=\delta_{i,j},\ i=0,\dots,m,\ j=0,\dots,r, \tag{6}$$

and

$$\forall f\in B_C\ \forall j\in\{1,\dots,m\}\ |\nu_j(f)|\le 1-|\nu_0(f)|. \tag{7}$$

Let us verify that the following measures satisfy these conditions. We define $\nu_j,\ j=1,\dots,m$ as atomic measures with atoms at $x_0^*+\sum_{i=1}^m\theta_i x_i^*,\ \theta_i=\pm1$ satisfying $\nu_j(x_0^*+\sum_{i=1}^m\theta_i x_i^*)=2^{-m}\theta_j$ and ν_0 as an atomic measure satisfying $\nu_0(x_0^*+\sum_{i=1}^m\theta_i x_i^*)=2^{-m}$.

Condition (6) follows from the fact that the sequences $\{x_0,\dots,x_r\}$, $\{x_0^*,\dots,x_r^*\}$ are biorthogonal.

Let us verify condition (7). Denote by $\mathbb{I}$ the function that is identically 1 on $S(X^*)$. Let $f\in B_C$, $j\in\{1,\dots,m\}$. Then $\nu_j(-f)=\nu_j(\mathbb{I}-f)\le$ (since the function $\mathbb{I}-f$ is nonnegative) $\le\nu_0(\mathbb{I}-f)=1-\nu_0(f)$. (Here we explicitly use the fact that the spaces are real.)

This proves (7) in the case when $\nu_j(f)$ is negative and $\nu_0(f)$ is positive. It remains to observe that:

A. Since we may consider $-f$ instead of f it is enough to prove (7) for functions with positive $\nu_0(f)$.

B. For each function $f\in B_C$ and $j\in\{1,\dots,m\}$ there exists a function $f_j\in B_C$ such that $\nu_0(f_j)=\nu_0(f)$ and $\nu_j(f_j)=-\nu_j(f)$.

We introduce $Q_2:C\to Y$ by the equality

$$Q_2(f)=\nu_0(f)y_0+\sum_{i=1}^m\alpha_i\nu_i(f)y_{k-m-1+i}.$$

Condition (7) implies that there exists $\{\alpha_i\}_{i=1}^m$ such that $\alpha_i\ne0$ for every i and $||Q_2||\le1$.

Condition (6) implies that for $x\in X$ we have

$$Q_2(x)=x_0^*(x)y_0+\sum_{i=1}^m\alpha_i x_i^*(x)y_{k-m-1+i}. \tag{8}$$

Now, let $Q=\frac{1}{2}(Q_1+Q_2)$. Our estimates for $||Q_1||$ and $||Q_2||$ immediately imply $||Q||\le1$. On the other hand, equations (5) and (8) imply that $Q(x_0)=y_0$. Hence, $||Q|_X||\ge1$ and $||Q||=||Q|_X||=1$.

Also, from (5) and (8) we get for every $x\in X$,

$$Q(x)=x_0^*(x)y_0+\sum_{i=1}^{k-m-1}\frac{1}{2||\mu_{m+i}||}x_{m+i}^*(x)y_i + \sum_{i=1}^m\frac{1}{2}\alpha_i x_i^*(x)y_{k-m-1+i}.$$

Since the sequences $\{y_0,\dots,y_s\}$ and $\{x_0^*,\dots,x_r^*\}$ are linearly independent, it follows that $Q|_X$ is of rank k. □

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