PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 11, Pages 3417–3423
S 0002-9939(01)05938-X
Article electronically published on April 2, 2001

INCOMPRESSIBLE SURFACES IN LINK COMPLEMENTS

YING-QING WU

(Communicated by Ronald A. Fintushel)

ABSTRACT. We generalize a theorem of Finkelstein and Moriah and show that if a link L has a $2n$-plat projection satisfying certain conditions, then its complement contains some closed essential surfaces. In most cases these surfaces remain essential after any totally nontrivial surgery on L.

A link L in S^3 has a $2n$-plat projection for some n, as shown in Figure 1, where a box on the i-th row and j-th column consists of 2 vertical strings with an a_{ij} left-hand half twist; in other words, it is a rational tangle of slope $1/a_{ij}$. See for example [BZ]. Let n be the number of boxes in the even rows, so there are $n-1$ boxes in the odd rows. Let m be the number of rows in the diagram. It was shown by Finkelstein and Moriah [FM1], [FM2] that if $n \geq 3$, $m \geq 5$, and if $|a_{ij}| \geq 3$ for all i, j, then the link exterior $E(L) = S^3 - \text{Int}N(L)$ contains some essential planar surfaces, which can be tubed on one side to obtain closed incompressible surfaces in $E(K)$. In this note we will prove a stronger version of this theorem, showing that $E(L)$ contains some essential surfaces if $n \geq 3$, the boxes at the two ends of the odd rows have $|a_{ij}| \geq 3$, and $a_{ij} \neq 0$ for the boxes which are not on the ends of the rows. We allow $a_{ij} = 0$ for boxes at the ends of the even rows, and there is no restriction on m, the number of rows in the diagram. The argument here provides a much simpler proof to the above theorem of Finkelstein and Moriah. In [FM2] that theorem was applied to show that if L is a knot, then all surgeries on L contain essential surfaces. Corollary 2 below generalizes this to the case when L has multiple components, with a mild restriction that each component of L intersects some "allowable" spheres.

We first give some definitions. Let $\alpha = \alpha(a_1, \dots, a_m)$ be an arc running monotonically from the top to the bottom of the $2n$-plat, such that α is disjoint from the boxes, and on the i-th row there are a_i boxes on the left of α. See Figure 1 for the arc $\alpha(1,1,1,2,2)$. The arc α is an *allowable path* if (i) each row has at least one box on each side of α, and (ii) α intersects L at $m+1$ points (so α intersects L once when passing from one row to another). Note that the leftmost allowable path is $\alpha(1, \dots, 1)$, which has on its left one box from each row.

Given an allowable path $\alpha = \alpha(a_1, \dots, a_m)$, we can connect the two ends of α by an arc β disjoint from the projection of L to form a circle, then cap it off by two disks, one on each side of the projection plane, to get a sphere $S = S(a_1, \dots, a_m)$,

Received by the editors February 22, 2000 and, in revised form, March 27, 2000.
1991 *Mathematics Subject Classification.* Primary 57N10, 57M25.
Key words and phrases. Incompressible surfaces, $2n$-plat projections, Dehn surgery.
The author was supported in part by NSF grant #DMS 9802558.

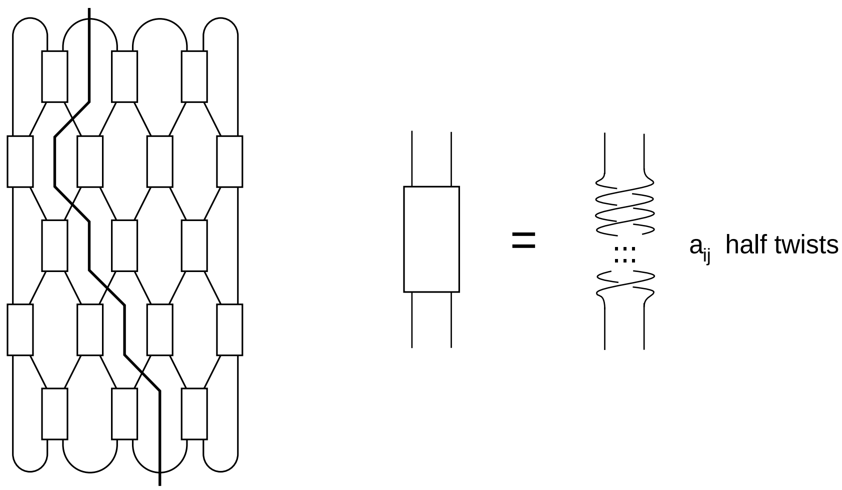

FIGURE 1.

