PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 11, Pages 3133–3141
S 0002-9939(01)05927-5
Article electronically published on April 9, 2001

POLYNOMIALS IN $\mathbb{R}[x, y]$ THAT ARE SUMS OF SQUARES IN $\mathbb{R}(x, y)$

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(Communicated by Lance W. Small)

Abstract. A positive semidefinite polynomial $f \in \mathbb{R}[x, y]$ is said to be $\Sigma(m, n)$ if f is a sum of m squares in $\mathbb{R}(x, y)$, but no fewer, and f is a sum of n squares in $\mathbb{R}[x, y]$, but no fewer. If f is not a sum of polynomial squares, then we set $n = \infty$.

It is known that if $m \leq 2$, then $m = n$. The Motzkin polynomial is known to be $\Sigma(4, \infty)$. We present a family of $\Sigma(3, 4)$ polynomials and a family of $\Sigma(3, \infty)$ polynomials. Thus, a positive semidefinite polynomial in $\mathbb{R}[x, y]$ may be a sum of three rational squares, but not a sum of polynomial squares. This resolves a problem posed by Choi, Lam, Reznick, and Rosenberg.

1. Introduction

A polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ is said to be *positive semidefinite* (psd) if $f(a_1, \ldots, a_d) \geq 0$ for all $(a_1, \ldots, a_d) \in \mathbb{R}^d$. In Artin's solution of Hilbert's 17th problem, he proved that every positive semidefinite polynomial in $\mathbb{R}[x_1, \ldots, x_d]$ is a sum of squares in $\mathbb{R}(x_1, \ldots, x_d)$ [Ar]. In fact, Pfister showed that 2^d squares always suffice [Pf]. It is interesting to study whether such polynomials are sums of squares in $\mathbb{R}[x_1, \ldots, x_d]$, and, if so, how many are needed. Let $f \in \mathbb{R}[x_1, \ldots, x_d]$ be psd.

Definition 1. We say that f is $\Sigma(m, n)$ if

(1) f is a sum of m squares in $\mathbb{R}(x_1, \ldots, x_d)$, but no fewer, and
(2) f is a sum of n squares in $\mathbb{R}[x_1, \ldots, x_d]$, but no fewer.

A polynomial is $\Sigma(m, \infty)$ if (1) holds for m, but f is not a sum of squares in $\mathbb{R}[x_1, \ldots, x_d]$. For which values of $m, n \in \mathbb{Z}^+ \cup \{\infty\}$ do there exist $\Sigma(m, n)$ polynomials in $\mathbb{R}[x_1, \ldots, x_d]$? Several results are known.

Assume $f \in \mathbb{R}[x_1, \ldots, x_d]$ is psd and $\Sigma(m, n)$.

1. $m \leq n$, since $\mathbb{R}[x_1, \ldots, x_d] \subset \mathbb{R}(x_1, \ldots, x_d)$.
2. $m \leq 2^d$ [Pf].
3. (a) $m = 1 \iff n = 1$, since $\mathbb{R}[x_1, \ldots, x_d]$ is a UFD.
   (b) $m = 2 \iff n = 2$ by [CLRR, Theorem A], (a), and 1 above.
   (c) $n = 3 \implies m = 3$ by (a), (b), and 1.

Received by the editors May 19, 1999 and, in revised form, March 8, 2000.
2000 *Mathematics Subject Classification.* Primary 11E25, 12D15.
Key words and phrases. Positive semidefinite polynomial, sum of squares of polynomials.
This work formed part of the second author's dissertation.

4. If $d = 1$, then either f is $\Sigma(1,1)$ (if f is a square), or f is $\Sigma(2,2)$ (if f is not a square) [Lam, Chapter 8].
5. Let $d = 2$.
   (a) $m \leq 4$ by Pfister's inequality ([Pf] or see [La]).
   (b) Hilbert showed that if f is a polynomial of total degree 4, then f is $\Sigma(m,n)$ with $n \leq 3$ [Hi], and then $m = n$ by 3.
   (c) If f has degree 2 in either x or y, then $m \leq 3$ [CEP].
   (d) There exist polynomials which are $\Sigma(4,\infty)$ [CEP]. The Motzkin polynomial $x^2y^4 + x^4y^2 - 3x^2y^2 + 1$ is one such. Thus there are polynomials in two variables which are sums of squares of rational functions, but are not sums of squares of polynomials. This shows that the results in 5b and 5c do not hold for a general polynomial in $\mathbb{R}[x,y]$, and Pfister's inequality is best possible for $d = 2$.
   (e) Christie discovered a family of sixth degree polynomials which are $\Sigma(4,n)$ for some n [Ch]. We will show in Section 5 that $n = 4$, and correct a sign error appearing in [Ch].

