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ON THE BEREZIN-TOEPLITZ CALCULUS

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Abstract. We consider the problem of composing Berezin-Toeplitz operators on the Hilbert space of Gaussian square-integrable entire functions on complex n-space, $\mathbf{C}^n$. For several interesting algebras of functions on $\mathbf{C}^n$, we have $T_\varphi T_\psi = T_{\varphi\diamond\psi}$ for all φ, ψ in the algebra, where T_φ is the Berezin-Toeplitz operator associated with φ and $\varphi \diamond \psi$ is a "twisted" associative product on the algebra of functions. On the other hand, there is a C^∞ function φ for which T_φ is bounded but $T_\varphi T_\varphi \neq T_\psi$ for any ψ.

1. Introduction

For $z = (z_1, ..., z_n)$ in complex n-space, $\mathbf{C}^n$, with z_j in $\mathbf{C}$, $z \cdot w = z_1\overline{w_1} + ... + z_n\overline{w}_n$, consider the space $L^2(\mathbf{C}^n, d\mu)$ of Gaussian square-integrable complex-valued functions on $\mathbf{C}^n$, with $d\mu(z) = \exp\{-|z|^2/2\}dv(z)(2\pi)^{-n}$ with $dv(z)$ Lebesgue measure. The entire functions in $L^2(\mathbf{C}^n, d\mu)$ form a closed subspace $H^2(\mathbf{C}^n, d\mu)$ which arises naturally as a representation space of the Heisenberg group [B], [F], [BC1], [C]. On this (Segal-Bargmann) space, there are natural operators, formally introduced by Berezin [Be], defined densely for $\varphi(\cdot)$ with $\varphi(w)e^{w\cdot a}$ in $L^2(\mathbf{C}^n, d\mu)$ for all a in $\mathbf{C}^n$, by

$$(T_\varphi f)(z) = \int_{\mathbf{C}^n} e^{z\cdot w/2}\varphi(w)f(w)d\mu(w).$$

The (possibly unbounded) operator T_φ is called the *Berezin-Toeplitz operator* associated to φ. Note that $H^2(\mathbf{C}^n, d\mu)$ is a Bergman space with reproducing kernel function $e^{z\cdot a/2}$ for the functional of "evaluation at a" [B]. Note also that $T_\varphi = 0$ if and only if $\varphi = 0$ [F, p. 140].

The operators T_φ are closely related to pseudodifferential operators on $L^2(\mathbf{R}^n, dv)$. For φ bounded, and somewhat more generally, the relation is given by

$$B^{-1}T_\varphi B = W_{\beta_\varphi}$$

where B is the Bargmann isometry [Gu], W_β is the Weyl operator on $L^2(\mathbf{R}^n, dv)$ given by

$$(W_\beta\, g)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \beta(\xi, \tfrac{x+y}{2})e^{i(x-y)\cdot\xi}g(y)dyd\xi,$$

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and

$$\beta_\varphi(\xi, x) = \pi^{-n} \int_{\mathbf{C}^n} \varphi(w) e^{-|w-(x-i\xi)|^2} dv(w).$$

The operators T_φ might, therefore, be expected to share many of the properties of pseudodifferential operators. It is not easy to demonstrate a complete equivalence, partly because β_φ is a "very smoothed" version of φ. The analytic structure of $H^2(\mathbf{C}^n, d\mu)$ also enters the picture so that, for example,

$$T_\varphi T_{z_j} = T_{\varphi z_j}.$$

Moreover, the available function-theoretic machinery on $H^2(\mathbf{C}^n, d\mu)$ is relatively rudimentary, limited primarily to the Bergman space structure and the structure inherited as a representation space of the Heisenberg group.

In this note, we deal with the composition problem: is there a function $\varphi \diamond \psi$ so that

$$(*) \qquad T_\varphi T_\psi = T_{\varphi \diamond \psi}?$$

As a consequence of representation-theoretic results in [C], we do have $(*)$ for a reasonably large class of bounded φ, ψ and there is an explicit formula for $\varphi \diamond \psi$. The *same* "Moyal-type" formula also holds for a large class of *unbounded* φ, ψ (with unbounded $T_\varphi, T_\psi, T_{\varphi\diamond\psi}$) – precisely, φ, ψ can be arbitrary polynomials in $\{z_j, \overline{z}_j : 1 \leq j \leq n\}$.

