PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 11, Pages 3253–3263
S 0002-9939(01)05924-X
Article electronically published on April 9, 2001

SPECTRAL SYNTHESIS FOR $A(G)$ AND SUBSPACES OF $VN(G)$

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(Communicated by Dale Alspach)

Abstract. Let G be a locally compact group, $A(G)$ the Fourier algebra of G and $VN(G)$ the von Neumann algebra generated by the left regular representation of G. We introduce the notion of X-spectral set and X-Ditkin set when X is an $A(G)$-invariant linear subspace of $VN(G)$, thus providing a unified approach to both spectral and Ditkin sets and their local variants. Among other things, we prove results on unions of X-spectral sets and X-Ditkin sets, and an injection theorem for X-spectral sets.

Introduction

Let G be a locally compact group and $A(G)$ the Fourier algebra of G. Recall that, when G is abelian, $A(G)$ is isometrically isomorphic (by means of the Fourier transform) to $L^1(\widehat{G})$, the L^1-algebra of the dual group $\widehat{G}$ of G. $A(G)$ is a regular commutative Banach algebra with spectrum $\Delta(A(G)) = G$. Thus, associated to every closed subset E of G, is a largest and a smallest ideal, $I(E)$ and $J(E)$, of $A(G)$ with zero set equal to E. More precisely,

$$I(E) = \{u \in A(G) : u(x) = 0 \text{ for all } x \in E\}$$

and

$$J(E) = \{u \in A(G) \cap C_c(G) : u \,\text{vanishes on a neighbourhood of}\, E\}.$$

Then $J(E) \subseteq I \subseteq I(E)$ for every ideal I of $A(G)$ with zero set E. E is called a *spectral set* or *set of synthesis* if $I(E) = \overline{J(E)}$, and E is said to be a *Ditkin set* if $u \in \overline{uJ(E)}$ for every $u \in I(E)$. Of course, every Ditkin set is a spectral set. In addition, there are what might be referred to as local variants of these notions. Roughly speaking, they arise by replacing $I(E)$ by $I(E) \cap C_c(G)$.

Since Malliavin's [15] famous discovery that, given any non-discrete locally compact abelian group G, there is a closed subset of G which fails to be a (local) spectral set for $A(G)$, there has been much effort on producing sets of synthesis and Ditkin sets. Specifically, so-called injection and projection theorems for spectral sets and Ditkin sets as well as results about unions of such sets have been established.

Received by the editors June 19, 1999 and, in revised form, March 10, 2000.
2000 *Mathematics Subject Classification.* Primary 43A45, 43A46, 43A30, 22D15.
Supported by NATO collaborative research grant CRG 940184. The first author has also been supported by a travel grant from the German Research Foundation (DFG), and the second author is also supported by NSERC grant A7679.

Now, let $VN(G)$ denote the von Neumann algebra generated by the left regular representation of G on $L^2(G)$. Then $VN(G)$ is, in a natural manner, the Banach space dual of $A(G)$ and an $A(G)$-module. For a closed subset E of G, the property that E be a spectral or Ditkin set (respectively, local spectral set or local Ditkin set) can be expressed in terms of $VN(G)$ (respectively, the set of operators in $VN(G)$ with compact support). These observations lead us to consider arbitrary $A(G)$-invariant linear subspaces X of $VN(G)$ and to introduce the notions of X-spectral set and X-Ditkin set, thereby in particular providing a unified approach to both spectral and Ditkin sets and their local variants.

The main results of Section 2 then say that if E and F are closed subsets of G such that $E \cap F$ is X-Ditkin, then $E \cup F$ is an X-spectral set if and only if E and F are X-spectral sets (Theorem 2.9) and analogously for X-Ditkin sets (Theorem 2.10). In Section 3 we concentrate on establishing an injection theorem for X-spectral sets. Thus let H be a closed subgroup of G and X an $A(G)$-invariant linear subspace of $VN(G)$. The adjoint of the restriction map from $A(G)$ onto $A(H)$ assigns to X an appropriate $A(H)$-invariant linear subspace X_H of $VN(H)$. Then a closed subset E of H is an X-spectral set for $A(G)$ if and only if it is an X_H-spectral set for $A(H)$ (Theorem 3.4). Our results extend the existing ones, and some of our proofs in the general case are even more transparent.

1. Preliminaries

Let G be a locally compact group, $L^1(G)$ the convolution algebra of integrable functions on G and $C^*(G)$ the enveloping C^*-algebra of $L^1(G)$. The *Fourier-Stieltjes algebra* and the *Fourier algebra* of G, $B(G)$ and $A(G)$, have been introduced by Eymard [7]. Let $P(G)$ denote the set of all continuous positive definite functions on G, and let $P^1(G) = \{u \in P(G) : u(e) = 1\}$. Then $B(G)$ is the linear span of $P(G)$ and can be identified with the dual of $C^*(G)$ by means of the pairing $\langle u, f\rangle = \int f(x)u(x)dx$, for $f \in L^1(G)$ and $u \in B(G)$ [7, p. 192]. With pointwise multiplication and the dual norm, $B(G)$ is a commutative semisimple Banach algebra [7, Proposition 2.16].

The closed ideal $A(G)$ of $B(G)$ generated by all compactly supported functions in $B(G)$ turns out to be just the set of coefficients of the left regular representation ρ of G on $L^2(G)$ [7]. That is, $u \in A(G)$ if and only if there are f and g in $L^2(G)$ so that $u(x) = \langle \rho(x)f, g\rangle$ for all $x \in G$. The spectrum of $A(G)$ can be identified with G (point evaluations of functions in $A(G)$) [7, Théorème 3.34], and $A(G)$ is regular in the sense that given any compact subset C of G and closed subset E of G such that $C \cap E = \emptyset$, there exists $u \in A(G) \cap C_c(G)$ such that $u(x) = 1$ for all $x \in C$ and $u(y) = 0$ for all $y \in E$ [7, Lemme 3.2].

