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A NOTE ON EXTENSIONS OF ASYMPTOTIC DENSITY

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Abstract. By a density we mean any extension of the asymptotic density to a finitely additive measure defined on all sets of natural numbers. We consider densities associated to ultrafilters on ω and investigate two additivity properties of such densities. In particular, we show that there is a density ν for which $L_1(\nu)$ is complete.

1. Introduction

We denote the set of natural numbers by ω ($= \{0, 1, 2, \dots\}$), and often regard any $n \in \omega$ as the set $\{0, 1, \dots, n-1\}$. The symbol $\mathcal{P}(\omega)$ stands for the family of all subsets of ω. Recall that the asymptotic density of a set $A \subseteq \omega$, denoted here by $d(A)$, is defined as

$$d(A) = \lim_{n\to\infty} \frac{|A \cap n|}{n},$$

provided this limit exists. Note that, while d is finitely additive, the domain of d is not an algebra. Several authors have considered extensions of asymptotic density to a finitely additive measure defined on a certain algebra of sets; see for instance Buck [2], Maharam [6], and Mekler [7]. In the sequel, any finitely additive ν defined on $\mathcal{P}(\omega)$ and extending d will be called a *density*.

Following Buck [2] and Mekler [7], we shall consider a certain additivity property of densities. Say that a density ν has property $\mathbf{AP}(*)$ if for every increasing sequence $(A_i)_{i\geq 1} \subseteq \mathcal{P}(\omega)$ there is a set $B \in \mathcal{P}(\omega)$ such that

(i) $A_i \subseteq^* B$ for every i;
(ii) $\nu(B) = \lim_i \nu(A_i)$.

Here and below, we write $A \subseteq^* B$ to denote that the set $A \setminus B$ is finite; in this case we also say that A is *almost included* in B.

Suppose now that a σ-algebra $\mathcal{F}$ of subsets of an arbitrary space X is given, and that ν is a *finitely additive* finite measure defined on $\mathcal{F}$. One can consider the following natural weakening of $\mathbf{AP}(*)$: Say that ν has property $\mathbf{AP}(\text{null})$ if for every increasing sequence $(A_i)_{i\geq 1} \subseteq \mathcal{F}$ there is a set $B \in \mathcal{F}$ such that

(i)′ $\nu(A_i \setminus B) = 0$ for every i;
(ii) $\nu(B) = \lim_i \nu(A_i)$.

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Property **AP**(null), which is a weak version of continuity from below, characterizes those finitely additive measures ν (defined on a σ-algebra $\mathcal{F}$), for which the space $L_1(\nu)$ is complete in the "usual" metric; see [5] and [4] (and see [1] for the theory of L_1 spaces over finitely additive measures).

The authors of [5] asked whether there exists a density with property **AP**(null). We give below a positive answer to that question and present some related results, some of which build on ideas from [7].

Every free ultrafilter $\mathcal{U}$ on ω defines, in a natural way, a certain density $\nu_{\mathcal{U}}$ (see section 2). We shall show that $\nu_{\mathcal{U}}$ has property **AP**(null) whenever $\mathcal{U}$ contains a set which is thin enough. Our next result yields a short proof of a result due to Mekler [7]—we show that $\nu_{\mathcal{U}}$ has property **AP**(*) provided $\mathcal{U}$ is a P-point ultrafilter. We also prove that there is a $\mathcal{U}$ giving a density $\nu_{\mathcal{U}}$ which has property **AP**(null) but fails to have **AP**(*). Finally, we explain that, roughly speaking, one cannot retrieve properties of $\mathcal{U}$ from corresponding properties of $\nu_{\mathcal{U}}$. In particular, $\nu_{\mathcal{U}}$ can have property **AP**(*) even if $\mathcal{U}$ is not a P-point.

2. Ultrafilters and densities

Let $\mathcal{U}$ be a free ultrafilter on a space X. Given a bounded function $\alpha : X \to \mathbf{R}$, we write $a = \mathcal{U}\text{-}\lim_x \alpha(x)$ if $\{x : \ |a_x - a| < \varepsilon\} \in \mathcal{U}$ for every positive ε. If $\mathcal{U}$ is a free ultrafilter on ω, then it is routine to check that the formula

$$\nu_{\mathcal{U}}(A) = \mathcal{U}\text{-}\lim_n \frac{|A \cap n|}{n},$$

defines a finitely additive extension of the asymptotic density to the power set of ω (that is, $\nu_{\mathcal{U}}$ is a density in our terminology). The averaging process involved in densities can also be expressed in terms of the Cesàro matrix, as explained in [7].