called an *allowable sphere.* S cuts (S^3, L) into two tangles (B, T) and (B', T'), where (B, T) denotes the one on the left hand side of S. Let $P = P(a_1, \dots, a_m)$ be the planar surface $S \cap E(L)$, which cuts $E(L)$ into two pieces $X = X(a_1, \dots, a_m)$ and $X' = X'(a_1, \dots, a_m)$, with $X = B \cap E(L)$ the one on the left of P. Let $F = F(a_1, \dots, a_m)$ be the surface obtained by tubing P on the left hand side; in other words, F is the component of ∂X containing P, pushed slightly into the interior of $E(L)$. Similarly, denote by $F' = F'(a_1, \dots, a_m)$ the surface obtained by tubing P on the right hand side.

Recall that a properly embedded surface F in a 3-manifold M is an *essential surface* if it is incompressible, ∂-incompressible, and is not boundary parallel. We define a surface F on the boundary of M to be *essential* if it is incompressible, $M \neq F \times I$, and there is no compressing disk of ∂M which intersects F at a single essential arc in F. Thus if F is properly embedded in M, then it is essential if and only if after cutting along F the two copies of F are essential in the resulting manifold. A 3-manifold M is *∂-irreducible* if ∂M is incompressible in M. Given a set A in M, denote by $N(A)$ a regular neighborhood of A in M.

Theorem 1. *Suppose L has a $2n$-plat projection such that* (i) $n \geq 3$*;* (ii) $a_{ij} \neq 0$ *for $j \neq 0, n$; and* (iii) $|a_{ij}| \geq 3$ *for i odd and $j = 0$ or $n-1$. Let $S = S(a_1, \dots, a_m)$ be an allowable sphere. Then $E(L)$ is irreducible, and the surfaces $F = F(a_1, \dots, a_m)$ and $F' = F'(a_1, \dots, a_m)$ are essential in $E(L)$.*

Let $L = L_1 \cup \dots \cup L_k$ be a k component link, let $r = (r_1, \dots, r_k)$ be a set of slopes on $\partial N(L)$, with r_i a slope on $\partial N(L_i)$. Then $L(r)$ denotes the r-Dehn surgery on L, which is the manifold obtained by gluing k solid tori $V_1, \dots, V_k$ to $E(L)$ so that each r_i is identified with a meridian disk of V_i. The surgery and the slope r are *totally nontrivial* if no r_i is the meridian slope of L_i.

Corollary 2. *Let L be as in Theorem* 1*. If each component of L intersects some allowable sphere, then $L(r)$ is a Haken manifold for all totally nontrivial r, and the surfaces F and F' in Theorem* 1 *remain incompressible in $L(r)$.*

Remark. (1) It is easy to see that $F = F(a_1, \dots, a_m)$ being incompressible implies that $P = P(a_1, \dots, a_m)$ is an essential planar surface in $E(L)$. With a similar proof to that of Theorem 1 one can show that P is essential even if the condition $|a_{ij}| \geq 3$ in (iii) of Theorem 1 is replaced by $|a_{ij}| \geq 2$. This generalizes the main theorem of [FM1].

(2) When $n \leq 2$, the link is a 2-bridge link, so by [HT] $E(L)$ contains no closed essential surface. Hence the assumption $n \geq 3$ in Theorem 1 is necessary.

(3) By definition of $2n$-plat projection, the number of rows m is odd. If $m = 1$ the link is a composite link, and our assumption implies that it is nonsplit. In this case $E(L)$ is irreducible, and the surfaces in the theorem are swallow-follow tori, which are essential. Therefore the theorem is true for $m = 1$. We may thus assume that $m \geq 3$ in the proof of Theorem 1.

