CONDITIONAL WEAK COMPACTNESS
IN VECTOR-VALUED FUNCTION SPACES

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ABSTRACT. Let $E$ be an ideal of $L^0$ over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ and let $E'$ be the Köthe dual of $E$ with $\text{supp } E' = \Omega$. Let $(X, \| \cdot \|_X)$ be a real Banach space, and $X^*$ the topological dual of $X$. Let $E(X)$ be a subspace of the space $L^0(X)$ of equivalence classes of strongly measurable functions $f : \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\| f(\cdot) \|_X$ belongs to $E$. For a subset $H$ of $E(X)$ for which the set $\{ \| f(\cdot) \|_X : f \in H \}$ is $\sigma(E, E')$-bounded the following statement is equivalent to conditional $\sigma(E(X), E'(X^*))$-compactness: the set $\{ \| f(\cdot) \|_X : f \in H \}$ is conditionally $\sigma(E, E')$-compact and $\{ \int_A f(\omega) d\mu : f \in H \}$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Applications to Orlicz-Bochner spaces are given.

1. INTRODUCTION AND PRELIMINARIES

Given a dual pair $(L, K)$, a subset $A$ of $L$ is said to be conditionally $\sigma(L, K)$-compact whenever each sequence in $A$ contains a $\sigma(L, K)$-Cauchy subsequence (cf. [1] p. 100]). The problem of characterizing relatively sequentially $\sigma(L^p(X), L^q(X^*))$-compact subsets of Lebesgue-Bochner spaces $L^p(X)$ (where $1 \leq p < \infty$ and $q$ conjugate to $p$) over a finite measure space was considered by F. Bombal [4] and J. Batt and W. Hiermeyer [5]. Moreover, F. Bombal characterized relatively sequentially $\sigma(L^\infty(X), L^\infty(X^*))$-compact subsets of Orlicz-Bochner spaces $L^{\phi}(X)$ [4, Theorem 3]. C. Abott, E. Bator, R. Bilyeu and P. Lewis [7] obtained the following characterization of conditionally $\sigma(L^1(X), L^\infty(X^*))$-compact subsets of $L^1(X)$.

**Theorem 1.1** (cf. [7, Theorem 2.5]). Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Then for a norm bounded subset $H$ of $L^1(X)$ the following statements are equivalent:

(i) $H$ is conditionally $\sigma(L^1(X), L^\infty(X^*))$-compact.

(ii) a) The subset $\{ \| f(\cdot) \|_X : f \in H \}$ of $L^1$ is uniformly integrable.

b) The set $\{ \int_A f(\omega) d\mu : f \in H \}$ is conditionally weakly compact in $X$ for each $A \in \Sigma$.

In this paper, by making use of Theorem 1.1 we characterize conditionally $\sigma(E(X), E'(X^*))$-compact subsets of $E(X)$, where $E$ is an ideal of $L^0$ over a $\sigma$-finite measure space.

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Now we establish notation and terminology (see [AB], [KA]).

Let \((\Omega, \Sigma, \mu)\) be a complete \(\sigma\)-finite measure space and let \(L^0\) denote the space of equivalence classes of all \(\Sigma\)-measurable functions defined and finite a.e. on \(\Omega\). Let \(\chi_A\) stand for the characteristic function of a set \(A\) and \(\mathbb{N}\) denote the set of all natural numbers. Let \(E\) be an ideal of \(L^0\) with \(\text{supp } E = \Omega\), and let \(E'\) stand for the Köthe dual of \(E\), i.e.,

\[
E' = \{ v \in L^0 : \int_{\Omega} |u(\omega)v(\omega)| d\mu < \infty \text{ for all } u \in E \}.
\]

We assume that \(\text{supp } E' = \Omega\).

Let \((X, \| \cdot \|_X)\) be a real Banach space, and let \(S_X\) and \(B_X\) denote the unit sphere and the closed unit ball in \(X\), resp. Let \(X^*\) stand for the Banach dual of \(X\). By \(L^0(X)\) we denote the set of equivalence classes of all strongly \(\Sigma\)-measurable functions \(f : \Omega \to X\). For \(f \in L^0(X)\) let us set \(\bar{f}(\omega) = \|f(\omega)\|_X\) for \(\omega \in \Omega\). Let

\[
E(X) = \{ f \in L^0(X) : \bar{f} \in E \}.
\]

By \(\sigma(E(X), E'(X^*))\) we will denote the weak topology on \(E(X)\) with respect to the dual system \(\langle E(X), E'(X^*) \rangle\) under the natural duality \(\langle f, g \rangle = \int f(\omega), g(\omega) \rangle d\mu\) for \(f \in E(X), g \in E'(X^*)\).