On the other hand, we will exhibit a φ (unbounded, but C^∞), for which T_φ is a bounded operator but $T_\varphi T_\varphi$ cannot be approximated in norm by bounded Berezin-Toeplitz operators. Thus, there is a genuine limitation on our ability to compose Berezin-Toeplitz operators.

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2. Composition of Berezin-Toeplitz operators

For C^∞ functions φ, ψ we consider the (formal) twisted product

$$(**) \qquad \varphi \diamond \psi = \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)(\overline{\partial}^k \psi)$$

where $k = (k_1, ..., k_n)$ with k_j non-negative integers, and

$$\begin{aligned} \partial_j &= \tfrac{\partial}{\partial z_j}, \ \overline{\partial}_j = \tfrac{\partial}{\partial \overline{z}_j}, \\ \partial^k &= \partial_1^{k_1} ... \partial_n^{k_n}, \ \overline{\partial}^k = \overline{\partial}_1^{k_1} ... \overline{\partial}_n^{k_n}, \\ |k| &= k_1 + k_2 + ... + k_n, \\ k! &= k_1! k_2! ... k_n!. \end{aligned}$$

In the cases we will consider, the sum in $(**)$ will converge.

The first case we consider arises from representation-theoretic considerations of the Heisenberg group [C]. We consider φ, ψ in the "smooth Bochner algebra" $B_a(\mathbf{C}^n)$ which consists of all Fourier-Stieltjes transforms of compactly supported, regular, bounded complex-valued Borel measures on $\mathbf{C}^n$. More precisely, let

$$\chi_a(z) = \exp\{i \ Im(z \cdot a)\}.$$

Then $B_a(\mathbf{C}^n)$ consists of all functions

$$\hat{\sigma}(z) = \int_{\mathbf{C}^n} \chi_a(z) \ d\sigma(a)$$

where σ is a compactly supported, regular, bounded complex-valued Borel measure. It is well known that such functions are bounded, uniformly continuous, with bounded derivatives of all orders.

As our first positive result, we have

Theorem 1. *For φ, ψ in $B_a(\mathbf{C}^n)$, $\varphi \diamond \psi$ is also in $B_a(\mathbf{C}^n)$ and $T_\varphi T_\psi = T_{\varphi\diamond\psi}$. The series in $(**)$ converges uniformly and absolutely.*

Proof. In [C], it was shown that for $\varphi = \hat{\sigma}$, $\psi = \hat{\tau}$ in $B_a(\mathbf{C}^n)$,

$$T_\varphi T_\psi = T_{(\sigma\diamond\tau)\widehat{}}.$$

Here, we **defined** $\sigma \diamond \tau$ for all ϕ in $C_0(\mathbf{C}^n)$ by

$$\int_{\mathbf{C}^n} \phi(c)\ d(\sigma \diamond \tau)(c) = \int_{\mathbf{C}^n} \int_{\mathbf{C}^n} \phi(a+b) e^{b\cdot a/2} d\sigma(a)\ d\tau(b)$$

so that

$$(***) \qquad (\sigma \diamond \tau)\widehat{}(z) = \int_{\mathbf{C}^n} \int_{\mathbf{C}^n}\ \chi_{a+b}(z) e^{b\cdot a/2} d\sigma(a) d\tau(b)$$

is in $B_a(\mathbf{C}^n)$.

Expanding $e^{b\cdot a/2}$ in MacLaurin series in $(***)$ gives

$$\begin{aligned}
(\sigma \diamond \tau)\widehat{}(z) &= \sum_{s=0}^{\infty} \sum \tfrac{1}{s!}\tfrac{1}{2^s} \sum_{1\le j_i \le n} \int \overline{a}_{j_1}...\overline{a}_{j_s} \chi_a(z)\, d\sigma(a) \int b_{j_1}...b_{j_s}\, \chi_b(z) d\tau(b) \\
&= \sum_{s=0}^{\infty} \tfrac{1}{s!}\tfrac{1}{2^s} \sum_{1\le j_i\le n} 2^s (\partial_{j_1}...\partial_{j_s}\varphi)(-2)^s(\overline{\partial}_{j_1}...\overline{\partial}_{j_s}\psi) \\
&= \sum_{s=0}^{\infty} \tfrac{(-2)^s}{s!} \sum_{1\le j_i\le n} (\partial_{j_1}...\partial_{j_s}\varphi)(\overline{\partial}_{j_1}...\overline{\partial}_{j_s}\psi) \\
&= \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)(\overline{\partial}^k \psi)
\end{aligned}$$

and it is clear that the series converges uniformly and absolutely. Comparison with $(**)$ shows that

$$T_\varphi T_\psi = T_{\varphi\diamond\psi}$$

and completes the proof.