(4) In Corollary 2, each component of L intersects some allowable sphere if and only if no component of L is on the left of $S(1, \dots, 1)$ or on the right of $S(n-2, n-1, \dots, n-2)$, which is equivalent to that a_{i1} and $a_{j,n-1}$ are odd for some odd i, j.

(5) The results remain true if we replace the twist tangles with rational tangles of slopes p_{ij}/a_{ij} with a_{ij} satisfying the conditions in the theorem, or certain kinds of more complicated tangles. However in this case the link diagram would not be in $2n$-plat form.

A p/q *rational tangle* is a pair (B, T), where B is a "pillow case" in $\mathbf{R}^3$ with corner points $(0, \pm 1, \pm 1)$, and T is obtained by taking 2 arcs of slope p/q on ∂B connecting the four conner points of the pillow case, then pushing the interior of the arcs into the interior of B. The xz-plane intersects ∂B in a circle C of slope ∞, called a *vertical circle* on ∂B. Each component of $\partial B - C$ contains two points of ∂T. We need the following result about rational tangles.

Lemma 3. *Suppose (B, T) is a p/q rational tangle, and C a vertical circle on ∂B. Let $X = B - \mathrm{Int}N(T)$, and let P be a component of $(\partial B \cap X) - C$.*

(i) *if $q \geq 1$, then P is incompressible in X;*

(ii) *if $q \geq 2$, then $\partial X - C$ is incompressible in X;*

(iii) *if $q \geq 3$, then any compressing disk of ∂X intersects P at least twice.*

Proof. (ii) Notice that when attaching a 2-handle to X along the curve C, the manifold X_C is the exterior of a 2-bridge link associated to the rational number p/q, which is nontrivial and nonsplit when $q \geq 2$. In particular, ∂X_C is incompressible. If D is a compressing disk of ∂X disjoint from C, then since X is a handlebody of genus 2, we can find a nonseparating compressing disk D' which is still disjoint from C. But then D' would remain a compressing disk in X_C, a contradiction.

(i) If $q \geq 2$ this follows from (ii) and the fact that P is a subsurface of $\partial X - C$ whose complement contains no disk components. If $q = 1$, X is a product $P \times I$, and the result is obvious.

(iii) By (i) P is incompressible, which also implies that $\partial X - P$ is incompressible because any simple loop on $\partial X - P$ is isotopic to one in P. By [Wu, Lemma 2.1] there is no compressing disk of X intersecting P at a single essential arc. □

The following lemma is well-known. The proof is an easy innermost circle outermost arc argument, and will be omitted.

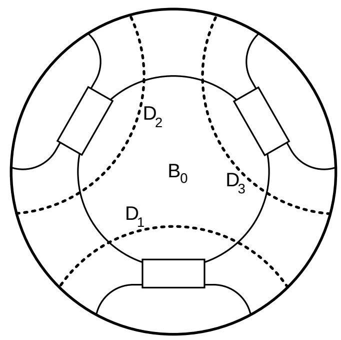

FIGURE 2.

Lemma 4. *Let F be an essential surface in a compact orientable* 3*-manifold M. If $M' = M - \text{Int}N(F)$ is irreducible, and no compressing disk of $\partial M'$ is disjoint from the two copies of F on $\partial M'$, then M is irreducible and ∂-irreducible.*

We now proceed to prove Theorem 1. In the following, we will assume that L is a link as in Theorem 1. By the remark above, we may assume $m \geq 3$.

Lemma 5. *The manifold $X = X(1, \ldots, 1)$ is irreducible and ∂-irreducible.*

Proof. Consider the tangle (B, T) on the left of S. By an isotopy of (B, T) we can untwist the boxes in T which lie on the even rows of the projection of L, so the tangle (B, T) is equivalent to the one shown in Figure 2, where each box corresponds to the first box on an odd row of the projection of L; hence there are $k = (m + 1)/2 \geq 2$ boxes ($k = 3$ in Figure 2). Let $D_1, \ldots, D_k$ be the disks represented by the dotted lines in Figure 2, which cuts (B, T) into $k + 1$ subtangles $(B_0, T_0), \ldots, (B_k, T_k)$, where (B_0, T_0) is the one in the middle, which intersects all the D_i. Let $P_i = D_i \cap X$ be the twice punctured disk in X corresponding to D_i. They cut X into $X_0, \ldots, X_k$, with $X_i = B_i - \text{Int}N(T_i)$ the tangle space of (B_i, T_i).