The following characterization of conditional \(\sigma(E, E')\)-compactness is needed.

**Proposition 1.2** ([N], Theorem 1.1). For a \(\sigma(E, E')\)-bounded subset \(A\) of \(E\) the following statements are equivalent:

(i) \(A\) is conditionally \(\sigma(E, E')\)-compact.

(ii) For each \(v \in E'\) the subset \(\{uv : u \in A\}\) of \(L^1\) is uniformly integrable.

(iii) The functional \(p_A\) on \(E'\) defined by \(p_A(v) = \sup_{u \in A} \int |u(\omega)v(\omega)| d\mu\) is an order continuous Riesz seminorm.

2. Conditionally \(\sigma(E(X), E'(X^*))\)-compact sets in \(E(X)\)

Let \(ca(\Omega, \Sigma)\) stand for the Riesz space of countably additive set functions \(\nu\) on \(\Sigma\). For a sequence \((A_n)\) in \(\Sigma\) we write \(A_n \downarrow_{\mu} \emptyset\) whenever \(A_n \downarrow \emptyset\) and \(\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0\) (that is, \(A_n \downarrow\) and \(\mu(A_n \cap A) \to 0\) for each \(A \in \Sigma\) with \(\mu(A) < \infty\)).

The following well-known result characterizes uniformly \(\mu\)-continuous sets in \(ca(\Omega, \Sigma)\).

**Lemma 2.1.** For a subset \(K\) of \(ca(\Omega, \Sigma)^+\) the following statements are equivalent:

(i) \(K\) is uniformly \(\mu\)-continuous (i.e., \(\lim_{n \to \infty} (\sup_{\nu \in K} \nu(A_n)) = 0\) as \(A_n \downarrow_{\mu} \emptyset\)).

(ii) For each \(\eta > 0\) there exist \(\delta > 0\) and \(A_0 \in \Sigma\) with \(\mu(A_0) < \infty\) such that \(\nu(A) \leq \eta\) and \(\nu(\Omega \setminus A_0) \leq \eta\) for all \(A \in \Sigma\) with \(\mu(A) \leq \delta\) and all \(\nu \in K\).

We shall need the following technical result.

**Proposition 2.2.** Let \(K\) be a subset of \(ca(\Omega, \Sigma)^+\) such that each \(\nu \in K\) is \(\mu\)-continuous. Assume that \(K\) is not uniformly \(\mu\)-continuous. Then there exist a pairwise disjoint sequence \((B_n)\) in \(\Sigma\), a number \(\varepsilon_0 > 0\) and a sequence \((\nu_n)\) in \(K\) such that \(\nu_n(B_n) > \varepsilon_0\) for all \(n \in \mathbb{N}\).
Proof. In view of Lemma 2.1 there exists \( \varepsilon_0 > 0 \) such that either there exist a sequence \( (A_n) \) in \( \Sigma \) and a sequence \( (\nu^1_n) \) in \( \mathcal{K} \) such that
\[
\mu(A_n) \to 0 \quad \text{and} \quad \nu^1_n(A_n) > 2\varepsilon_0
\]
or there exists a sequence \( (\nu^2_n) \) in \( \mathcal{K} \) such that
\[
\nu^2_n(\Omega \setminus \Omega_n) > 2\varepsilon_0
\]
whenever \( \Omega_n \uparrow \Omega \) and \( \mu(\Omega_n) < \infty \) for \( n \in \mathbb{N} \).

Assume that condition (1) holds. Then arguing as in [BL, p. 546] one can find a pairwise disjoint sequence \( (B_n) \) in \( \Sigma \) and a subsequence \( (\nu^1_{k_n}) \) of \( (\nu^1_n) \) such that \( \nu^1_{k_n}(B_n) \geq \varepsilon_0 \). Let \( \nu_n = \nu^1_{k_n} \) for \( n \in \mathbb{N} \).

