CONDITIONAL WEAK COMPACTNESS
IN VECTOR-VALUED FUNCTION SPACES

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Abstract. Let $E$ be an ideal of $L^0$ over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ and let $E'$ be the Köthe dual of $E$ with $\text{supp } E' = \Omega$. Let $(X, \| \cdot \|_X)$ be a real Banach space, and $X^*$ the topological dual of $X$. Let $E(X)$ be a subspace of the space $L^0(X)$ of equivalence classes of strongly measurable functions $f: \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\| f(\cdot) \|_X$ belongs to $E$. For a subset $H$ of $E(X)$ for which the set $\{ \| f(\cdot) \|_X : f \in H \}$ is $\sigma(E, E')$-bounded the following statement is equivalent to conditional $\sigma(E(X), E'(X^*))$-compactness: the set $\{ \| f(\cdot) \|_X : f \in H \}$ is conditionally $\sigma(E, E')$-compact and $\int_A f(\omega) \, d\mu : f \in H$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$.

Applications to Orlicz-Bochner spaces are given.

1. INTRODUCTION AND PRELIMINARIES

Given a dual pair $\langle L, K \rangle$, a subset $A$ of $L$ is said to be conditionally $\sigma(L, K)$-compact whenever each sequence in $A$ contains a $\sigma(L, K)$-Cauchy subsequence (cf. [MN] p. 100). The problem of characterizing relatively sequentially $\sigma(L^p(X), L^q(X^*))$-compact subsets of Lebesgue-Bochner spaces $L^p(X)$ (where $1 \leq p < \infty$ and $q$ conjugate to $p$) over a finite measure space was considered by F. Bombal [B1] and J. Batt and W. Hiermeyer [BH] Theorem 2.1]. Moreover, F. Bombal characterized relatively sequentially $\sigma(L^p(X), L^{q'}(X^*))$-compact subsets of Orlicz-Bochner spaces $L^p(X)$ [B2] Theorem 3]. C. Abott, E. Bator, R. Bilyeu and P. Lewis [ABBL] obtained the following characterization of conditionally $\sigma(L^1(X), L^{\infty}(X^*))$-compact subsets of $L^1(X)$.

Theorem 1.1 (cf. [ABBL Theorem 2.5]). Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Then for a norm bounded subset $H$ of $L^1(X)$ the following statements are equivalent:

(i) $H$ is conditionally $\sigma(L^1(X), L^{\infty}(X^*))$-compact.

(ii) a) The subset $\{ \| f(\cdot) \|_X : f \in H \}$ of $L^1$ is uniformly integrable.

b) The set $\int_A f(\omega) \, d\mu : f \in H$ is conditionally weakly compact in $X$ for each $A \in \Sigma$.

In this paper, by making use of Theorem 1.1 we characterize conditionally $\sigma(E(X), E'(X^*))$-compact subsets of $E(X)$, where $E$ is an ideal of $L^0$ over a $\sigma$-finite measure space.

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Now we establish notation and terminology (see [AB], [KA]).
Let $(Ω, Σ, μ)$ be a complete $σ$-finite measure space and let $L^0$ denote the space of equivalence classes of all $Σ$-measurable functions defined and finite a.e. on $Ω$. Let $χ_A$ stand for the characteristic function of a set $A$ and let $ℕ$ denote the set of all natural numbers. Let $E$ be an ideal of $L^0$ with supp $E = Ω$, and let $E'$ stand for the Köthe dual of $E$, i.e.,

$$E' = \{ v \in L^0 : \int_Ω |u(ω)v(ω)|dμ < ∞ \text{ for all } u \in E \}.$$  

We assume that supp $E' = Ω$.

Let $(X, \| \cdot \|_X)$ be a real Banach space, and let $S_X$ and $B_X$ denote the unit sphere and the closed unit ball in $X$, resp. Let $X^*$ stand for the Banach dual of $X$. By $L^0(X)$ we denote the set of equivalence classes of all strongly $Σ$-measurable functions $f : Ω \to X$. For $f \in L^0(X)$ let us set $\bar{f}(ω) = \|f(ω)\|_X$ for $ω \in Ω$. Let

$$E(X) = \{ f \in L^0(X) : f(ω) \in E \text{ for all } ω \in Ω \}.$$  

By $σ(E(X), E'(X^*))$ we will denote the weak topology on $E(X)$ with respect to the dual system $\langle E(X), E'(X^*) \rangle$ under the natural duality $\langle f, g \rangle = \int_Ω (f(ω), g(ω))dμ$ for $f \in E(X)$, $g \in E'(X^*)$.