Let $VN(G)$ denote the closure in the weak operator topology of the linear span of $\{\rho(x) : x \in G\}$ in $\mathcal{B}(L^2(G))$, the algebra of bounded linear operators on $L^2(G)$. Then $A(G)$ is the unique predual of the von Neumann algebra $VN(G)$ [7, Théorème 3.10], and for $T \in VN(G)$ and $u \in A(G)$, we write $\langle T, u\rangle$ for the value of T at u. There is a natural action of $A(G)$ on $VN(G)$ given by

$$\langle v \cdot T, u\rangle = \langle T, vu\rangle , \ T \in VN(G), u, v \in A(G).$$

We now have to introduce various $A(G)$-invariant subspaces of $VN(G)$. To start with, recall from [7, Chapitre 4] that for $T \in VN(G)$, the support of T, $\operatorname{supp} T$, is the closed subset of G consisting of all $x \in G$ such that $\rho(x)$ is the weak*-limit of some net $(u_\alpha \cdot T)_\alpha, u_\alpha \in A(G)$. Then $\operatorname{supp}(u \cdot T) \subseteq \operatorname{supp} u \cap \operatorname{supp} T$

[7, Proposition 4.8]. In [9], $UC(\widehat{G})$ was defined to be the closed linear span of $\{u \cdot T : T \in VN(G), u \in A(G)\}$. When G is abelian, $UC(\widehat{G})$ is precisely the C^*-algebra of bounded uniformly continuous functions on the dual group $\widehat{G}$ of G (whence the notation in the general case). As noted by Herz, $UC(\widehat{G})$ is the norm closure of $UC_c(\widehat{G})$, the set of operators in $VN(G)$ with compact support. In particular, $UC(\widehat{G})$ is a C^*-subalgebra of $VN(G)$.

Moreover, let $C^*_\delta(G)$ be the C^*-subalgebra of $VN(G)$ generated by all operators $\rho(x), x \in G$, and let $C^*_\rho(G)$ be the norm closure of $\{\rho(f) : f \in L^1(G)\} \subseteq VN(G)$.

The collection of operators T in $VN(G)$ for which the set $\{u \cdot T : u \in P^1(G) \cap A(G)\}$ is relatively norm compact (weakly compact) is denoted $AP(\widehat{G})$ ($WAP(\widehat{G})$). Then both $AP(\widehat{G})$ and $WAP(\widehat{G})$ are closed, $A(G)$-invariant subspaces of $VN(G)$. When G is abelian, $AP(\widehat{G}) = C^*_\delta(G)$ and $WAP(\widehat{G})$ are the spaces of continuous almost periodic and continuous weakly almost periodic functions on $\widehat{G}$, respectively. In general, we have the following inclusions:

(i) $C^*_\delta(G) \subseteq AP(\widehat{G}) \subseteq WAP(\widehat{G})$;

(ii) $C^*_\delta(G) \cup C^*_\rho(G) \subseteq UC(\widehat{G})$;

(iii) If G is discrete, then $UC(\widehat{G}) \subseteq AP(\widehat{G})$;

(iv) If G is amenable, then $WAP(\widehat{G}) \subseteq UC(\widehat{G})$ (see [13, 18, 4] for details and problems related to these inclusions).

2. Spectral sets, Ditkin sets and their unions

To begin with, we recall from [3, 5, 11, 14] the local variants of the notions of set of synthesis and Ditkin set. A closed subset E of G is said to be a *local spectral set* or *set of local synthesis* if $I(E) \cap C_c(G) \subseteq \overline{J(E)}$ or, equivalently, if $\langle T, u\rangle = 0$ for every $u \in I(E) \cap C_c(G)$ whenever $T \in VN(G)$ is such that $\operatorname{supp} T \subseteq E$. E is called a *local Ditkin set* if $u \in \overline{uJ(E)}$ for each $u \in I(E) \cap C_c(G)$. Clearly, every local Ditkin set is a local spectral set.

Lemma 2.1. *Let E be an open and closed subset of G. Then E is of local synthesis. If, in addition, E has the property that $u \in \overline{uA(G)}$ for each $u \in I(E)$, then E is of synthesis.*

Proof. Let $T \in VN(G)$ such that $\operatorname{supp} T \subseteq E$, and let $u \in I(E) \cap C_c(G)$. Choose $v \in A(G)$ such that $v = 1$ on $\operatorname{supp} u$. Since E is open, $\operatorname{supp} u \cap \operatorname{supp} T = \emptyset$ and hence $u \cdot T = 0$. Thus

$$\langle T, u\rangle = \langle T, uv\rangle = \langle u \cdot T, v\rangle = 0.$$

Now suppose that there exists a net $(v_\alpha)_\alpha$ in $A(G)$ such that $uv_\alpha \to u$. Then, as in the previous case $u \cdot T = 0$ and hence

$$\langle T, u\rangle = \lim_\alpha \langle T, uv_\alpha\rangle = \lim_\alpha \langle u \cdot T, v_\alpha\rangle = 0,$$

as required. □

For $A(G)$ (more generally, any semisimple regular commutative Banach algebra A), it is customary to say that spectral synthesis (respectively, local spectral synthesis) holds for A whenever every closed subset of the spectrum $\Delta(A)$ is a set of synthesis (respectively, set of local synthesis).