Note that the case $d = 1$ has been characterized completely. In the case $d = 2$, however, there are still open questions. Result 3b above indicates that if f is a sum of two rational squares, then f is a sum of two polynomial squares. However, if f is a sum of four rational squares, f need not even be a sum of a finite number of polynomial squares, as the Motzkin polynomial illustrates.

In this paper, we address the case in which f is a sum of three rational squares. We exhibit a family of $\Sigma(3,\infty)$ polynomials, as well as a family of $\Sigma(3,4)$ polynomials. These families provide a negative answer to problem 3 posed in [CLRR, p. 254] for the case $d = 2$:

> If $f \in A = \mathbb{R}[x_1, \ldots, x_d]$ is a sum of three squares in $\mathbb{R}(x_1, \ldots, x_d)$, is f then a sum of (three) squares in A?

This problem also appears in [Ge]. It is pointed out in [Ge, Corollary 2 and the succeeding Remarks] that these examples provide a simple way to construct unimodular $\mathbb{R}[x_1,x_2]$-lattices of rank 3 that are not extended from $\mathbb{R}$, and thus an elementary proof of a result of Knus and Ojanguren [KnO].

It follows easily that the answer to the question of [CLRR] is also negative for $d \geq 2$ by observing that $f(x_1,x_2) \in \mathbb{R}[x_1, \ldots, x_d]$ and that $f(x_1,x_2)$ remains $\Sigma(3,n)$ over $\mathbb{R}[x_1, \ldots, x_d]$. One can also show easily that if $f(x_1,x_2)$ is $\Sigma(3,n)$, then $g(x_1, \ldots x_d) = (x_3 + \ldots + x_d)^2 f(x_1,x_2)$ is $\Sigma(3,n)$ as well.

2. Basic lemmas

Several of the results in this section are special cases of results appearing in [CLR]. See especially Proposition 2.3, Theorem 2.4, and Theorem 3.5 of [CLR].

We begin with three lemmas from linear algebra with some short proofs included for convenience.

Lemma 1. *Let $v_1, \ldots, v_n \in \mathbb{R}^m$ and let $A = [v_1, \ldots, v_n]$ be the $m \times n$ matrix whose j^{th} column is v_j. Let $M = A^TA$ be the $n \times n$ matrix whose (i,j)-entry is $v_i \cdot v_j$. Then the following statements hold:*

(a) *rank* $M = \dim(\mathit{span}\{v_1, \ldots, v_n\})$.
(b) *If* $\det M \neq 0$, *then* $\det M > 0$.

Proof. (a) We have $\ker M = \ker A$ since $Mx = 0$ implies $(Ax)^T(Ax) = 0$, and this implies $Ax = 0$. Therefore,

$$\begin{aligned}\text{rank } M &= \dim(\text{im } M) = n - \dim(\ker M) = n - \dim(\ker A)\\ &= \dim(\text{im } A) = \dim(\text{span}\{v_1, \dots, v_n\}).\end{aligned}$$

(b) If $\det M \neq 0$, then $n = \text{rank } M = \dim(\text{span}\{v_1, \dots, v_n\})$ and so the vectors $v_1, \dots, v_n$ are linearly independent. Since A^TA is symmetric, there is a nonsingular $n \times n$ matrix P such that $P^T(A^TA)P = D$, where $D = \text{diag}(d_1, \dots, d_n)$ is a diagonal matrix. Then $\det M$ and $\det D$ have the same sign since $\det D = \det(P)^2 \det(A^TA)$. Let e_i be the i^{th} standard basis vector. We have

$$d_i = e_i^T D e_i = e_i^T(P^T(A^TA)P)e_i = (APe_i)^T(APe_i) > 0,$$

since APe_i is a nonzero vector (using the facts that $e_i \neq 0, P$ is invertible, and the columns of A are linearly independent). We see that $\det D > 0$. □