The following characterization of conditional $σ(E, E')$-compactness is needed.

**Proposition 1.2** ([N2] Theorem 1.1). For a $σ(E, E')$-bounded subset $A$ of $E$ the following statements are equivalent:

(i) $A$ is conditionally $σ(E, E')$-compact.

(ii) For each $v \in E'$ the subset $\{ uv : u \in A \}$ of $L^1$ is uniformly integrable.

(iii) The functional $p_A$ on $E'$ defined by $p_A(v) = \sup_{u \in A} \int_Ω |u(ω)v(ω)|dμ$ is an order continuous Riesz seminorm.

2. **Conditionally $σ(E(X), E'(X^*))$-compact sets in $E(X)$**

Let $ca(Ω, Σ)$ stand for the Riesz space of countably additive set functions $ν$ on $Σ$. For a sequence $(A_n)_n$ in $Σ$ we write $A_n \searrow_μ θ$ whenever $A_n \downarrow$ and $μ(\bigcap_{n=1}^∞ A_n) = 0$ (that is, $A_n \downarrow$ and $μ(A_n ∩ A) → 0$ for each $A \in Σ$ with $μ(A) < ∞$).

The following well-known result characterizes uniformly $μ$-continuous sets in $ca(Ω, Σ)$.

**Lemma 2.1.** For a subset $K$ of $ca(Ω, Σ)^+$ the following statements are equivalent:

(i) $K$ is uniformly $μ$-continuous (i.e., $\lim sup_{n} ν(A_n) = 0$ as $A_n \searrow_μ θ$).

(ii) For each $η > 0$ there exist $δ > 0$ and $A_0 ∈ Σ$ with $μ(A_0) < ∞$ such that $ν(A) ≤ η$ and $ν(Ω \setminus A_0) ≤ η$ for all $A ∈ Σ$ with $μ(A) ≤ δ$ and all $ν ∈ K$.

We shall need the following technical result.

**Proposition 2.2.** Let $K$ be a subset of $ca(Ω, Σ)^+$ such that each $ν ∈ K$ is $μ$-continuous. Assume that $K$ is not uniformly $μ$-continuous. Then there exist a pairwise disjoint sequence $(B_n)_n$ in $Σ$, a number $ε_0 > 0$ and a sequence $(ν_n)_n$ in $K$ such that $ν_n(B_n) > ε_0$ for all $n ∈ ℕ$. 

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Proof. In view of Lemma 2.1 there exists \( \varepsilon_0 > 0 \) such that either there exist a sequence \((A_n)\) in \( \Sigma \) and a sequence \((\nu_n^1)\) in \( \mathcal{K} \) such that
\[
\mu(A_n) \rightarrow 0 \quad \text{and} \quad \nu_n^1(A_n) > 2\varepsilon_0
\]
or there exists a sequence \((\nu_n^2)\) in \( \mathcal{K} \) such that
\[
\nu_n^2(\Omega \setminus \Omega_n) > 2\varepsilon_0
\]
whenever \( \Omega_n \uparrow \Omega \) and \( \mu(\Omega_n) < \infty \) for \( n \in \mathbb{N} \).

Assume that condition (1) holds. Then arguing as in [BL, p. 546] one can find a pairwise disjoint sequence \((B_n)\) in \( \Sigma \) and a subsequence \((\nu_{k_n}^1)\) of \((\nu_n^1)\) such that \(\nu_{k_n}^1(B_n) \geq \varepsilon_0\). Let \( \nu_n = \nu_{k_n}^1 \) for \( n \in \mathbb{N} \).