We want to show that $\bigcup P_i$ is essential in X. Since each (B_i, T_i), $i \geq 1$, is a twist tangle with at least 3 twists, by Lemma 3, the surface P_i is essential in X_i. Now consider X_0. Put $Q = \partial B_0 - \bigcup D_i$. If D is a compressing disk of Q in X_0, then it is a disk in B_0 disjoint from $T_0 \cup (\bigcup D_i)$; but since $T_0 \cup (\bigcup D_i)$ is connected, this would imply that one side of D is disjoint from all D_i, hence ∂D is a trivial curve on Q, which is a contradiction. Therefore Q is incompressible in X_0. Assume there is a disk D in X_0 such that $\partial D \cap (\bigcup P_i)$ has only one component. Since each string of T_0 has ends on different D_i, we see that $\partial D \cap \partial N(T_0) = \emptyset$, so $\partial D \cap (\bigcup P_i)$ is either a proper arc in some D_i which separates the two points of T_0 on D_i, or it is a circle bounding a disk on D_i containing exactly one point of T_0, or ∂D can be isotoped into Q. The first two cases are impossible because then D would be a disk in B_0 disjoint from T_0 and yet each component of $\partial B_0 - \partial D$ contains an odd number of endpoints of T_0. The third case contradicts the incompressibility of Q. This completes the proof that $\bigcup P_i$ is an essential surface in X.

Notice that all X_i are handlebodies, and hence irreducible. Since Q is incompressible in X_0, and by Lemma 3 the surfaces $\partial X_i - P_i \subset \partial X_i - \partial D_i$ are incompressible in X_i for $i \geq 1$, it follows from Lemma 4 that X is irreducible and ∂-irreducible. $\square$

Lemma 6. *The manifold $X = X(a_1, \dots, a_m)$ associated to an allowable sphere $S(a_1, \dots, a_m)$ is irreducible and ∂-irreducible.*

Proof. There is a sequence of allowable spheres $S_1, \dots, S_{k+1}$, such that $S_1 = S(1, \dots, 1)$, $S_{k+1} = S(a_1, \dots, a_m)$, and the noncommon part of S_i, S_{i+1} bounds a single box in the projection of L, that is, $S_i \cup S_{i+1} - \mathrm{Int}(S_i \cap S_{i+1}) = \partial \widetilde{B}_i$ for some twist tangle $(\widetilde{B}_i, \widetilde{T}_i)$ with $a \neq 0$ left hand half-twists. Let (B_i, T_i) be the tangle on the left of S_i, and let $X_i = B_i - \mathrm{Int}N(T_i)$. Similarly, let $\widetilde{X}_i = \widetilde{B}_i - \mathrm{Int}N(\widetilde{T}_i)$. Thus $X = X_{k+1} = X_k \cup_P \widetilde{X}_k$, where $P = X_k \cap \widetilde{X}_k$ is a twice punctured disk. By Lemma 5, X_1 is irreducible and ∂-irreducible, and by induction on the length of the sequence we may assume that X_k is irreducible and ∂-irreducible. Clearly P is incompressible and ∂-incompressible on the X_k side. If $|a| \geq 3$, then, by Lemma 3, P is also incompressible and ∂-incompressible on the $\widetilde{X}_k$ side, hence P is an essential surface in X. Since $\partial \widetilde{X}_k - P$ is also incompressible in $\widetilde{X}_k$, and since X_k and $\widetilde{X}_k$ are irreducible, it follows that $X = X_k \cup_P \widetilde{X}_k$ is irreducible and ∂-irreducible. Also, if $|a| = 1$, then $\widetilde{X}_k$ is a product $P \times I$, so $X_{k+1} \cong X_k$, and the result follows.