Proposition 2.2. *Let G be an arbitrary locally compact group. Then*

(i) *Local spectral synthesis holds for $A(G)$ if and only if G is discrete.*

(ii) *Spectral synthesis holds for $A(G)$ if and only if G is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.*

Proof. Suppose first that local spectral synthesis holds for $A(G)$. Using the fact that this property is inherited by quotient groups and by closed subgroups, it has already been shown that G must be totally disconnected (see [14] and [8]). Fix a compact open subgroup K of G and suppose that K is infinite. Then, by a theorem of Zelmanov [21, Theorem 2], K contains an infinite abelian (closed) subgroup H. Now, local spectral synthesis, and hence spectral synthesis, holds for $A(H)$, contradicting Malliavin's theorem [15]. Thus K is finite, whence G is discrete.

The converse follows from Lemma 2.1.

For (ii), notice first that, if synthesis holds for G, then $u \in \overline{uA(G)}$ for each $u \in A(G)$. Indeed, denoting by E the zero set of u , we have $Z(uA(G)) = E$, whence $\overline{uA(G)} = I(E)$. Again, the reverse is a consequence of Lemma 2.1. □

We proceed by introducing the notions that are fundamental to our investigation.

Definition 2.3. Let X be an $A(G)$-invariant linear subspace of $VN(G)$. A closed subset E of G is called an *X-spectral set* or set of *X-synthesis* if $T \in X$ and $\operatorname{supp} T \subseteq E$ implies that $T \in I(E)^{\perp}$.

E is called an *X-Ditkin set* for $A(G)$ if for every $T \in X$ and $u \in I(E)$ there exists a net $(v_\alpha)_\alpha$ in $J(E)$ such that

$$\langle v_\alpha \cdot T, u\rangle \;\to\; \langle T, u\rangle.$$

Remark 2.4. (a) We shall frequently use the following simple fact. Suppose that E is X-Ditkin. Then, given $T \in X$ and $u \in I(E)$, there exists $v \in J(E)$ such that $\langle T, u\rangle = \langle T, vu\rangle$. To see this, choose $v = 0$ when $\langle T, u\rangle = 0$, and if $\langle T, u\rangle \neq 0$, notice that $\{\langle T, uv\rangle : v \in J(E)\} = \mathbb{C}$.

(b) Every X-Ditkin set E is an X-spectral set. Indeed, if $T \in X$ such that $\operatorname{supp} T \subseteq E, u \in I(E)$ and $v \in J(E)$ such that $\langle T, u\rangle = \langle T, vu\rangle$, then $\langle T, u\rangle = 0$ since $\operatorname{supp} T \subseteq E$ and $vu \in J(E)$ implies that $\langle T, vu\rangle = 0$.

Lemma 2.5. *Let E be a closed subset of G. Then*

(i) *E is of local synthesis if and only if E is of $UC_c(\widehat{G})$-synthesis.*

(ii) *E is of synthesis if and only if E is of $VN(G)$-synthesis.*

Proof. We show (i), the proof of (ii) being similar (in fact, easier).

Suppose first that E is of local synthesis, and let $T \in UC_c(\widehat{G})$ such that $\operatorname{supp} T \subseteq E$. Choose $v \in A(G) \cap C_c(G)$ such that $v = 1$ on some open neighbourhood V of $\operatorname{supp} T$. Then $u - uv = 0$ on V for all $u \in A(G)$, and this implies $v \cdot T - T = 0$ (see [7, Proposition 4.8]). Now, since $\operatorname{supp} T \subseteq E$, we have $\langle T, u\rangle = 0$ for all $u \in I(E) \cap C_c(G)$. It follows that, for $u \in I(E)$,

$$\langle T, u\rangle \;=\; \langle v \cdot T, u\rangle \;=\; \langle T, vu\rangle \;= 0,$$

since $vu \in I(E) \cap C_c(G)$.

Conversely, suppose that E is of $UC_c(\widehat{G})$-synthesis, and let $u \in I(E) \cap C_c(G)$. We have to show that $\langle T, u\rangle = 0$ whenever $T \in VN(G)$ annihilates $J(E)$. Choose $v \in A(G) \cap C_c(G)$ such that $v = 1$ on $\operatorname{supp} u$. Then $v \cdot T \in UC_c(\widehat{G})$, and hence $v \cdot T$ annihilates $I(E)$. Thus $\langle T, u\rangle \;=\; \langle T, vu\rangle \;=\; \langle v \cdot T, u\rangle \;=\; 0.$ □

Lemma 2.6. *Let E be a closed subset of G. Then*
(i) *E is a local Ditkin set if and only if E is $UC_c(\widehat{G})$-Ditkin.*
(ii) *E is a Ditkin set if and only if E is $VN(G)$-Ditkin.*

Proof. Suppose first that E is a Ditkin set (respectively, a local Ditkin set), and let $u \in I(E)$ and $T \in VN(G)$ (respectively, $T \in UC_c(\widehat{G})$). If $T \in UC_c(\widehat{G})$, then choose $v \in A(G) \cap C_c(G)$ such that $v \cdot T = T$. Now, by hypothesis, there exists a net $(v_\alpha)_\alpha$ in $J(E)$ such that

$$v_\alpha u \to u \quad \text{and} \quad v_\alpha(uv) \to uv$$

in $A(G)$, respectively. It follows that

$$\langle v_\alpha \cdot T - T, u \rangle = \langle T, v_\alpha u - u \rangle \to 0$$

in the first case, whereas in the second case

$$\langle v_\alpha \cdot T - T, u \rangle = \langle v_\alpha \cdot (v \cdot T) - v \cdot T, u \rangle = \langle T, v_\alpha uv - uv \rangle \to 0.$$