Now assume that condition (2) holds. Let \( \Omega_n = \Omega \setminus \Omega_n \) for \( n \in \mathbb{N} \). Then \( \nu_n(\Omega_n) \to 0 \) for each \( n \in \mathbb{N} \). Let \( l_1 = 1 \) and choose \( l_n \in \mathbb{N} \) such that \( l_2 > l_1, \nu^2_{l_1}(C_{l_2}) < \varepsilon_0 \). Then choose \( l_3 \in \mathbb{N} \) such that \( l_3 > l_2 \) and \( \nu^1_{l_3}(C_{l_3}) < \varepsilon_0 \).

Continuing this process inductively we can find an increasing sequence \( (l_n) \) in \( \mathbb{N} \) such that \( \nu^2_{l_n}(C_{l_{n+1}}) < \varepsilon_0 \). Let \( B_n = C_{l_n} \setminus C_{l_{n+1}} \) for \( n \in \mathbb{N} \). Then \( (B_n) \) is a disjoint sequence and since \( B_n \) is a non-negative \( \mu \)-continuous countably additive set function on \( X \) for each \( n \in \mathbb{N} \), by making use of (2) we obtain that \( \nu^2_{l_n}(B_n) \geq \nu^2_{l_{n+1}}(C_{l_n}) - \nu^2_{l_{n+1}}(C_{l_{n+1}}) > 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0 \). Put \( \nu_n = \nu^2_{l_n} \) for \( n \in \mathbb{N} \). \( \square \)

For a subset \( H \) of \( E(X) \) let \( \bar{H} = \{ \bar{f} : f \in H \} \).

Now we are ready to state our main result.

Theorem 2.3. Let \( H \) be a subset of \( E(X) \) such that the subset \( \bar{H} \) of \( E \) is \( \sigma(E, E') \)-bounded. Then the following statements are equivalent:

(i) \( H \) is conditionally \( \sigma(E(X), E'(X^*)) \)-compact.

(ii) a) \( \bar{H} \) is conditionally \( \sigma(E, E') \)-compact.

b) \( \left\{ \int_A f(\omega)d\mu : f \in H \right\} \) is a conditionally weakly compact subset of \( X \) for each \( A \in \Sigma, \mu(A) < \infty \) with \( \chi_A \in E' \).

Proof. (i) \( \Rightarrow \) (ii) To prove that (a) holds, in view of Proposition 1.2 it is enough to show that for each \( 0 \leq v \in E' \) the subset \( \{ \bar{f}v : f \in H \} \) of \( L^1 \) is uniformly integrable. Assume on the contrary that there exists \( 0 \leq v_0 \in E' \) such that the set \( \{ \bar{f}v_0 : f \in H \} \) is not uniformly integrable. For each \( f \in H \) set \( \nu_f(A) = \int_A \bar{f}(\omega)v_0(\omega)d\mu \) for \( A \in \Sigma \). Then \( \nu_f \) is a non-negative \( \mu \)-continuous countably additive set function on \( \mu \) but the family \( \{ \nu_f : f \in H \} \) is not uniformly \( \mu \)-continuous. Hence in view of Proposition 2.2 there exist a pairwise disjoint sequence \( (B_n) \) in \( \Sigma \), a sequence \( (f_n) \) in \( H \), and a number \( \varepsilon_0 > 0 \) such that \( \nu_{f_n}(B_n) = \int_{B_n} \bar{f}_n(\omega)v_0(\omega)d\mu > \varepsilon_0 \) for each \( n \in \mathbb{N} \). Clearly \( v_0f_n \in L^1(X) \), so in view of [BM, Theorem 1.1.4]

\[
\nu_{f_n}(B_n) = \| \chi_{B_n}v_0\bar{f}_n \|_{L^1} = \| \chi_{B_n}v_0f_n \|_{L^1(X)} = \sup_{B_n} \left\{ \int_{B_n} \langle v_0(\omega)f_n(\omega), g(\omega) \rangle d\mu : g \in L^\infty(X^*), \| g \|_{L^\infty(X^*)} \leq 1 \right\}.
\]