Lemma 2. *Let M be an $n \times n$ symmetric matrix that is positive semidefinite of rank r. Then there exist vectors $v_1, \dots, v_n \in \mathbb{R}^r$ such that $M = (v_i \cdot v_j)$.*

Proof. Since M is positive semidefinite, an invertible matrix P exists such that $P^TMP = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$, where $\lambda_1, \dots, \lambda_r > 0$. Thus,

$$\begin{aligned}M &= (P^T)^{-1}\text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)P^{-1}\\ &= \left[\text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_r}, 0 \dots, 0)P^{-1}\right]^T \left[\text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_r}, 0 \dots, 0)P^{-1}\right]\\ &= L^TL,\end{aligned}$$

where L is an $n \times n$ matrix with nonzero entries occurring only in its upper $r \times n$ submatrix. Let $v_1, \dots, v_n$ be the vectors in $\mathbb{R}^r$ corresponding to the first r entries of the columns of L. Then $M = (v_i \cdot v_j)$. □

Lemma 3. *Let M be an $n \times n$ symmetric matrix, and let d_i be either the upper $i \times i$ principal minor for all $i, 1 \leq i \leq n$, or let d_i be the lower $i \times i$ principal minor for all $i, 1 \leq i \leq n$.*

(a) *If $d_i > 0, 1 \leq i \leq n$, then M is positive definite.*
(b) *If $d_1, \dots, d_{n-1} > 0$ and $d_n = 0$, then M is positive semidefinite of rank $n - 1$.*

Proof. See [HK, Theorem 6, p. 328] for a proof. □

The next two lemmas deal with special polynomials $f \in \mathbb{R}[x, y]$ described in the statement of Lemma 4.

Lemma 4. *Let $f(x, y) \in \mathbb{R}[x, y]$ be a polynomial of total degree 6. Assume that f contains no monomials of the form x^i or y^i for $i > 0$. Suppose that $f(x, y) = \sum_{k=1}^t p_k^2(x, y)$ with $p_k \in \mathbb{R}[x, y], t \in \mathbb{Z}^+$. Then each $p_k = a_kxy^2 + (b_kx^2 + c_kx)y + d_k$ for some $a_k, b_k, c_k, d_k \in \mathbb{R}$, and $1 \leq k \leq t$.*

Proof. Since f has total degree 6, each p_k may have total degree at most 3. Write each p_k as

$$p_k = a_ky^3 + b_kxy^2 + c_kx^2y + d_kx^3 + e_ky^2 + f_kxy + g_kx^2 + h_ky + i_kx + j_k.$$

Consider the vectors $A = (a_1, a_2, \ldots, a_t)^T, B = (b_1, b_2, \ldots, b_t)^T, \ldots, J = (j_1, j_2, \ldots, j_t)^T \in \mathbb{R}^t$. The coefficient of y^6 in f is $A \cdot A$ and is zero since y^6 does not appear in f; therefore, each $a_k = 0$. Similarly, each $d_k = 0$. Now

$$p_k = b_k xy^2 + c_k x^2 y + e_k y^2 + f_k xy + g_k x^2 + h_k y + i_k x + j_k.$$

In order to obtain the y^4 term in f, we require either a product of a linear term and a cubic term, or a quadratic and a quadratic. The remaining cubic terms in p_k all contain the factor x, and only one quadratic term (y^2) doesn't contain the factor x. This implies that the coefficient of y^4 in f is $E \cdot E$. Since f doesn't contain the monomial y^4, each $e_k = 0$. Similarly, each $g_k = 0$. Now we have

$$p_k = b_k xy^2 + c_k x^2 y + f_k xy + h_k y + i_k x + j_k.$$

Since the coefficients of x^2 and y^2 in f are both zero, it follows that $H \cdot H = I \cdot I = 0$, so the p_k reduce to

$$p_k = b_k xy^2 + c_k x^2 y + f_k xy + j_k.$$

Relabelling gives the lemma. □

Lemma 4 is a special case of Theorem 8 in [Mo, 217].