Our second case consists of φ, ψ arbitrary polynomials in $\{z_j, \overline{z}_j : 1 \le j \le n\}$. Here, the operators T_φ, T_ψ are unbounded and we need to be a little more careful. Nevertheless, we have for $\varphi \diamond \psi$ given by $(**)$,

Theorem 2. *For φ, ψ polynomials in $(z_1, ..., z_n, \overline{z}_1, ..., \overline{z}_n)$, we have $T_\varphi T_\psi$ defined on a dense domain consisting of linear combinations of functions of the form $\{p(z)e^{z\cdot a} : a \in \mathbf{C}^n$ and $p(z)$ polynomial in $(z_1, ..., z_n)\}$. On this domain*

$$T_\varphi T_\psi = T_{\varphi\diamond\psi}$$

and $\varphi \diamond \psi$ is polynomial in the $z_j, \overline{z}_j$.

Proof. Clearly, $T_{\overline{z}_j} = 2\partial_j$ and it is now easy to check that $T_\varphi\ p(z)e^{z\cdot a} = q(z)e^{z\cdot a}$ where p, q are polynomial in $z_1, ..., z_n$. The proof of the composition formula is inductive, in several steps.

We note first that, for φ polynomial in $\{z_j, \overline{z}_j : 1 \leq j \leq n\}$, $T_\varphi T_{\overline{z}_j} = T_{\varphi\diamond\overline{z}_j}$ implies $T_{\varphi\diamond|z_j|^2} = T_\varphi T_{|z_j|^2}$. This is because

$$\begin{aligned} T_\varphi T_{|z_j|^2} &= (T_\varphi T_{\overline{z}_j})T_{z_j} \\ &= T_{(\varphi\diamond\overline{z}_j)z_j} = T_{\varphi\diamond|z_j|^2}. \end{aligned}$$

Next, we check inductively that $T_\varphi T_{\overline{z}_j} = T_{\varphi\diamond\overline{z}_j}$ for all φ polynomial in $\{z_j, \overline{z}_j : 1 \leq j \leq n\}$. It is enough to consider φ monomial. Assume the result for φ of fixed degree (φ constant is trivial). The inductive step is:

$$\begin{aligned} T_{\varphi z_k}T_{\overline{z}_j} &= T_\varphi T_{\overline{z}_j}T_{z_k} \\ &= T_{(\varphi\diamond\overline{z}_j)z_k} \\ &= T_{\varphi z_k\diamond\overline{z}_j}, \quad k \neq j, \\ T_{\varphi z_j}T_{\overline{z}_j} &= T_\varphi(T_{z_j}T_{\overline{z}_j}) \\ &= T_\varphi(T_{|z_j|^2} - 2I) \\ &= T_\varphi T_{|z_j|^2} - T_{2\varphi} \\ &= T_{\varphi\diamond|z_j|^2 - 2\varphi} \\ &= T_{\varphi z_j\diamond\overline{z}_j}, \\ T_{\overline{z}_k\varphi}T_{\overline{z}_j} &= T_{\overline{z}_k}(T_\varphi T_{\overline{z}_j}) = T_{\overline{z}_k(\varphi\diamond\overline{z}_j)} \\ &= T_{\varphi\overline{z}_k\overline{z}_j - 2\overline{z}_k(\partial_j\varphi)} \\ &= T_{\overline{z}_k\varphi\diamond\overline{z}_j}. \end{aligned}$$

Thus, $T_\varphi T_{\overline{z}_j} = T_{\varphi\diamond\overline{z}_j}$ for all φ.