Hence one can produce a sequence \( (g_n) \) in \( L^\infty(X^*) \) with \( \| g_n \|_{L^\infty(X^*)} \leq 1, \chi_{\Omega \setminus B_n}g_n = 0 \) and such that
\[
(1) \quad \int_{B_n} \langle v_0(\omega)f_n(\omega), g_n(\omega) \rangle d\mu > \varepsilon_0.
\]
Set \( g_0 = \sum_{n=1}^{\infty} g_n \). Then \( g_0 \in L^0(X^*) \) and \( \|g_0\|_{L^\infty(X^*)} \leq 1 \). Clearly \( v_0 g_0 \in E'(X^*) \), so \( \chi_A v_0 g_0 \in E'(X^*) \) for each \( A \in \Sigma \). In view of the assumption (i) there exists a \( \sigma(E(X), E'(X^*)) \)-Cauchy subsequence \( (f_{k_n}) \) of \( (f_n) \) so for each \( A \in \Sigma \), \( \lim_n \int_A (f_{k_n}(\omega), v_0(\omega) g_0(\omega)) d\mu \) exists. Setting \( \mu_n(A) = \int_A (f_{k_n}(\omega), v_0(\omega) g_0(\omega)) d\mu \) for \( A \in \Sigma \), in view of Nikodym’s convergence theorem (see [Z, Chap. 7]), \( \{\mu_n : n \in \mathbb{N}\} \) is uniformly countably additive on \( \Sigma \). Hence there exists \( m_0 \in \mathbb{N} \) such that for \( m \geq m_0 \), \( \sup_n |\mu_n(B_{k_m})| \leq \varepsilon_0 \) (see [D, Chap. 7, Theorem 10]). Hence for each \( m \geq m_0 \) we get

\[
|\mu_m(B_{k_m})| = \left| \int_{B_{k_m}} (f_{k_m}(\omega), v_0(\omega) g_{k_m}(\omega)) d\mu \right| \\
= \left| \int_{B_{k_m}} (v_0(\omega) f_{k_m}(\omega), g_{k_m}(\omega)) d\mu \right| \leq \varepsilon_0
\]

which contradicts (1). This contradiction establishes that (a) holds.

To show that (b) holds, take \( A \in \Sigma \) with \( \chi_A \in E' \), and let \( (f_n) \) be a sequence in \( H \). Set \( g = \chi_A x^* \) where \( x^* \in S_{X^*} \). Then \( g \in E'(X^*) \) and by assumption (i) there exists a subsequence \( (f_{k_n}) \) of \( (f_n) \) such that \( \lim_n \int_A (f_{k_n}(\omega), g(\omega)) d\mu \) exists. Since \( f(\omega), g(\omega)) d\mu = x^* ( \int_A f_{k_n}(\omega) d\mu ) \), the set \( \left\{ \int f(\omega) d\mu : f \in H \right\} \) is conditionally weakly compact in \( X \).

(ii) \( \Rightarrow \) (i) Let \( (f_n) \) be a sequence in \( H \). Since \( \sup E' = \Omega \) there exists a sequence \( \{\Omega_m\} \) in \( \Sigma \) such that \( \Omega_m \uparrow \Omega \) and \( \mu(\Omega_m) < \infty \), \( \chi_{\Omega_m} \in E' \) for \( m \in \mathbb{N} \) (see [Z Theorem 86.2]). Setting \( A_m = \Omega \setminus \Omega_m \) for \( m \in \mathbb{N} \), we see that \( A_m \searrow \emptyset \). Given \( m \in \mathbb{N} \) we have \( \sup_n \int_{A_m} f_n(\omega) d\mu = c_m < \infty \), because \( \chi_{\Omega_m} \in E' \) and \( \int_{A_m} \).

In view of the above observation there exists a \( \sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*)) \)-Cauchy subsequence \( (\chi_{\Omega_m} f_{k_n}) \) of \( (\chi_{\Omega_m} f_n) \). Next, there exists a \( \sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*)) \)-Cauchy subsequence \( (\chi_{\Omega_m} f_{k_n}) \) of \( (\chi_{\Omega_m} f_{k_n}) \). It follows that the diagonal sequence \( (f_{k_n}) \) has the property that for each \( m \in \mathbb{N} \), \( (\chi_{\Omega_m} f_{k_n}) \) is a \( \sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*)) \)-Cauchy sequence. Put \( h_n = f_{k_n} \) for \( n \in \mathbb{N} \).