Definition 2. Given

$$\begin{aligned} f(x,y) = {} & S_{11} x^2 y^4 + 2S_{12} x^3 y^3 + 2S_{13} x^2 y^3 + 2S_{14} xy^2 + S_{22} x^4 y^2 \\ & + 2S_{23} x^3 y^2 + 2S_{24} x^2 y + S_{33} x^2 y^2 + 2S_{34} xy + S_{44}, \end{aligned}$$

we define the 4×4 matrix $M_f = (S_{ij})$, where $S_{ij} = S_{ji}$.

Remark. If f is a polynomial satisfying the hypotheses of Lemma 4, then

$$M_f = \begin{bmatrix} A \cdot A & A \cdot B & A \cdot C & A \cdot D \\ B \cdot A & B \cdot B & B \cdot C & B \cdot D \\ C \cdot A & C \cdot B & C \cdot C & C \cdot D \\ D \cdot A & D \cdot B & D \cdot C & D \cdot D \end{bmatrix}$$

(after the relabelling occurring in the proof).

Lemma 5. *Let $f \in \mathbb{R}[x, y]$ be a polynomial of the form in Definition 2. Let $r =$ rank M_f.*

(a) *If f is a sum of t polynomial squares, then M_f is positive semidefinite and $r \leq t$.*
(b) *If M_f is positive semidefinite, then f is a sum of r polynomial squares, and no fewer.*

Proof. (a) Since f is a sum of t polynomial squares, Lemma 4 and the Remark imply there exist vectors $A, B, C, D \in \mathbb{R}^t$ such that $M_f = [A, B, C, D]^T [A, B, C, D]$. This shows that M_f is positive semidefinite. By Lemma 1(a), $r = \text{rank } M_f = \dim(\text{span}\{A, B, C, D\}) \leq t$.

(b) Since M_f is a 4×4 positive semidefinite matrix of rank r, Lemma 2 implies there exist vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^r$ such that $M_f = (v_i \cdot v_j)$. Let $v_i = (e_{i1}, \ldots, e_{ir})^T$, $1 \leq i \leq 4$. Then

$$f = \sum_{k=1}^{r} \left[e_{1k} xy^2 + (e_{2k} x^2 + e_{3k} x) y + e_{4k} \right]^2,$$

since $v_i \cdot v_j$ is the (i, j) entry of M_f. (See Definition 2 and the Remark.) If f were a sum of fewer than r polynomial squares, then part (a) would be contradicted. $\square$

3. Families of $\Sigma(3,4)$ and $\Sigma(3,\infty)$ polynomials in $\mathbb{R}[x,y]$

Theorem 1. *The family of polynomials*

$$\begin{aligned}P_\beta(x,y) &= (9\beta^2+1)(\beta^2+1)^2x^2y^4 + 2\beta(9\beta^2-7)(\beta^2+1)x^3y^3 \\ &\quad + \left[(9\beta^2+1)(\beta^2+1)x^4 + 8(3\beta^2+1)x^2\right]y^2 + 1\end{aligned}$$

is $\Sigma(3,4)$ *for all* $\beta \in \mathbb{R}, \beta^2 \neq \dfrac{1}{15}$, *and is* $\Sigma(3,3)$ *for* $\beta^2 = \dfrac{1}{15}$.

Proof. A straightforward calculation shows that

$$P_\beta(x,y) = \frac{F_1^2}{(x^2+1)^2} + \frac{F_2^2}{(x^2+1)^2} + \frac{F_3^2}{(x^2+1)^2},$$

where

$$F_1(x,y) = 2(\beta^2+1)x^2y^2 + 4\beta x^3y - (x^2-1),$$

$$F_2(x,y) = \left[(7\beta^2-1)x^3 + (5\beta^2+1)x\right]y^2 + 2\beta\left(x^4-x^2\right)y + 2x,$$

$$F_3(x,y) = \beta\left[(3\beta^2-5)x^3 + (3\beta^2-1)x\right]y^2 + \left[(3\beta^2+1)x^4 + (3\beta^2+5)x^2\right]y.$$