Now assume that condition (2) holds. Let \( C_n = \Omega \setminus \Omega_n \) for \( n \in \mathbb{N} \). Then \( C_n \setminus \mu = \emptyset \), so \( \nu(C_n) \rightarrow 0 \) for each \( \nu \in \mathcal{K} \). Let \( l_1 = 1 \) and choose \( l_2 \in \mathbb{N} \) such that \( l_2 > l_1 \), \( \nu_{l_2}^1(C_{l_2}) < \varepsilon_0 \). Then choose \( l_3 \in \mathbb{N} \) such that \( l_3 > l_2 \) and \( \nu_{l_3}^2(C_{l_3}) < \varepsilon_0 \).

Continuing this process inductively we can find an increasing sequence \((l_n)\) in \( \mathbb{N} \) such that \( \nu_{l_n}^2(C_{l_n+1}) < \varepsilon_0 \). Let \( B_n = C_{l_n} \setminus C_{l_n+1} \) for \( n \in \mathbb{N} \). Then \((B_n)\) is a disjoint sequence and since \( B_n = C_{l_n} \setminus C_{l_n+1} \) for \( n \in \mathbb{N} \), by making use of (2) we obtain that \( \nu_{l_n}^2(B_n) = \nu_{l_n}^2(C_{l_n}) - \nu_{l_n}^2(C_{l_n+1}) > 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0 \). Put \( \nu_n = \nu_{l_n}^2 \) for \( n \in \mathbb{N} \). \( \square \)

For a subset \( H \) of \( E(X) \) let \( \tilde{H} = \{ \tilde{f} : f \in H \} \).

Now we are ready to state our main result.

**Theorem 2.3.** Let \( H \) be a subset of \( E(X) \) such that the subset \( \tilde{H} \) of \( E \) is \( \sigma(E, E') \)-bounded. Then the following statements are equivalent:

(i) \( H \) is conditionally \( \sigma(E(X), E'(X^*)) \)-compact.

(ii) a) \( \tilde{H} \) is conditionally \( \sigma(E, E') \)-compact.

b) \( \left\{ \int_A f(\omega)d\mu : f \in H \right\} \) is a conditionally weakly compact subset of \( X \) for each \( A \in \Sigma \), \( \mu(A) < \infty \) with \( \chi_A \in E' \).

**Proof.** (i) \( \Rightarrow \) (ii) To prove that (a) holds, in view of Proposition 1.2 it is enough to show that for each \( 0 \leq v \in E' \) the subset \( \{ \tilde{f}v : f \in H \} \) of \( L^1 \) is uniformly integrable. Assume on the contrary that there exists \( 0 \leq v_0 \in E' \) such that the set \( \{ \tilde{f}v_0 : f \in H \} \) is not uniformly integrable. For each \( f \in H \) set \( \nu_f(A) = \int_A \tilde{f}(\omega)v_0(\omega)d\mu \) for \( A \in \Sigma \). Then \( \nu_f \) is a non-negative \( \mu \)-continuous countably additive set function on \( \Sigma \) but the family \( \{ \nu_f : f \in H \} \) is not uniformly \( \mu \)-continuous. Hence in view of Proposition 2.2 there exist a pairwise disjoint sequence \((B_n)\) in \( \Sigma \), a sequence \((f_n)\) in \( H \), and a number \( \varepsilon_0 > 0 \) such that \( \nu_{f_n}(B_n) \rightarrow \int_{B_n} \tilde{f}_n(\omega)v_0(\omega)d\mu > \varepsilon_0 \) for each \( n \in \mathbb{N} \). Clearly \( v_0f_n \in L^1(X) \), so in view of [BL, Theorem 1.1.(4)]

\[
\nu_{f_n}(B_n) = \|\chi_{B_n}v_0\tilde{f}_n\|_{L^1} = \|\chi_{B_n}v_0f_n\|_{L^1(X)}
\]

\[
= \sup \left\{ \left| \int_{B_n} (v_0(\omega)f_n(\omega), g(\omega))d\mu \right| : g \in L^\infty(X^*), \|g\|_{L^\infty(X^*)} \leq 1 \right\}.
\]