Conversely, suppose that E is $UC_c(\widehat{G})$-Ditkin, and let $T \in VN(G)$ and $u \in I(E) \cap C_c(G))$. Choose $v \in A(G) \cap C_c(G)$ such that $v = 1$ on $\operatorname{supp} u$. Since $v \cdot T \in UC_c(\widehat{G})$, there exists a net $(v_\alpha)_\alpha$ in $J(E)$ such that

$$\langle v_\alpha \cdot (v \cdot T), u\rangle \to \langle v \cdot T, u\rangle,$$

and hence, since $vu = u$,

$$\langle T, v_\alpha u\rangle \to \langle T, u\rangle.$$

Thus $\langle T, u \rangle = 0$ whenever T annihilates $uJ(E)$, as required. The proof that E is Ditkin if it is $VN(G)$-Ditkin is even simpler. ☐

Parts (i) of Lemmas 2.5 and 2.6 were shown earlier in [5, Proposition 7 and Proposition 9]. However, we have included the simple arguments for completeness.

Let $M(G)$ be the algebra of finite regular Borel measures on G, and for $\mu \in M(G)$, let $\rho(\mu)$ denote the left regular representation operator given by

$$\langle \rho(\mu) f, g\rangle = \int_G \langle \rho(x) f, g\rangle d\mu(x), \;\; f, g \in L^2(G).$$

Proposition 2.7. *Let G be any locally compact group. Then every closed subset of G is $\rho(M(G))$-Ditkin.*

Proof. Let E be a closed subset of G, and let $u \in I(E), \mu \in M(G)$ and $\varepsilon > 0$ be given. Let $K = \{x \in G : |u(x)| \geq \varepsilon\}$, then K is compact and $K \cap E = \emptyset$. Since $A(G)$ is regular, there exists $v \in A(G) \cap C_c(G)$ such that $v(x) = 1$ for all $x \in K, 0 \leq v \leq 1$ and $v(x) = 0$ for all x in some neighbourhood of E. Then $v \in J(E)$ and

$$\begin{aligned} |\langle v \cdot \rho(\mu), u\rangle - \langle \rho(\mu), u\rangle| &= \left| \int_G (v(x) - 1) u(x) d\mu(x) \right| \\ &\leq \int_{G \setminus K} |1 - v(x)| \cdot |u(x)| d|\mu|(x) \leq \varepsilon \|\mu\|. \end{aligned}$$

This shows that E is $\rho(M(G))$-Ditkin. ☐

Proposition 2.8. *Let G be a locally compact group such that $u \in \overline{uA(G)}$ for every $u \in A(G)$, and let E be a closed subset of G.*

(i) *If E is a $UC(\widehat{G})$-spectral set, then E is a spectral set.*

(ii) *If E is $UC(\widehat{G})$-Ditkin, then E is a Ditkin set.*

Proof. Choose a net $(u_\alpha)_\alpha$ in $A(G)$ such that $\|u_\alpha u - u\| \to 0$.

Suppose first that E is $UC(\widehat{G})$-spectral, and let $u \in I(E)$ and $T \in VN(G)$ such that $\operatorname{supp} T \subseteq E$. Then, since $u_\alpha \cdot T \in UC(\widehat{G})$ and $\operatorname{supp} u_\alpha \cdot T \subseteq E$, we have that $\langle u_\alpha \cdot T, u\rangle = 0$. Hence $\langle T, u\rangle = 0$ because $\langle T, u_\alpha u\rangle \to \langle T, u\rangle$.

Now, let E be $UC(\widehat{G})$-Ditkin and let $T \in VN(G)$ and $u \in I(E)$. We have to show that there exists $v \in J(E)$ such that $\langle T, u\rangle = \langle T, vu\rangle$. We can assume that $\langle T, u\rangle \neq 0$. Since $u_\alpha \cdot T \in UC(\widehat{G})$, for each α there is a $v_\alpha \in J(E)$ such that

$$\langle u_\alpha \cdot T, u\rangle = \langle u_\alpha \cdot T, v_\alpha u\rangle.$$

Since $\langle T, u_\alpha u\rangle \to \langle T, u\rangle$, $\langle T, u\rangle \neq 0$ and $u_\alpha v_\alpha \in J(E)$, we must have that $\{\langle T, vu\rangle : v \in J(E)\} = \mathbb{C}$, whence $\langle T, u\rangle = \langle T, vu\rangle$ for some $v \in J(E)$. □

In the case where G is amenable, part (i) of Proposition 2.8 has been shown, in a completely different way, in [13, Proposition 7.4].

We now turn to the question of how spectral sets and Ditkin sets behave under forming unions. To start with, it is worthwhile to mention that Atzmon [1] has given an example of a regular commutative Banach algebra A with unit and of two sets of synthesis in $\Delta(A)$, the union of which fails to be of synthesis.

Now, let H be a locally compact abelian group. While the union of two Ditkin sets in $\widehat{H} = \Delta(L^1(H))$ is Ditkin, one of the main unsettled questions (even for $H = \mathbb{Z}$) is whether the union of two spectral sets is again a spectral set (see [3] for a survey on this). In the more general context of Fourier algebras $A(G)$, the current state-of-the-art regarding unions is as follows. Let E and F be closed subsets of G, and assume that $E \cap F$ is Ditkin. Then $E \cup F$ is a spectral set if and only if E and F are spectral sets [20, Theorem 4], and the analogous result holds for Ditkin sets [20, Theorem 1]. Warner points out Reiter's influence in the formulation and proof of Theorem 4. In fact, Reiter [17, p. 557] proved the analogue of Theorem 4 for the case that $E \cap F = \emptyset$. Also, Herz [10, Theorem 6.4] proved one direction of Theorem 4. In the sequel we generalize Theorems 4 and 1 of [20] to X-spectral sets and X-Ditkin sets.