Thus, P_β is a sum of three rational squares. Let $M = M_{P_\beta}$. Then

$$M = \begin{bmatrix} (\beta^2+1)^2(9\beta^2+1) & \beta(9\beta^2-7)(\beta^2+1) & 0 & 0 \\ \beta(9\beta^2-7)(\beta^2+1) & (9\beta^2+1)(\beta^2+1) & 0 & 0 \\ 0 & 0 & 8(3\beta^2+1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\det M = 8(3\beta^2+1)(15\beta^2-1)^2(\beta^2+1)^2$. If $\beta^2 \neq 1/15$, then M is positive definite by Lemma 3(a) and so P_β is a sum of four polynomial squares, and no fewer, by Lemma 5(b). If $\beta^2 = 1/15$, then M is positive semidefinite of rank 3 by Lemma 3(b) (applied to the lower $i \times i$ minors) and so P_β is a sum of three polynomial squares, and no fewer, by Lemma 5(b). Since Result 3b in the Introduction implies P_β could not be a sum of two rational squares, it follows that P_β is $\Sigma(3,4)$ if $\beta^2 \neq 1/15$, and is $\Sigma(3,3)$ if $\beta^2 = 1/15$. $\square$

Here is an expression that yields a representation of P_β as a sum of four polynomial squares if $\beta^2 \neq 1/15$, and a sum of three polynomial squares if $\beta^2 = 1/15$.

$$\begin{aligned}P_\beta(x,y) &= \frac{1}{9\beta^2+1}\left[\beta\left(9\beta^2-7\right)x^2y + \left(\beta^2+1\right)\left(9\beta^2+1\right)xy^2\right]^2 \\ &\quad + \frac{1}{9\beta^2+1}\left[\left(15\beta^2-1\right)x^2y\right]^2 + 8\left(3\beta^2+1\right)x^2y^2 + 1^2.\end{aligned}$$

Example 1. $P_0 = x^2y^4 + x^4y^2 + 8x^2y^2 + 1$ is $\Sigma(3,4)$.

$$P_0 = \frac{1}{(1+x^2)^2}\left([2x^2y^2 - x^2 + 1]^2 + [(x^3-x)y^2 - 2x]^2 + [(x^4+5x^2)y]^2\right)$$

and is clearly a sum of four polynomial squares.

Theorem 2. *The family of polynomials*

$$Q_\beta = \frac{1}{2}x^2y^4 - (2x^3 - \beta x^2)y^3 + \left[(\beta^2+2)x^4 + 4\beta x^3 + (\beta^2+9)x^2 + 2\beta x\right]y^2 - (8\beta x^2 - 4x)y + 4(\beta^2+1)$$

is $\Sigma(3,\infty)$ *for all* β *satisfying* $0 < |\beta| < 2$, *is* $\Sigma(3,4)$ *for all* β *satisfying* $|\beta| > 2$, *and is* $\Sigma(3,3)$ *for* $|\beta| = 0, 2$.

Proof. A calculation shows that

$$Q_\beta = \frac{G_1^2 + G_2^2 + G_3^2}{(x^2+1)^2},$$

where

$$G_1(x,y) = x^2y^2 + \left(\beta x^4 + x^3 + \beta x^2 + x\right)y - 2(x^2-1),$$

$$G_2(x,y) = \frac{1}{2}(x^3 - x)y^2 - \left(x^4 + \beta x^3 + x^2 + \beta x\right)y - 4x,$$

$$G_3(x,y) = \frac{1}{2}(x^3 + x)y^2 - (x^4 + 5x^2)y + 2\beta(x^2+1).$$

Let $M = M_{Q_\beta}$. Then

$$M = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2}\beta & \beta \\ -1 & \beta^2+2 & 2\beta & -4\beta \\ \frac{1}{2}\beta & 2\beta & \beta^2+9 & 2 \\ \beta & -4\beta & 2 & 4(\beta^2+1) \end{bmatrix}$$

and a calculation gives $\det M = 8\beta^2(\beta^2 - 4)$.

If Q_β is a sum of polynomial squares, then Lemma 5(a) implies M is positive semidefinite, and then Lemma 2 implies that M satisfies the hypotheses of Lemma 1. In particular, if $\det M \neq 0$, then $\det M > 0$ by Lemma 1(b). It follows that Q_β is not a sum of polynomial squares if $0 < |\beta| < 2$. If $|\beta| > 2$, then M is positive definite, and thus has rank 4, since the upper 2×2 principal minor of M equals $\beta^2/2$ and the upper 3×3 principal minor of M equals $\beta^4/4$. It follows that Q_β is a sum of four polynomial squares, and no fewer, by Lemma 5(b). If $|\beta| = 0$ or 2, then Lemmas 3(b) and 5(b) imply that Q_β is a sum of three polynomial squares, and no fewer. (For $\beta = 0$, apply Lemma 3(b) to the lower $i \times i$ minors.) As before, Result 3b in the Introduction implies that Q_β is not a sum of two rational squares. This completes the proof of Theorem 2. ☐