It remains to prove the lemma for the case $|a| = 2$. In this case there is a disk D in $\widetilde{X}_k$ which intersects P in a single arc γ, cutting $(\widetilde{X}_k, P)$ into a pair $(A \times I, A \times \partial I)$, where A is an annulus. Thus

$$X = X_k \cup_P \widetilde{X}_k = (X_k \cup_{\gamma \times I} (D \times I)) \cup_{A \times \partial I} (A \times I) \cong X_k \cup_{A \times \partial I} (A \times I).$$

Since a compressing disk of $\partial(A \times I)$ intersects $A \times \partial I$ at least twice, by the same argument as above, one can show that $A \times \partial I$ is essential in X, and X is irreducible and ∂-irreducible. □

Proof of Theorem 1. Let F, F' be the surfaces in the theorem, isotoped slightly to be disjoint from each other. Then $F \cup F'$ cuts $E(L)$ into three parts: The component on the left of F is homeomorphic to X, the one on the right of F' is homeomorphic to X', and the one X'' between F and F' is the union of $P \times I$ and $Q \times I$, where Q is the set of tori in $\partial E(L)$ which intersect ∂P. We have shown in Lemma 6 that X is irreducible and ∂-irreducible, and because of symmetry, so is X'. Now X'' can be cut into $F \times I$ along some (essential) meridional annuli in $Q \times I$, hence by Lemma 4 it is irreducible and ∂-irreducible. Since F and F' have genus at least 2, they are not boundary parallel. It follows that $F \cup F'$ is essential in X, and X is irreducible. □

Proof of Corollary 2. Let $S_1, \dots, S_k$ be a set of disjoint allowable spheres, so that $S_1 = S(1, \dots, 1)$, $S_k = S(n-2, n-1, \dots, n-2)$, and there is only one box of the projection of L between S_i and S_{i+1}. These spheres are similar to those in the proof of Lemma 6, except that they are now mutually disjoint, so the manifold between S_i and S_{i+1} is a product $S^2 \times I$.

Let F_i be the essential surfaces corresponding to S_i, as defined before Theorem 1, isotoped slightly so that they are disjoint from each other. Also, isotope F'_k to be disjoint from F_k. Then the set of $k+1$ surfaces $F_1, F_2, \dots, F_k, F'_k$ cuts $E(L)$ into $k+2$ components $Y_0, \dots, Y_{k+1}$, where Y_0 is the manifold $X(1, \dots, 1)$ on the left of F_1, $Y_{k+1} = X'(n-2, n-1, \dots, n-2)$ is the manifold on the right of F_{k+1}, Y_k is between F_k and F'_k, and for $1 \leq i \leq k-1$, Y_i is between F_i and F_{i+1}. Since all the F_i and F'_k are essential, we see that Y_i are all irreducible and ∂-irreducible. We need to show that the manifold $\widehat{Y}_i$ obtained from Y_i by Dehn filling on its toroidal boundary components (if any), with slopes the corresponding subset of r,

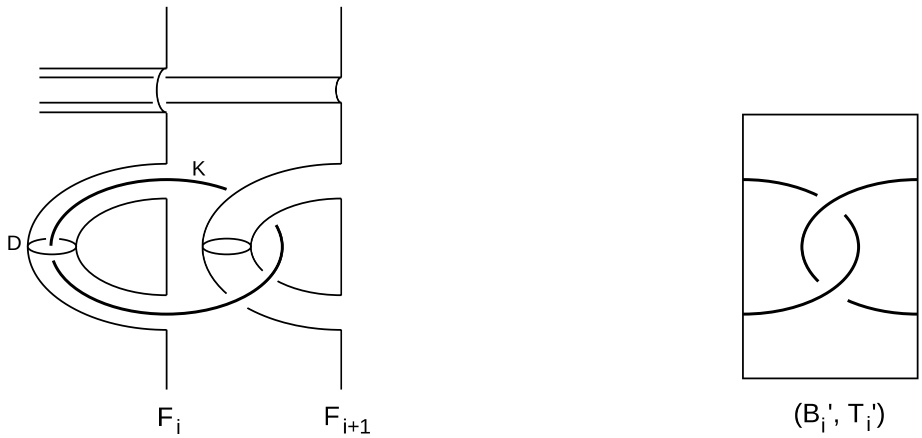

FIGURE 3.

is still irreducible and ∂-irreducible. The result will then follow by gluing the pieces together along F_i and F_k'.