Next, for arbitrary φ we consider $T_\varphi T_\psi$ and do induction on the degree of ψ. We can assume ψ is monomial. Assume the result for all φ and for ψ of fixed degree (ψ constant is trivial). The inductive step is, first,

$$T_\varphi T_{\psi z_j} = (T_\varphi T_\psi)T_{z_j} = T_{(\varphi\diamond\psi)z_j} = T_{\varphi\diamond\psi z_j}.$$

We must also consider

$$T_\varphi T_{\overline{z}_j\psi} = (T_\varphi T_{\overline{z}_j})T_\psi.$$

By the first part of the proof,

$$T_\varphi T_{\overline{z}_j} = T_{\varphi\diamond\overline{z}_j}$$

and by the inductive hypothesis

$$T_{\varphi\diamond\overline{z}_j}T_\psi = T_{(\varphi\diamond\overline{z}_j)\diamond\psi}.$$

Thus, we need only check that

$$\varphi \diamond \overline{z}_j\psi = (\varphi \diamond \overline{z}_j) \diamond \psi.$$

This is a direct calculation. We note that

$$\varphi \diamond \overline{z}_j = \varphi \overline{z}_j - 2(\partial_j \varphi)$$

so

$$\begin{aligned}(\varphi \diamond \overline{z}_j) \diamond \psi &= \varphi \overline{z}_j \diamond \psi - 2(\partial_j \varphi) \diamond \psi \\ &= \sum_k \frac{(-2)^{|k|}}{k!} \overline{z}_j (\partial^k \varphi)(\overline{\partial}^k \psi) \\ &\quad -2 \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \partial_j \varphi)(\overline{\partial}^k \psi).\end{aligned}$$

Using

$$\overline{\partial}^k (\overline{z}_j \psi) = \overline{z}_j (\overline{\partial}^k \psi) + k_j (\overline{\partial}^{k-\delta_j} \psi)$$

where

$$k - \delta_j = (k_1, k_2, ..., k_j - 1, k_{j+1}, ..., k_n),$$

we see that

$$\begin{aligned}\varphi \diamond \overline{z}_j \psi &= \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi)(\overline{\partial}^k \overline{z}_j \psi) \\ &= \sum_k \frac{(-2)^{|k|}}{k!} \overline{z}_j (\partial^k \varphi)(\overline{\partial}^k \psi) \\ &\quad + \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi) k_j (\overline{\partial}^{k-\delta_j} \psi).\end{aligned}$$

Thus, we need only check that

$$\sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \varphi) k_j (\overline{\partial}^{k-\delta_j} \psi) = -2 \sum_k \frac{(-2)^{|k|}}{k!} (\partial^k \partial_j \varphi)(\overline{\partial}^k \psi).$$

Reindexing the sum on the left by $\ell = k - \delta_j$ completes the proof.

Remark. Since $\overline{z}_j \diamond \psi = \overline{z}_j \psi$, the identity

$$\varphi \diamond \overline{z}_j \psi = (\varphi \diamond \overline{z}_j) \diamond \psi$$

follows from the reasonably well-known associativity of $\diamond$ [G]. Our computational proof has the advantage of giving associativity of $\diamond$ as an immediate corollary of Theorem 2 since

$$T_{\varphi \diamond (\psi \diamond \gamma)} = T_\varphi (T_\psi T_\gamma) = (T_\varphi T_\psi) T_\gamma = T_{(\varphi \diamond \psi) \diamond \gamma}.$$

3. T_φ with $T_\varphi T_\varphi \neq T_\psi$ for any ψ

In this section, we produce the promised obstruction to composition of Berezin-Toeplitz operators. We use some calculations from [BC2] and we begin with a needed improvement of [BC2, Theorem 17]. In this section, we work on $H^2(\mathbf{C}, d\mu)$ $(n = 1)$. Here, the Bergman reproducing kernel function for evaluation at z is just

$$K(w, z) = e^{w\overline{z}/2}$$

and it follows that

$$k_z(w) = K(w, z)/\sqrt{K(z, z)} = e^{w\overline{z}/2 - |z|^2/4}$$

is a unit vector in $H^2(\mathbf{C}, d\mu)$. We consider the unitary operator

$$(R_a f)(z) = f(az)$$

on $H^2(\mathbf{C}, d\mu)$ for $|a| = 1$.