Here is an expression that yields a representation of Q_β as a sum of four polynomial squares if $|\beta| > 2$, and a sum of three polynomial squares if $|\beta| = 0, 2$.

$$Q_\beta = \frac{1}{2}\left[xy^2 - 2x^2y + \beta xy + 2\beta\right]^2 + \frac{1}{2\beta^2+4}\left[2\beta x^2y + (\beta^2-2)xy - (2\beta^2+4)\right]^2 + \frac{1}{(\beta^2+2)(3\beta^2+4)}\left[(\beta^3+4\beta)x^2y + (6\beta^2+8)xy\right]^2 + \frac{2\beta^2(\beta^2-4)}{3\beta^2+4}x^4y^2.$$

Example 2. $Q_1 = \frac{1}{2}x^2y^4 - (2x^3 - x^2)y^3 + (3x^4 + 4x^3 + 10x^2 + 2x)y^2 - 4(2x^2 - x)y + 8$ is $\Sigma(3,\infty)$.

$$Q_1 = \frac{1}{(1+x^2)^2}\left(G_1^2 + G_2^2 + G_3^2\right),$$

where

$$G_1 = x^2y^2 + (x^4 + x^3 + x^2 + x)y - 2(x^2 - 1),$$

$$G_2 = \frac{1}{2}(x^3 - x)y^2 - (x^4 + x^3 + x^2 + x)y - 4x,$$

$$G_3 = \frac{1}{2}(x^3 + x)y^2 - (x^4 + 5x^2)y + 2(x^2 + 1).$$

4. More general families

The polynomials P_β and Q_β come from 3-parameter families of polynomials. We specified two of the parameters to simplify the computations; there are potentially many more possibilities for $\Sigma(3,4)$ and $\Sigma(3,\infty)$ polynomials. The families are

$$\begin{aligned} P_{\alpha\beta\gamma} &= (9\beta^2+\gamma^2)(\beta^2+\gamma^2)^2x^2y^4 + 4\alpha\beta(9\beta^2-7\gamma^2)(\beta^2+\gamma^2)x^3y^3 \\ &\quad + 4\alpha^2(9\beta^2+\gamma^2)(\beta^2+\gamma^2)x^4y^2 + 32\alpha^2\gamma^2(3\beta^2+\gamma^2)x^2y^2 + 16\alpha^4\gamma^2, \end{aligned}$$

represented by

$$P_{\alpha\beta\gamma} = \frac{F_1^2 + F_2^2 + F_3^2}{(1+x^2)^2},$$

where

$$\begin{aligned} F_1 &= 2\gamma\left[(\beta^2+\gamma^2)x^2y^2 + 4\alpha\beta x^3y - 2\alpha^2(x^2-1)\right], \\ F_2 &= \left[(\gamma^3 - 7\beta^2\gamma)x^3 - (5\beta^2\gamma + \gamma^3)x\right]y^2 - 4\alpha\beta\gamma(x^4 - x^2)y - 8\alpha^2\gamma x, \\ F_3 &= \beta\left[(3\beta^2 - 5\gamma^2)x^3 + (3\beta^2 - \gamma^2)x\right]y^2 + 2\alpha\left[(3\beta^2+\gamma^2)x^4 + (3\beta^2 + 5\gamma^2)x^2\right]y, \end{aligned}$$

and

$$\begin{aligned} Q_{\alpha\beta\gamma} &= \left(\gamma^2 + \frac{1}{64}\alpha^2\right)x^2y^4 - \left(\alpha\gamma x^3 - \frac{1}{4}\alpha\beta x^2\right)y^3 \\ &\quad + \left[\left(\beta^2 + 4\gamma^2 + \frac{1}{16}\alpha^2\right)x^4 + 8\beta\gamma x^3 + \left(\beta^2 + 4\gamma^2 + \frac{1}{2}\alpha^2\right)x^2 + 4\beta\gamma x\right]y^2 \\ &\quad - (2\alpha\beta x^2 - 2\alpha\gamma x)y + \left(\frac{1}{4}\alpha^2 + 4\beta^2\right), \end{aligned}$$