Theorem 2.9. *Let G be a locally compact group and X an $A(G)$-invariant linear subspace of $VN(G)$. Suppose that E_1 and E_2 are closed subsets of G such that $E_1 \cap E_2$ is X-Ditkin. Then $E_1 \cup E_2$ is an X-spectral set if and only if both E_1 and E_2 are X-spectral sets.*

Proof. Suppose first that E_1 and E_2 are X-spectral sets, and let $T \in X$ such that $\operatorname{supp} T \subseteq E_1 \cup E_2$ and $u \in I(E_1 \cup E_2)$. Since $E_1 \cap E_2$ is X-Ditkin, there exists $v \in J(E_1 \cap E_2)$ such that $\langle T, u\rangle = \langle v \cdot T, u\rangle$. Since v has compact support disjoint from $E_1 \cap E_2$, there are compact sets F_1 and F_2 such that

$$\operatorname{supp}(v \cdot T) = F_1 \cup F_2 \quad \text{and} \quad F_j \subseteq E_j \setminus (E_1 \cap E_2)\, (j = 1, 2).$$

Now there exist $v_j \in A(G) \cap C_c(G)$, $j = 1, 2$, such that $v_j = 1$ on a neighbourhood of F_j and $\operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \emptyset$. Then $(v_1 + v_2)v \cdot T = v \cdot T$ since $v_1 + v_2 = 1$ on some neighbourhood of $F_1 \cup F_2$. Moreover,

$$\operatorname{supp}(v_j v) \cdot T \subseteq E_j \quad \text{and} \quad (v_j v) \cdot T \in X \ (j = 1, 2),$$

since X is $A(G)$-invariant. Since E_1 and E_2 are of X-synthesis, it follows that $\langle (v_j v) \cdot T, u\rangle = 0$ for $j = 1, 2$, and hence $\langle v \cdot T, u\rangle = 0$, as was to be shown.

Conversely, suppose that $E_1 \cup E_2$ is of X-synthesis, and let $T \in X$ such that $\operatorname{supp} T \subseteq E_1$ and $u \in I(E_1)$ be given. As above, since $E_1 \cap E_2$ is X-Ditkin, $\langle T, u\rangle = \langle T, vu\rangle$ for some $v \in J(E_1 \cap E_2)$. Since $\operatorname{supp}(v \cdot T)$ is a compact set contained in $E_1 \setminus E_2$, there exists $w \in A(G)$ so that $w = 1$ on a compact neighbourhood of $\operatorname{supp}(v \cdot T)$ and $w = 0$ on E_2. It follows that

$$wu \in I(E_1 \cup E_2) \quad \text{and} \quad v \cdot T = (wv) \cdot T.$$

Now, since $v \cdot T \in X, \operatorname{supp}(v \cdot T) \subseteq E_1 \cup E_2$ and $E_1 \cup E_2$ is of X-synthesis,

$$\langle T, u\rangle = \langle v \cdot T, u\rangle = \langle v \cdot T, wu\rangle = 0.$$

This shows that E_1 is an X-spectral set. In the same way it is shown that E_2 is an X-spectral set. □

Theorem 2.10. *Let G and X be as in Theorem 2.9, and let E and F be closed subsets of G such that $E \cap F$ is an X-Ditkin set. Then $E \cup F$ is an X-Ditkin set if and only if both E and F are X-Ditkin sets.*

Proof. If E and F are X-Ditkin, then so is $E \cup F$, without assuming that $E \cap F$ is X-Ditkin. Indeed, given $u \in I(E \cup F)$ and $T \in X$, there exist $v \in J(E)$ such that $\langle T, u\rangle = \langle T, uv\rangle$ and then $w \in J(F)$ such that $\langle T, uv\rangle = \langle T, (uv)w\rangle$. Thus $vw \in J(E \cup F)$ and

$$\langle T, u\rangle = \langle T, u(vw)\rangle,$$

as required.

Conversely, suppose that $E \cap F$ and $E \cup F$ are both X-Ditkin, and let $T \in X$ and $u \in I(E)$. Since $E \cap F$ is X-Ditkin, there exists $v \in J(E \cap F)$ such that $\langle T, u\rangle = \langle T, vu\rangle$. Let $C = F \cap \operatorname{supp}(vu)$, a compact set disjoint from E. Thus there exists $w \in A(G) \cap C_c(G)$ such that $w = 0$ on a neighbourhood of E and $w = 1$ on C. So, in particular, $w \in J(E)$. Let $u' = vu - vwu$, then $u' \in I(E \cup F)$ since $w = 1$ on C. Since $E \cup F$ is an X-Ditkin set, there exists $v' \in J(E \cup F)$ such that $\langle T, u'\rangle = \langle T, u'v'\rangle$. It follows that $(v - vw)v' + vw \in J(E)$ and

$$\langle T, u\rangle = \langle T, vu\rangle = \langle T, u'\rangle + \langle T, uvw\rangle$$

$$= \langle T, u'v'\rangle + \langle T, uvw\rangle = \langle T, u((v - vw)v' + vw)\rangle.$$

This shows that E is X-Ditkin, and similarly for F. □

The following corollary strengthens the second statement of Lemma 2.1 and hence complements part (ii) of Proposition 2.2.