Theorem 3. *For $|a| = 1$ and $\operatorname{Re} a < 0$, we have*

$$\|R_a - T_\psi\| \geq 1$$

*for **all** ψ such that $\psi K(\cdot, z)$ is in $L^2(\mathbf{C}, d\mu)$ for every z in $\mathbf{C}$.*

Proof. We consider

$$\begin{aligned} \|T_\psi - R_a\| &\geq |\langle T_\psi k_z, R_a k_z\rangle - \langle R_a k_z, R_a k_z\rangle| \\ &\geq |\langle T_\psi k_z, R_a k_z\rangle - 1| . \end{aligned}$$

Now,

$$\langle T_\psi k_z, R_a k_z\rangle = \langle \psi \chi_z, K(\cdot, (1+\overline{a})z)\rangle \; e^{-|z|^2/2}$$

so we have

$$\begin{aligned} |\langle T_\psi k_z, R_a k_z\rangle| &\leq e^{-|z|^2/2} \|\psi\| \sqrt{K((1+\overline{a})z, (1+\overline{a})z)} \\ &\leq \|\psi\| \, e^{-|z|^2/2} e^{|1+a|^2|z|^2/4} \\ &\leq \|\psi\| \, e^{|z|^2 \operatorname{Re} a/2}. \end{aligned}$$

Since $\operatorname{Re} a < 0$, we see that

$$|\langle T_\psi k_z, R_a k_z\rangle| \to 0$$

as $|z| \to \infty$. Thus, $\|T_\psi - R_a\| \geq 1$.

The function φ will be chosen to have the form $\varphi(z) = e^{\lambda|z|^2}$ where $\operatorname{Re} \lambda < \frac{1}{4}$ so that T_φ makes sense.

Lemma. *For $\lambda = \frac{1}{5} + i\frac{2}{5}$ and $\varphi(z) = e^{\lambda|z|^2}$, we have T_φ unitary with*

$$T_\varphi T_\varphi = a R_a$$

for $a = \overline{(1-2\lambda)^2} = -\frac{7}{25} + i\frac{24}{25}$.

Proof. $\operatorname{Re} \lambda < \frac{1}{4}$ and calculations outlined in [BC2, p. 582] show that T_φ is diagonal in the basis

$$e_k = (2^k k!)^{-1/2} z^k, \quad k = 0, 1, ...,$$

for $H^2(\mathbf{C}, d\mu)$, with

$$T_\varphi e_k = (1-2\lambda)^{-(k+1)} e_k.$$

Now

$$\lambda = \tfrac{1}{5} + i\tfrac{2}{5}$$

and so

$$\begin{aligned} T_\varphi T_\varphi e_k &= \overline{(1-2\lambda)^{2(k+1)}} e_k \\ &= a^{k+1} e_k. \end{aligned}$$

But

$$a R_a e_k = a^{k+1} e_k$$

and we are done.

We now have the promised

Theorem 4. *For* $\lambda = \frac{1}{5} + i\frac{2}{5}$ *and* $a = \overline{(1-2\lambda)^2} = -\frac{7}{25} + \frac{24}{25}i$, *with* $\varphi(z) = e^{\lambda|z|^2}$,

$$\|T_\varphi T_\varphi - T_\psi\| \geq 1$$

for *<u>all</u>* ψ *such that* $\psi K(\cdot, z)$ *is in* $L^2(\mathbf{C}, d\mu)$ *for every* z *in* $\mathbf{C}$.

Proof. Direct combination of Theorem 3 and the Lemma.

Remark. In fact, for $\varphi(z) = e^{\lambda|z|^2}$, $(**)$ yields

$$\varphi \diamond \varphi = e^{\mu|z|^2},$$

where $\mu = 2\lambda(1-\lambda)$. Thus, for $\lambda = \frac{1}{5} + i\frac{2}{5}$, we have $\mu = \frac{16}{25} + i\frac{12}{25}$ and $e^{\mu|z|^2} f(z)$ can**not** be in $L^2(\mathbf{C}, d\mu)$ for **any** $f \neq 0$ in $H^2(\mathbf{C}, d\mu)$.

4. Remarks

There is a considerable space between Theorems 1 and 2 and Theorem 4. It does not seem easy to lift the known much stronger positive results directly over from the setting of pseudodifferential operators. It does seem likely that $(**)$ provides a composition formula for Berezin-Toeplitz operators in a setting substantially larger than those of Theorems 1 and 2. For non-C^∞ φ, ψ or even for general C^∞ φ, ψ, the problem of determining whether there is a $\varphi \diamond \psi$ with $T_\varphi T_\psi = T_{\varphi \diamond \psi}$, as well as the form of $\varphi \diamond \psi$, remains open.