represented by

$$Q_{\alpha\beta\gamma} = \frac{G_1^2 + G_2^2 + G_3^2}{(1+x^2)^2},$$

where

$$G_1 = \frac{1}{4}\alpha x^2y^2 + (\beta x^4 + 2\gamma x^3 + \beta x^2 + 2\gamma x)y - \frac{1}{2}\alpha(x^2 - 1),$$

$$G_2 = \frac{1}{8}\alpha(x^3 - x)y^2 - \left(2\gamma x^4 + \beta x^3 + 2\gamma x^2 + \beta x\right)y - \alpha x,$$

$$G_3 = \gamma(x^3 + x)y^2 - \frac{1}{4}\alpha(x^4 + 5x^2)y + 2\beta(x^2 + 1).$$

The family P_β was obtained by setting $\alpha = \frac{1}{2}$ and $\gamma = 1$ in $P_{\alpha\beta\gamma}$. The family Q_β was obtained by setting $\alpha = 4$ and $\gamma = \frac{1}{2}$ in $Q_{\alpha\beta\gamma}$. An (α,β,γ) making $M = [A,B,C,D]^T[A,B,C,D]$ positive definite of rank 4 would yield a $\Sigma(3,4)$

polynomial. For example, one could choose $(\alpha, \beta, \gamma) = (8, 0, 0)$ in $Q_{\alpha\beta\gamma}$. To obtain a $\Sigma(3, \infty)$ polynomial, it is sufficient to take (α, β, γ) such that $\det M < 0$.

5. Christie's polynomials

Christie [Ch, Section 6] presents the family of functions

$$F(x, y) = x(x + \mu)^3 y^2 + \left(1 + \frac{1}{4}\nu^3 xy^2\right)^2$$

as $\Sigma(4, n)$ polynomials for certain special values of μ and ν (see below).

Perhaps the simplest way to see that this is in error is to examine his example for $\mu = 3$ and $\nu = 10$. In this case, $F(x, y)$ is not psd:

$$F(-1/200, 1) = \frac{-114921799}{200^4}.$$

According to Christie's development, it seems that the desired family is

$$\begin{aligned} F(x, y) &= x(x + \mu)^3 y^2 + \left(1 - \frac{1}{4}\nu^3 xy^2\right)^2 \\ &= \frac{1}{16}\nu^6 x^2 y^4 + \left[x^4 + 3\mu x^3 + 3\mu^2 x^2 + \left(\mu^3 - \frac{1}{2}\nu^3\right) x\right] y^2 + 1, \end{aligned}$$

where μ and ν are integers satisfying $0 < \mu < \nu$, $3|\mu$, $\mu = (-3)^n\lambda$ for some n, $3 \nmid \lambda$, $\lambda \equiv -\nu (\text{mod } 3)$, and $3\mu(\nu - \mu)$ and $\nu^2 + \nu\mu + \mu^2$ are not integer squares [Ch]. The proof in Christie's paper (with the sign changed) applies to this family. The polynomials $F(x, y)$ are $\Sigma(4, 4)$ since

$$\begin{aligned} F(x, y) = &\left[\frac{1}{2}\sqrt{\mu^3(\nu^3 - \mu^3)} xy^2\right]^2 + \left(x^2 y + \frac{3}{2}\mu xy\right)^2 + \left(\frac{\sqrt{3}}{2}\mu xy\right)^2 \\ &+ \left[\left(\frac{1}{2}\mu^3 - \frac{1}{4}\nu^3\right) xy^2 + 1\right]^2. \end{aligned}$$

Although it is difficult to show that $F(x, y)$ is not a sum of three rational squares, it is easy to show that $F(x, y)$ is not a sum of three polynomial squares by using Lemma 5(b) and the other ideas in this paper.

The sign error in Christie's paper was also noted in the review of his article in Math Reviews (Vol. 54, # 289). Macé constructed many more examples of polynomials $F(x, y)$ that are $\Sigma(4, 4)$ in [Ma] by extending the methods of Christie and filling a gap in one of the proofs of a result in [Ch].

Acknowledgments

We thank Bruce Reznick and the referee for their very helpful comments.

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