Corollary 2.11. *Suppose that $A(G)$ has an approximate identity, and let E be an open and closed subset of G. Then E is a Ditkin set.*

Proof. Since, by hypothesis, the empty set is a Ditkin set for $A(G)$, the claim follows by applying Theorem 2.10 to E and $F = G \setminus E$. □

3. The injection theorem for spectral sets

In this section we are going to prove an injection theorem for X-spectral sets. Regarding the classical injection theorems for L^1-algebras of locally compact abelian groups we refer to [18, Chapter 7, Theorems 3.8 and 4.5].

To prepare for the setting of the injection theorem, let H be a closed subgroup of the locally compact group G, and let

$$r : A(G) \to A(H),\ u \to r(u)$$

be the restriction map. r is norm decreasing [11, Theorem 1a] and surjective. More precisely, given $v \in A(H)$, there exists $u \in A(G)$ such that $r(u) = v$ and $||u||_{A(G)} = ||v||_{A(H)}$ [11, Theorem 1b]; [16, Theorem 4.21]. Thus the adjoint map

$$r^* : VN(H) \to VN(G),\ \langle r^*(S), u\rangle = \langle S, r(u)\rangle,$$

$u \in A(G), S \in VN(H)$, is injective. In what follows we let ρ_G and ρ_H denote the regular representation of G and H, respectively. Let $VN_H(G)$ denote the weak*-closure of the linear span of $\{\rho_G(h) : h \in H\}$. Then $VN_H(G)$ is a von Neumann algebra.

Lemma 3.1. *r^* is a w^*-w^*-continuous isomorphism from $VN(H)$ onto $VN_H(G)$.*

Proof. Clearly, r^* is w^*-w^*-continuous. To see that r^* is a homomorphism, we first observe that if $x \in H$, then $r^*(\rho_H(x)) = \rho_G(x)$. Hence r^* preserves products on $D = \langle \rho_H(x) : x \in H\rangle$. Since multiplication in a von Neumann algebra is separately continuous in the w^*-topology, it follows that $r^*(ST) = r^*(S)r^*(T)$ for any $S, T \in VN(H)$. Now r^* also preserves involution. Indeed, if $T \in D$, then clearly $r^*(T^*) = (r^*(T))^*$. If $T \in VN(H)$ is arbitrary, let $(T_\alpha)_\alpha$ be a net in the linear span of D such that $T_\alpha \to T$ in the w^*-topology. Then $T_\alpha^* \to T^*$ in the w^*-topology, and hence

$$r^*(T^*) = \lim_\alpha r^*(T_\alpha^*) = \lim_\alpha (r^*(T_\alpha))^* = (r^*(T))^*.$$

To see that r^* is surjective, it suffices to show that $X = r^*(VN(H))$ is w^*-closed in $VN(G)$. Since r^* is a $*$-homomorphism of the C^*-algebra $VN(H)$ into the C^*-algebra $VN(G)$, X must be norm-closed. By the open mapping theorem, $X_1 = \{T \in X : \|T\| \leq 1\}$ is contained in $r^*(VN(H)_\delta)$ for some $\delta > 0$, where

$$VN(H)_\delta = \{S \in VN(H) : \|S\| \leq \delta\}.$$

We claim that X_1 is w^*-closed. For that, let $(T_\alpha)_\alpha$ be a net in X_1 such that $T_\alpha \to T$ in the w^*-topology. Then, for each α, there exists $S_\alpha \in VN(H)_\delta$ such that $r^*(S_\alpha) = T_\alpha$. After passing to a subnet if necessary, we can assume that $S_\alpha \to S$ in the w^*-topology for some $S \in VN(H)_\delta$. Since r^* is w^*-w^*-continuous, it follows that $T_\alpha = r^*(S_\alpha) \to r^*(S)$ and hence that $T = r^*(S) \in X$. As $VN(G)_1$ is w^*-closed, $T \in X_1$. Thus X_1 is w^*-closed, and hence X must be w^*-closed by the Krein-Šmulian theorem [6, p. 429, Theorem 7]. $\square$

Lemma 3.2. (i) $r^*(UC_c(\widehat{H})) = UC_c(\widehat{G}) \cap VN_H(G)$.

(ii) $r^*(UC(\widehat{G})) = UC(\widehat{G}) \cap VN_H(G)$.

(iii) $r^*(\rho_H(M(H))) = \rho_G(M(G)) \cap VN_H(G)$.

(iv) $r^*(C_\delta^*(H)) = C_\delta^*(G) \cap VN_H(G)$.

(v) $r^*(AP(\widehat{H})) = AP(\widehat{G}) \cap VN_H(G)$.

(vi) $r^*(WAP(\widehat{H})) = WAP(\widehat{G}) \cap VN_H(G)$.

Proof. (i) Let $T \in UC_c(\widehat{H})$. There exists $w \in A(H) \cap C_c(H)$ such that $w \cdot T = T$. Next, there exists $v \in A(G) \cap C_c(G)$ extending w. In fact, choose $u_1 \in A(G)$ such that $r(u_1) = w$ and $u_2 \in A(G) \cap C_c(G)$ such that $u_2 = 1$ on the compact set $\operatorname{supp} w$. Then $v = u_1 u_2 \in A(G) \cap C_c(G)$ and $r(v) = w$. It follows that, for all $u \in A(G)$,

$$\langle r^*(T), u\rangle \;=\; \langle r^*(w \cdot T), u\rangle \;=\; \langle w \cdot T, r(u)\rangle \;=\; \langle T, r(vu)\rangle$$

$$= \langle r^*(T), vu\rangle \;=\; \langle v \cdot r^*(T), u\rangle,$$

whence $r^*(T) = v \cdot r^*(T) \in UC_c(\widehat{G}) \cap VN_H(G)$ (Lemma 3.1).