Theorems 1 and 2 can be extended to the natural family of Gaussian measures on $\mathbf{C}^n$ which provide representation spaces for the Heisenberg group [C]. For $d\mu_r(z) = (\frac{r}{\pi})^n e^{-r|z|^2} dv(z)$ with $r > 0$ and $H^2(\mathbf{C}^n, d\mu_r)$ as before, we have Bergman kernels

$$K_r(w, z) = e^{rw \cdot z}$$

and Berezin-Toeplitz operators on $H^2(\mathbf{C}^n, d\mu_r)$

$$(T_\varphi^{(r)} f)(z) = \int_{\mathbf{C}^n} e^{rz \cdot w} \varphi(w)\, f(w)\, d\mu_r(w).$$

Then minor modifications yield

Theorem 1′. *For* φ, ψ *in* $B_a(\mathbf{C}^n), \varphi \diamond_r \psi$ *is also in* $B_a(\mathbf{C}^n)$ *for*

$$(\dagger) \qquad \varphi \diamond_r \psi = \sum_k (\tfrac{-1}{r})^{|k|} \tfrac{1}{k!} (\partial^k \varphi)\ (\overline{\partial}^k \psi)$$

and $T_\varphi^{(r)} T_\psi^{(r)} = T_{\varphi \diamond_r \psi}^{(r)}$. *The series in* $(\dagger)$ *converges uniformly and absolutely. Moreover, for* $r > 1$

$$\| \varphi \diamond_r \psi - \sum_{|k| \leq K} (\tfrac{-1}{r})^{|k|} \tfrac{1}{k!} (\partial^k \varphi)(\overline{\partial}^k \psi) \|_\infty \leq \frac{1}{r^{K+1}} C(\varphi, \psi, K)$$

for $C(\varphi, \psi, K)$ *a constant independent of* r.

Theorem 2′. *For* φ, ψ *polynomials in* $(z_1, ..., z_n,\ \overline{z}_1, ..., \overline{z}_n)$, *we have* $T_\varphi^{(r)} T_\psi^{(r)}$ *defined on a dense domain consisting of linear combinations of functions of the form* $\{p(z)e^{z \cdot a} : a \in \mathbf{C}^n$ *and* $p(z)$ *polynomial in* $(z_1, ..., z_n)\}$. *On this domain*

$$T_\varphi^{(r)} T_\psi^{(r)} = T_{\varphi \diamond_r \psi}^{(r)}$$

for $\varphi \diamond_r \psi$ *given by* $(\dagger)$ *and* $\varphi \diamond_r \psi$ *is polynomial in the* $z_j, \overline{z}_j$.

While Theorems 1 and 2 provide some basis for optimism about the development of a reasonably extensive Berezin-Toeplitz calculus on $\mathbf{C}^n$, the situation is considerably less promising on the classical Bergman space of the disc, $H^2(\mathbf{D}, \frac{dA}{\pi})$, where $\mathbf{D} = \{z \in \mathbf{C} : \ |z| < 1\}$ and $\frac{dA}{\pi}$ is normalized Lebesgue area measure. In this case, the Bergman kernel function is just $K(z,w) = (1 - z\overline{w})^{-2}$ and the Berezin-Toeplitz operator T_φ on $H^2(\mathbf{D}, \frac{dA}{\pi})$ is given by

$$(T_\varphi f)(z) = \int_{\mathbf{D}} K(z,w)\, \varphi(w)\, f(w) \frac{dA(w)}{\pi}.$$

Direct calculation shows, first, that

$$T_z T_{\overline{z}} = T_{1+\log|z|^2}.$$

Moreover,

$$T_{z^2} T_{\overline{z}^2} = T_{1+2\log|z|^2} + P_0$$

where $P_0 f = \int_{\mathbf{D}} f(z) \frac{dA(z)}{\pi}$ and $P_0 \neq T_\varphi$ for $\underline{\text{any}}$ φ. For asymptotic results on composition of Berezin-Toeplitz operators on $H^2(\mathbf{D}, \frac{dA}{\pi})$ see [KL].

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