Hence one can produce a sequence \((g_n)\) in \( L^\infty(X^*) \) with \( \|g_n\|_{L^\infty(X^*)} \leq 1, \chi_{\Omega \setminus B_n}g_n = 0 \) and such that
\[
\left| \int_{B_n} (v_0(\omega)f_n(\omega), g_n(\omega))d\mu \right| > \varepsilon_0.
\] \( \square \)
Set $g_0 = \sum_{n=1}^{\infty} g_n$. Then $g_0 \in L^0(X^*)$ and $\|g_0\|_{L^\infty(X^*)} \leq 1$. Clearly $v_0g_0 \in E'(X^*)$, so $\chi_Av_0g_0 \in E'(X^*)$ for each $A \in \Sigma$. In view of the assumption (i) there exists a $\sigma(E(X), E'(X^*))$-Cauchy subsequence $(f_{k_n})_n$ of $(f_n)_n$ so that each $A \in \Sigma$, 

$$\lim_{n} \int_A (f_{k_n}(\omega), v_0(\omega)g_0(\omega))d\mu$$

exists. Setting $\mu_n(A) = \int_A (f_{k_n}(\omega), v_0(\omega)g_0(\omega))d\mu$ for $A \in \Sigma$, in view of Nikodym’s convergence theorem (see [3] Chap. 7), $\{\mu_n : n \in \mathbb{N}\}$ is uniformly countably additive on $\Sigma$. Hence there exists $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $\sup |\mu_n(B_m)| \leq \varepsilon_0$ (see [3] Chap. 7, Theorem 10). Hence for each $m \geq m_0$ we get

$$|\mu_m(B_{k_m})| = \left| \int_{B_{k_m}} f_{k_m}(\omega), v_0(\omega)g_{k_m}(\omega))d\mu \right|$$

which contradicts (1). This contradiction establishes that (a) holds.

To show that (b) holds, take $A \in \Sigma$ with $\chi_A \in E'$, and let $(f_n)$ be a sequence in $H$. Set $g = \chi_{AX^*}$ where $x^* \in S_{X^*}$. Then $g \in E'(X^*)$ and by assumption (i) there exists a subsequence $(f_{k_n})_n$ of $(f_n)_n$ such that $\lim_{n} \int (f_{k_n}(\omega), g(\omega))d\mu$ exists. Since $\int (f_{k_n}(\omega), g(\omega))d\mu = x^* \int (f_{k_n}(\omega))d\mu$, the set $\left\{ \int f(\omega)d\mu : f \in H \right\}$ is conditionally weakly compact in $X$.

(ii) $\Rightarrow$ (i) Let $(f_n)$ be a sequence in $H$. Since $E' = \Omega$ there exists a subsequence $(\Omega_m)_n$ in $\Sigma$ such that $\Omega_m \uparrow \Omega$ and $\mu(\Omega_m) < \infty$, $\chi_{\Omega_m} \in E'$ for $m \in \mathbb{N}$ (see [3] Theorem 5.9). Setting $A_m = \Omega \setminus \Omega_m$ for $m \in \mathbb{N}$ we see that $A_m \setminus \chi_m \emptyset$. Given $m \in \mathbb{N}$ we have $\sup f_n(\omega)d\mu = c_m < \infty$, because $\chi_{\Omega_m} \in E'$ and $H$ is $\sigma(E, E')$-bounded. Hence $\{\chi_{\Omega_m} f_n : n \in \mathbb{N}\} \subset L^1_{\Omega_m}(X)$, and by assumption (a), $\{\chi_{\Omega_m} f_n : n \in \mathbb{N}\}$ is a uniformly integrable subset of $L^1_{\Omega_m}$. Combining this observation with (b), in view of Theorem 1.1 we see that $\{\chi_{\Omega_m} f_n : n \in \mathbb{N}\}$ is a conditionally $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-compact subset of $L^1_{\Omega_m}(X)$.

In view of the above observation there exists a $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-Cauchy subsequence $(\chi_{\Omega_m} f_{k_n})_n$ of $(\chi_{\Omega_m} f_n)$. Next, there exists a $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-Cauchy subsequence $(\chi_{\Omega_m} f_{k_n})_n$ of $(\chi_{\Omega_m} f_{k_n})_n$. It follows that the diagonal sequence $(f_{k_n})_n$ has the property that for each $m \in \mathbb{N}$ $(\chi_{\Omega_m} f_{k_n})$ is a $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-Cauchy sequence. Put $h_n = f_{k_n}$ for $n \in \mathbb{N}$.