Conversely, suppose that $S \in UC_c(\widehat{G}) \cap VN_H(G)$, and let $T \in VN(H)$ such that $r^*(T) = S$ (Lemma 3.1). Since $\operatorname{supp} S$ is compact, there exists $v \in A(G) \cap C_c(G)$ such that $v = 1$ on some open neighbourhood of $\operatorname{supp} S$. Then $v \cdot S = S$ and $r(v) \cdot T \in UC_c(\widehat{H})$. It follows that $r^*(r(v) \cdot T) = S$. Indeed, for all $u \in A(G)$,

$$\langle r^*(r(v) \cdot T), u\rangle = \langle r(v) \cdot T, r(u)\rangle = \langle T, r(vu)\rangle$$

$$= \langle r^*(T), vu\rangle \;=\; \langle v \cdot S, u\rangle \;=\; \langle S, u\rangle.$$

Thus $UC_c(\widehat{G}) \cap VN_H(G) \subseteq r^*(UC_c(\widehat{H}))$.

(ii) r^*, being a $*$-homomorphism of C^*-algebras, is a closed map. Therefore $r^*(UC(\widehat{H}))$ is a closed subalgebra of $VN(G)$ containing $UC_c(\widehat{G}) \cap VN_H(G)$ (by (i)). So (ii) holds.

(iii) Let $\mu \to \tilde{\mu}$ denote the embedding of $M(H)$ into $M(G)$ given by $\langle \tilde{\mu}, \varphi\rangle = \langle \mu, \varphi|H\rangle$ for $\mu \in M(H)$ and $\varphi \in C_0(G)$. Then, for $\mu \in M(H)$ and $u \in A(G)$,

$$\langle r^*(\rho_H(\mu)), \mu\rangle \;=\; \langle \rho_H(\mu), r(u)\rangle \;=\int_H u(t)d\mu(t)$$

$$= \int_G u(x)d\tilde{\mu}(x) = \;\langle \rho_G(\tilde{\mu}), u\rangle.$$

Thus $r^*(\rho_H(\mu)) = \rho_G(\tilde{\mu})$. In addition, $\rho_G(\tilde{\mu}) \in VN_H(G)$ since $\operatorname{supp}\tilde{\mu} = \operatorname{supp}\mu \subseteq H$. This shows $r^*(\rho_H(M(H))) \subseteq \rho_G(M(G)) \cap VN_H(G)$.

Conversely, let $\nu \in M(G)$ be such that $\rho_G(\nu) \in VN_H(G)$. Then (see [7, (4.7)])

$$\operatorname{supp}\nu = \operatorname{supp}\rho_G(\nu) \subseteq H.$$

It follows that $\mu \in M(H)$, the measure induced by ν, satisfies $\tilde{\mu} = \nu$ and hence $r^*(\rho_H(\mu)) = \rho_G(\nu)$. Thus $\rho_G(M(G)) \cap VN_H(G) \subseteq r^*(\rho_H(M(H)))$.

(iv) If $h \in H$, then clearly $r^*(\rho_H(h)) = \rho_G(h)$. Hence $r^*(C^*_\delta(H)) \subseteq C^*_\delta(G) \cap VN_H(G)$. Since r^* has closed range and $r^*(C^*_\delta(H))$ contains $\rho_G(h)$ for all $h \in H$, (iv) follows.

(v) Notice that $r(A(G) \cap P^1(G)) = A(H) \cap P^1(H)$ [11, Addendum to Theorem 1] and $u \cdot r^*(S) = r^*(r(u) \cdot S)$ for every $S \in VN(H)$ and $u \in A(G)$. Thus, if $S \in AP(\widehat{H})$, then

$$\{u \cdot r^*(S) : u \in A(G) \cap P^1(G)\} = r^*(\{v \cdot S : v \in A(H) \cap P^1(H)\})$$

is also relatively norm compact. So $r^*(S) \in AP(\widehat{G}) \cap VN_H(G)$.

Conversely, if $T \in AP(\widehat{G}) \cap VN_H(G)$, then there exists $S \in VN(H)$ such that $r^*(S) = T$. Now, given $v \in A(H) \cap P^1(H)$, there exists $u \in A(G) \cap P^1(G)$ such that $r(u) = v$, and then $r^*(v \cdot S) = u \cdot r^*(S) = u \cdot T$. It follows that

$$\{r^*(v \cdot S) : v \in A(H) \cap P^1(H)\} \subseteq \{u \cdot T : u \in A(G) \cap P^1(G)\},$$

which is relatively compact in the norm topology of $VN(G)$. Since r^* is isometric, $\{v \cdot S : v \in A(H) \cap P^1(H)\}$ is relatively compact in the norm topology on $VN(H)$.

(vi) is proved similarly. $\square$

Remark 3.3. (a) It should be noted that if H is a closed subgroup of G, then $r^*(C^*_{\rho_H}(H)) \subseteq C^*_{\rho_G}(G)$ if and only if H is open [2, Theorem 5.4].

(b) For any $A(G)$-invariant linear subspace X of $VN(G)$ and closed subgroup H of G, let

$$X_H = {r^*}^{-1}(X).$$

Then X_H is an $A(H)$-invariant linear subspace of $VN(H)$. Indeed, if $v \in A(H)$ and $u \in A(G)$ such that $r(u) = v$ and $S \in X_H$, then

$$r^*(v \cdot S) = u \cdot r^*(S) \in X.$$

Hence $v \cdot S = {r^*}^{-1}(u \cdot r^*(S))$.