Let \( g \in E'(X^*) \). For \( n \in \mathbb{N} \) let us put

\[
g_n(\omega) = \begin{cases} 
    g(\omega) & \text{if } \omega \in \Omega_m \text{ and } \|g(\omega)\|_{X^*} \leq n, \\
    0 & \text{elsewhere}.
\end{cases}
\]

Given \( \varepsilon > 0 \) there exist \( m_0 \in \mathbb{N} \) and \( \delta > 0 \) such that

\[
(2) \sup_n \int_{\Omega \setminus \Omega_{m_0}} \tilde{f}_n(\omega) \tilde{g}(\omega) d\mu \leq \frac{\varepsilon}{4} \quad \text{and} \quad \sup_n \int_A \tilde{f}_n(\omega) \tilde{g}(\omega) d\mu \leq \frac{\varepsilon}{4}
\]

where \( \tilde{f}_n(\omega) = f_n(\omega) \) for \( \omega \in \Omega \) and \( \tilde{g}(\omega) = g(\omega) \) for \( \omega \in \Omega \).
for each $A \in \Sigma$ with $\mu(A) \leq \delta$. For $\eta = \frac{\varepsilon}{4m_0}$ let

$$B_n = \{ \omega \in \Omega_{m_0} : \| g(\omega) - g_n(\omega) \|_{X^*} \geq \eta \}.$$ 

It is easy to observe that $B_n \downarrow \emptyset$, so $\mu(B_n) \to 0$. Choose $n_0 \in \mathbb{N}$ with $n_0 \geq m_0$ such that $\mu(B_{n_0}) \leq \delta$. Then by (2) we get

$$\sup_n \int_{B_{n_0}} \tilde{h}_n(\omega) \tilde{g}(\omega) d\mu \leq \frac{\varepsilon}{4}. \tag{3}$$

Hence, by (3) we have

$$\left| \int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega) \rangle d\mu \right| \leq \int_{\Omega_{m_0}} \tilde{h}_n(\omega) \| g(\omega) - g_{n_0}(\omega) \|_{X^*} d\mu \leq \int_{B_{n_0}} \tilde{h}_n(\omega) \| g(\omega) - g_{n_0}(\omega) \|_{X^*} d\mu + \int_{\Omega_{m_0} \setminus B_{n_0}} \tilde{h}_n(\omega) \| g(\omega) - g_{n_0}(\omega) \|_{X^*} d\mu \leq \int_{B_{n_0}} \tilde{h}_n(\omega) \tilde{g}(\omega) d\mu + \eta \int_{\Omega_{m_0}} \tilde{h}_n(\omega) d\mu \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4m_0} \cdot c_{m_0} = \frac{\varepsilon}{2} \tag{4}$$

Since $\int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega) \rangle d\mu \to a$ for some $a \in \mathbb{R}$, we can choose $n_1 \in \mathbb{N}$ such that for $n \geq n_1$

$$\left| \int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega) \rangle d\mu - a \right| \leq \frac{\varepsilon}{4}. \tag{5}$$

Thus by (2), (4) and (5) for $n \geq n_1$ we get

$$\left| \int_{\Omega} \langle h_n(\omega), g(\omega) \rangle d\mu - a \right| \leq \left| \int_{\Omega \setminus \Omega_{m_0}} \langle h_n(\omega), g(\omega) \rangle d\mu \right| + \left| \int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega) \rangle d\mu \right| + \left| \int_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega) \rangle d\mu - a \right| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$ 

This shows that $(h_n)$ is a $\sigma(E(X), E'(X^*))$-Cauchy subsequence of $(f_n)$, so $H$ is conditionally $\sigma(E(X), E'(X^*))$-compact. $\square$

Corollary 2.4. Assume that a Banach space $X$ contains no isomorphic copy of $\ell^1$, and let $H$ be a subset of $E(X)$ such that $H$ is $\sigma(E, E')$-bounded. Then the following statements are equivalent:

(i) $H$ is conditionally $\sigma(E(X), E'(X^*))$-compact.

(ii) $H$ is conditionally $\sigma(E, E')$-compact.

Proof. (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) In view of Theorem 2.3 it is enough to show that $\left\{ \int_A f(\omega) d\mu : f \in H \right\}$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$ such that $\chi_A \in E'$. In fact, let $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Hence $\sup_{f \in H} \| \int_A f(\omega) d\mu \| \leq \varepsilon$. 