Our assumption implies that Y_0 and Y_{k+1} are disjoint from $\partial E(L)$, hence $\widehat{Y}_i = Y_i$ for $i = 0, k+1$. Now Y_k is a regular neighborhood of $P \cup Q$, where $P = S_k \cap E(L)$, and Q is the set of tori in $\partial E(L)$ which intersect P. Since S is separating, each component Q_j of Q intersects ∂P at least twice, so there are two nonparallel essential annuli in Y_k, each having a boundary component on Q_j with meridional slope. Applying Menasco's theorem [Me] and Scharlemann's theorem [Sch] on each component of Q, we see that after any totally nontrivial Dehn filling on Q the manifold $\widehat{Y}_k$ is still irreducible and ∂-irreducible.

Now assume $1 \leq i \leq k-1$. Let (B_i', T_i') be the twist tangle between S_i and S_{i+1}. Notice that if the twist number a of T_i' is odd, then Y_i contains no component of $\partial E(L)$, so $\widehat{Y}_i = Y_i$ and we are done. If a is even, then the tangle (B_{i+1}, T_{i+1}) on the left of S_{i+1} may contain a loop K intersecting the twist tangle (B_i', T_i'), so Y_i may contain a single component Q of $\partial E(L)$.

Let $Y_i(m)$ be the manifold obtained by the trivial Dehn filling on Q. Then F_i has a compressing disk D in $Y_i(m)$ intersecting the core K of the Dehn filling solid torus only once, so K is not a cable knot in $Y_i(m)$. See Figure 3. It follows from [Sch] that after surgery the manifold $\widehat{Y}_i$ is irreducible. Also, by [CGLS, Theorem 2.4.3] $\widehat{Y}_i$ is ∂-irreducible if the surgery slope r_j on the torus Q intersects the meridian slope m at least twice. Now if r_j intersects m only once, then m is a longitude after the surgery, hence the manifold $\widehat{Y}_i$ is homeomorphic to the one obtained by cutting Y_i along the annulus $D \cap Y_i$, denoted by $\widetilde{Y}_i$. Now there is an annulus A in $B_{i+1} - \mathrm{Int} B_i$ (B_i is the ball on the left of S_i) separating the twist tangle (B_i', T_i') from the other arcs of L, which cuts $\widetilde{Y}_i$ into $\widetilde{X} \cong B_i' - \mathrm{Int} N(T_i')$ and some $G \times I$, where G is a subsurface of F_i with one boundary component. Clearly A is essential in $G \times I$. Since the twist number a is even, our assumption in Theorem 1 implies that $|a| \geq 2$. Hence by Lemma 3 the surface $\partial \widetilde{X} - A$ is incompressible in $\widetilde{X}$, which implies that A is essential in $\widetilde{X}$. It follows that $\widetilde{Y}_i$ is irreducible and ∂-irreducible. $\square$

References

[BZ] G. Burde and H. Zieschang, *Knots*, de Gruyter Studies in Math. 5, 1985. MR **87b:**57004

[CGLS] M. Culler, C. Gordon, J. Luecke and P. Shalen, *Dehn surgery on knots*, Annals Math. **125** (1987), 237–300. MR **88a:**57026; correction MR **89c:**57015

[FM1] E. Finkelstein and Y. Moriah, *Closed incompressible surfaces in knot complements*, Trans. Amer. Math. Soc. **352** (2000), 655–677. MR **2000c:**57007

[FM2] ———, *Tubed incompressible surfaces in knot and link complements*, Topology Appl. **96** (1999), 153–170. CMP 99:16

[HT] A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Inv. Math. **79** (1985), 225–246. MR **86g:**57003

[Me] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1984), 37–44. MR **86b:**57004

[Sch] M. Scharlemann, *Producing reducible 3-manifolds by surgery on a knot*, Topology **29** (1990), 481–500. MR **91i:**57003

[Wu] Y-Q. Wu, *The classification of nonsimple algebraic tangles*, Math. Ann. **304** (1996), 457–480. MR **97b:**57010

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242
E-mail address: wu@math.uiowa.edu