(c) If $S \in X_H$ and $\operatorname{supp} S \subseteq E$, then $\operatorname{supp} r^*(S) \subseteq E$. In fact, suppose that $x \in \operatorname{supp} r^*(S)$. Then there exists a net $(u_\alpha)_\alpha$ in $A(G)$ such that

$$r^*(r(u_\alpha) \cdot S) = u_\alpha \cdot r^*(S) \to \rho_G(x) = r^*(\rho_H(x)).$$

By Lemma 3.1, $r(u_\alpha) \cdot S \to \rho_H(x)$.

Theorem 3.4 (Injection theorem for X-spectral sets)**.** *Let X be an $A(G)$-invariant linear subspace of $VN(G)$. Let H be a closed subgroup of G and E a closed subset of H. Then E is an X-spectral set for $A(G)$ if and only E is an X_H-spectral set for $A(H)$.*

Proof. Suppose first that E is of X-synthesis, and let $S \in X_H = {r^*}^{-1}(X)$ such that $\operatorname{supp} S \subseteq E$. Then $r^*(S) \in X$ and $\operatorname{supp} r^*(S) \subseteq E$ (Remark 3.3). Then, by hypothesis, for every $u \in I(E)$,

$$0 = \langle r^*(S), u\rangle = \langle S, r(u)\rangle.$$

Since $r(I(E)) = \{w \in A(H) : w|E = 0\}$, it follows that E is of X_H-synthesis.

Conversely, suppose that E is of X_H-synthesis, and let $T \in X$ such that $\operatorname{supp} T \subseteq E$. Since H is a set of synthesis [19, Theorem 3], T annihilates $I(H)$. Thus there exists a unique $S \in X_H$ such that $r^*(S) = T$. Clearly, $\operatorname{supp} S \subseteq E$ also, and hence, by hypothesis, $\langle T, u\rangle = \langle S, r(u)\rangle = 0$ for all $u \in I(E)$, as required. $\square$

Remark 3.5. Applying Theorem 3.4, Lemma 3.2 and Lemma 2.5, we obtain, in particular, injection theorems for spectral sets and local spectral sets. The injection theorem for local spectral sets has previously been shown in [5, Proposition 8], whereas the injection theorem for spectral sets appears to be new.

References

[1] A. Atzmon, *On the union of sets of synthesis and Ditkin's condition in regular Banach algebras*, Bull. Amer. Math. Soc. **2** (1980), 317-320. MR **81b:**43007

[2] M.B. Bekka, E. Kaniuth, A.T. Lau, and G. Schlichting, *Weak*-closedness of subspaces of Fourier-Stieltjes algebras and weak*-continuity of the restriction map*, Trans. Amer. Math. Soc. **350** (1998), 2277-2296. MR **98h:**22004

[3] J. Benedetto, *Spectral synthesis*, Academic Press, 1975. MR **58:**29850b

[4] C. Chou, *Almost periodic operators in VN(G)*, Trans. Amer. Math. Soc. **317** (1990), 229-253. MR **90d:**43004

[5] A. Derighetti, *Quelques observations concernant les ensembles de Ditkin d'un groupe localement compact*, Monatsh. Math. **101** (1986), 95-113. MR **88c:**43006

[6] N. Dunford and J.T. Schwartz, *Linear operators. I*, Interscience Publishers, 1958. MR **22:**8302

[7] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181-236. MR **37:**4208

[8] B. Forrest, *Fourier analysis on coset spaces*, Rocky Mountain J. Math. **28** (1998), 173-190. MR **99h:**43020

[9] E.E. Granirer, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **189** (1974), 372-382. MR **49:**1017

[10] C. Herz, *The spectral theory of bounded functions*, Trans. Amer. Math. Soc. **94** (1960), 181-232. MR **24:**A1627

[11] C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier **23** (1973), 91-123. MR **50:**7956

[12] A.T. Lau, *Uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **251** (1979), 39-59. MR **80m:**43009

[13] A.T. Lau and V. Losert, *The C^*-algebra generated by operators with compact support on a locally compact group*, J. Funct. Anal. **112** (1993), 1-30. MR **94d:**22005

[14] N. Lohoué, *Remarques sur les ensembles de synthèse des algèbre de groupe localement compact*, J. Funct. Anal. **13** (1973), 185-194. MR **50:**14069

[15] P. Malliavin, *Impossibilité de la synthèse spectrale sur les groupes abéliens non compacts*, Inst. Hautes Ét. Sci. Publ. Math. **2** (1959), 61-68. MR **21:**5853

[16] J.R. McMullen, *Extensions of positive definite functions*, Mem. Amer. Math. Soc. **117**, 1972. MR **56:**3573

[17] H. Reiter, *Contributions to harmonic analysis. VI*, Ann. Math.(2) **77** (1963), 552-562. MR **27:**1778

[18] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Clarendon Press, 1968. MR **46:**5933

[19] M. Takesaki and N. Tatsuuma, *Duality and subgroups. II*, J. Funct. Anal. **11** (1972), 184-190. MR **52:**5865

[20] C.R. Warner, *A class of spectral sets*, Proc. Amer. Math. Soc. **57** (1976), 99-102. MR **53:**14025

[21] E.I. Zelmanov, *On periodic compact groups*, Israel J. Math. **77** (1992), 83-95. MR **94e:**20055

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