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<tr>
<th>Condition</th>
<th>Description</th>
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<tr>
<td>$\eta = \frac{\varepsilon}{4m_0}$</td>
<td>For $\omega \in \Omega_{m_0}$, $| g(\omega) - g_n(\omega) |_{X^*} \geq \eta$.</td>
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<tr>
<td>$B_n$</td>
<td>${ \omega : | g(\omega) - g_n(\omega) |_{X^*} \geq \eta }$.</td>
</tr>
<tr>
<td>$\mu(B_{n_0}) \leq \delta$</td>
<td>Choose $n_0 \geq m_0$ with $\mu(B_{n_0}) \leq \delta$.</td>
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<tr>
<td>$\sup_n \int_{B_{n_0}} \tilde{h}_n(\omega) \tilde{g}(\omega) d\mu \leq \frac{\varepsilon}{4}$</td>
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<tr>
<td>$\int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega) \rangle d\mu \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4m_0} \cdot c_{m_0} = \frac{\varepsilon}{2}$</td>
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sup ∫ f(ω)dμ < ∞, so in view of the Rosenthals ∋1-theorem [R] \[ \left\{ \int_A f(ω)dμ : f ∈ H \right\} \] is a conditionally weakly compact subset of X, as desired.

Assume now that (E, ∥·∥_E) is a Banach function space. Then the space E(X) provided with the norm \( \|f\|_{E(X)} := \|f\|_E \) is usually called a Köthe-Bochner space. The associated norm \( \cdot : E' \rightarrow \mathbb{R} \) on the Köthe dual E' can be defined as follows:

\[ \|v\|_{E'} = \sup \left\{ \left| \int \omega(v) d\mu \right| : u ∈ E, \|u\|_E ≤ 1 \right\}. \]

Clearly, for a subset H of E(X) the set \( \tilde{H} \) is \( \sigma(E, E') \)-bounded whenever

\[ \sup_{f ∈ H} \|f\|_{E(X)} < ∞. \]

Combining Corollary 2.4 and Proposition 1.2 we get:

**Corollary 2.5.** Let (E, ∥·∥_E) be a Banach function space, and assume that X contains no isomorphic copy of \( \ell^1 \). Then the following statements are equivalent:

(i) The associated norm \( \cdot : \|E' \rightarrow \mathbb{R} \) on E' is order continuous.

(ii) Every norm bounded set in E(X) is conditionally \( \sigma(E(X), E'(X^*)) \)-compact.

We now apply the previous results to Orlicz spaces (see [KR], [L] for more details).

By a Young function we mean a mapping \( \varphi : [0, ∞) \rightarrow [0, ∞) \) that is convex, vanishes only at 0 and \( \lim_{t → 0} \frac{\varphi(t)}{t} = 0, \lim_{t → ∞} \frac{\varphi(t)}{t} = ∞ \). Let \( L^\varphi \) be the Orlicz space associated with \( \varphi \) and provided with the Luxemburg norm \( \|u\|_\varphi := \inf \{ λ > 0 : \int_Ω \varphi(\|u(\omega)\|/λ)dμ ≤ 1 \} \). Then \( (L^\varphi)' = L^{\varphi^∗} \), where \( \varphi^∗ \) denotes the complementary Young function.

We say that a Young function \( \varphi \) increases more rapidly than another \( \psi \), in symbols \( \varphi ≺ \psi \), if for each \( c > 0 \) there is \( d > 1 \) such that \( c\varphi(t) ≤ \frac{1}{d} \psi(dt) \) for all \( t ≥ 0 \) (see [N]). Note that \( \varphi \) satisfies the \( \nabla_2 \)-condition iff \( \varphi ≺ \varphi \).

As a consequence of Corollary 2.4 and [N] Theorem 2.5 we get:

**Corollary 2.6.** Assume that X contains no isomorphic copy of \( \ell^1 \). Then for a norm bounded subset H of the Orlicz-Bochner space \( L^\varphi(X) \) the following statements are equivalent:

(i) H is conditionally \( \sigma(L^\varphi(X), L^{\varphi^∗}(X^*)) \)-compact.

(ii) There is a Young function \( \psi \) with \( \varphi ≺ \psi \) and such that \( H ⊂ L^φ(X) \) and \( \sup_{f ∈ H} \|f\|_{L^φ(X)} < ∞ \).

**Corollary 2.7.** Assume that X contains no isomorphic copy of \( \ell^1 \) and \( \varphi \) satisfies the \( \nabla_2 \)-condition. Then every norm bounded subset of \( L^\varphi(X) \) is conditionally \( \sigma(L^\varphi(X), L^{\varphi^∗}(X^*)) \)-compact.

**References**


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