Let $g \in E'(X^*)$. For $n \in \mathbb{N}$ let us put

$$g_n(\omega) = \begin{cases} g(\omega) & \text{if } \omega \in \Omega_m \text{ and } \|g(\omega)\|_{X^*} \leq n, \\ 0 & \text{elsewhere.} \end{cases}$$

Given $\varepsilon > 0$ there exist $m_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\sup_{n} \int_{\Omega \setminus \Omega_{m_0}} |f_n(\omega) - g_n(\omega)|d\mu \leq \frac{\varepsilon}{4} \text{ and } \sup_{n} \int_A |f_n(\omega) - g_n(\omega)|d\mu \leq \frac{\varepsilon}{4}.$$
for each $A \in \Sigma$ with $\mu(A) \leq \delta$. For $\eta = \frac{\varepsilon}{4m_0}$ let

$$B_n = \{\omega \in \Omega_{m_0} : \|g(\omega) - g_n(\omega)\|_{X^*} \geq \eta\}.$$  

It is easy to observe that $B_n \downarrow \emptyset$, so $\mu(B_n) \to 0$. Choose $n_0 \in \mathbb{N}$ with $n_0 \geq m_0$ such that $\mu(B_{m_0}) \leq \delta$. Then by (2) we get

$$\sup_{n} \int_{B_{m_0}} \tilde{h}_n(\omega) \tilde{g}(\omega) d\mu \leq \frac{\varepsilon}{4}. \tag{3}$$

Hence, by (3) we have

$$\left| \int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{m_0}(\omega) \rangle d\mu \right| \leq \int_{\Omega_{m_0}} \tilde{h}_n(\omega) \|g(\omega) - g_{m_0}(\omega)\|_{X^*} d\mu$$

$$\leq \int_{B_{m_0}} \tilde{h}_n(\omega) \|g(\omega) - g_{m_0}(\omega)\|_{X^*} d\mu + \int_{\Omega_{m_0} \setminus B_{m_0}} \tilde{h}_n(\omega) \|g(\omega) - g_{m_0}(\omega)\|_{X^*} d\mu$$

$$\leq \int_{B_{m_0}} \tilde{h}_n(\omega) \tilde{g}(\omega) d\mu + \eta \int_{\Omega_{m_0}} \tilde{h}_n(\omega) d\mu \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4m_0} \cdot c_{m_0} = \frac{\varepsilon}{2}. \tag{4}$$

Since $\int_{\Omega_{m_0}} \langle h_n(\omega), g_{m_0}(\omega) \rangle d\mu \xrightarrow{n} a$ for some $a \in \mathbb{R}$, we can choose $n_1 \in \mathbb{N}$ such that for $n \geq n_1$

$$\left| \int_{\Omega_{m_0}} \langle h_n(\omega), g_{m_0}(\omega) \rangle d\mu - a \right| \leq \frac{\varepsilon}{4}. \tag{5}$$

Thus by (2), (4) and (5) for $n \geq n_1$ we get

$$\left| \int_{\Omega} \langle h_n(\omega), g(\omega) \rangle d\mu - a \right|$$

$$\leq \left| \int_{\Omega \setminus \Omega_{m_0}} \langle h_n(\omega), g(\omega) \rangle d\mu \right| + \left| \int_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{m_0}(\omega) \rangle d\mu \right|$$

$$+ \left| \int_{\Omega_{m_0}} \langle h_n(\omega), g_{m_0}(\omega) \rangle d\mu - a \right| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$  

This shows that $(h_n)$ is a $\sigma(E(X), E'(X^*))$-Cauchy subsequence of $(f_n)$, so $H$ is conditionally $\sigma(E(X), E'(X^*))$-compact. \hfill \Box

**Corollary 2.4.** Assume that a Banach space $X$ contains no isomorphic copy of $\ell^1$, and let $H$ be a subset of $E(X)$ such that $H$ is $\sigma(E, E')$-bounded. Then the following statements are equivalent:

(i) $H$ is conditionally $\sigma(E(X), E'(X^*))$-compact.

(ii) $H$ is conditionally $\sigma(E, E')$-compact.

**Proof.** (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) In view of Theorem 2.3 it is enough to show that $\left\{ \int_A f(\omega) d\mu : f \in H \right\}$ is a conditionally weakly compact subset of $X$ for each $A \in \Sigma$ such that $\chi_A \in E'$. In fact, let $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Hence $\sup_{f \in H} \left\| \int_A f(\omega) d\mu \right\| \leq \int_A \sup_{f \in H} \left\| f(\omega) \right\| d\mu$.  

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\[ \sup_{f \in H} \int_A f(\omega) d\mu < \infty, \] so in view of the Rosenthal’s \( \ell^1 \)-theorem \cite{MR171865} \[
 \left\{ \int_A f(\omega) d\mu : f \in H \right\}
\] is a conditionally weakly compact subset of \( X \), as desired.

Assume now that \( (E, \| \cdot \|_E) \) is a Banach function space. Then the space \( E(X) \) provided with the norm \( \| f \|_{E(X)} := \| f \|_E \) is usually called a Köthe-Bochner space. The associated norm \( \| \cdot \|_{E'} \) on the Köthe dual \( E' \) can be defined as follows:

\[
\| v \|_{E'} = \sup_{u \in E} \left\{ \int \Omega u(\omega) v(\omega) d\mu : u \in E, \| u \|_E \leq 1 \right\}.
\]

Clearly, for a subset \( H \) of \( E(X) \) the set \( \tilde{H} \) is \( \sigma(E, E') \)-bounded whenever

\[
\sup_{f \in H} \| f \|_{E(X)} < \infty.
\]

Combining Corollary 2.4 and Proposition 1.2 we get:

**Corollary 2.5.** Let \( (E, \| \cdot \|_E) \) be a Banach function space, and assume that \( X \) contains no isomorphic copy of \( \ell^1 \). Then the following statements are equivalent:

(i) The associated norm \( \| \cdot \|_{E'} \) on \( E' \) is order continuous.

(ii) Every norm bounded set in \( E(X) \) is conditionally \( \sigma(E(X), E'(X^*)) \)-compact.

We now apply the previous results to Orlicz spaces (see \cite{MR2005887, MR1949485} for more details).

By a Young function we mean a mapping \( \varphi : [0, \infty) \to [0, \infty) \) that is convex, vanishes only at 0 and \( \lim_{t \to 0} \varphi(t) / t = 0 \), \( \lim_{t \to \infty} \varphi(t) / t = \infty \). Let \( L^\varphi \) be the Orlicz space associated with \( \varphi \) and provided with the Luxemburg norm \( \| u \|_\varphi := \inf \{ \lambda > 0 : \int \Omega \varphi(|u(\omega)|/\lambda) d\mu \leq 1 \} \). Then \( (L^\varphi)' = L^{\varphi^*} \), where \( \varphi^* \) denotes the complementary Young function.

We say that a Young function \( \psi \) increases more rapidly than another \( \varphi \), in symbols \( \varphi \prec \psi \), if for each \( c > 0 \) there is \( d > 1 \) such that \( c \varphi(t) \leq \frac{d}{t} \psi(dt) \) for all \( t \geq 0 \) (see \cite{MR1078064}). Note that \( \varphi \) satisfies the \( \nabla_2 \)-condition iff \( \varphi \prec \varphi \).

As a consequence of Corollary 2.4 and \cite{MR1078064} Theorem 2.5 we get:

**Corollary 2.6.** Assume that \( X \) contains no isomorphic copy of \( \ell^1 \). Then for a norm bounded subset \( H \) of the Orlicz-Bochner space \( L^\varphi(X) \) the following statements are equivalent:

(i) \( H \) is conditionally \( \sigma(L^\varphi(X), L^{\varphi^*}(X^*)) \)-compact.

(ii) There is a Young function \( \psi \) with \( \varphi \prec \psi \) and such that \( H \subset L^\psi(X) \) and \( \sup_{f \in H} \| f \|_{L^\psi(X)} < \infty \).

**Corollary 2.7.** Assume that \( X \) contains no isomorphic copy of \( \ell^1 \) and \( \varphi \) satisfies the \( \nabla_2 \)-condition. Then every norm bounded subset of \( L^\varphi(X) \) is conditionally \( \sigma(L^\varphi(X), L^{\varphi^*}(X^*)) \)-compact.

**References**


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