Recall that a free ultrafilter $\mathcal{U}$ is called a *P-point* if for every sequence (X_i) of elements of $\mathcal{U}$ one can find a set $Y \in \mathcal{U}$ which is almost contained in every X_i. It is well-known that P-points do exist under Martin's axiom; however, their nonexistence is also relatively consistent (see [8, Section VI.4]).

Given an infinite set $X \subseteq \omega$, we write

$$I_n^X = [\max(X \cap n), n) \cap \omega,$$

whenever $n \in X$. Say that a set X is *thin* if

$$\lim_{n \in X} \frac{|I_n^X|}{n} = 1.$$

In other words, a set X is thin if, enumerating X as $(n_k)_k$ in increasing order, we have $\lim_k n_k/n_{k+1} = 0$.

Lemma 1. *If an ultrafilter $\mathcal{U}$ contains a thin set X, then*

$$\nu_{\mathcal{U}}(A) = \mathcal{U}\text{-}\lim_n \frac{|A \cap I_n^X|}{n} = \mathcal{U}\text{-}\lim_n \frac{|A \cap I_n^X|}{|I_n^X|},$$

for every set A.

Proof. The second equation is obvious as X is thin and in $\mathcal{U}$. For the first equation, notice that

$$A \cap I_n^X \subseteq A \cap n \subseteq (A \cap I_n^X) \cup \max(X \cap n).$$

Therefore,

$$\frac{|A \cap I_n^X|}{n} \leq \frac{|A \cap n|}{n} \leq \frac{|A \cap I_n^X|}{n} + \frac{n - |I_n^X|}{n}.$$

Taking the limit along $\mathcal{U}$ and remembering that $X \in \mathcal{U}$ is thin, we obtain the first equation of the lemma.

Theorem 1. *If an ultrafilter $\mathcal{U}$ contains a thin set X, then $\nu_{\mathcal{U}}$ is a density having property* $\mathbf{AP}$(null).

Proof. Having an ultrafilter $\mathcal{U}$ and a thin set $X \in \mathcal{U}$ fixed, we write $\nu = \nu_{\mathcal{U}}$ and $I_n = I_n^X$ for simplicity. By Lemma 1 we have $\nu(A) = \mathcal{U}\text{-}\lim_n \frac{|A \cap I_n|}{n}$ for every A.

To check $\mathbf{AP}$(null) take an increasing sequence $(A_i)_{i \geq 1}$ and put $\alpha = \lim_i \nu(A_i)$. Passing to a subsequence we may assume that $\nu(A_i) \geq \alpha - 1/i$.

Find a decreasing sequence $(X_i)_{i \geq 1}$ of elements of $\mathcal{U}$ such that $X_1 \subseteq X$, $X_i \cap i = \emptyset$, and

$$\left| \frac{|A_i \cap I_n|}{|I_n|} - \nu(A_i) \right| < \frac{1}{i},$$

whenever $n \in X_i$.

Now we define a set B, separately on each segment I_n. Put $B \cap I_n = \emptyset$ for $n \in X \setminus X_1$ and $B \cap I_n = A_i \cap I_n$ for $n \in X_i \setminus X_{i+1}$.

Let $n \in X_i$; then $n \in X_j \setminus X_{j+1}$ for some $j \geq i$. It follows that

$$\alpha - \frac{2}{i} \leq \nu(A_j) - \frac{1}{j} \leq \frac{|B \cap I_n|}{|I_n|} = \frac{|A_j \cap I_n|}{|I_n|} \leq \nu(A_j) + \frac{1}{j} \leq \alpha + \frac{1}{i},$$

and hence for every $n \in X_i$ we have

$$\left| \frac{|B \cap I_n|}{|I_n|} - \alpha \right| < \frac{2}{i},$$

which gives $\nu(B) = \alpha$.

Now we check that $\nu(A_i \setminus B) = 0$. Indeed, if $n \in X_i$, then for some $j \geq i$ we have $B \cap I_n = A_j \cap I_n \supseteq A_i \cap I_n$. Thus $(A_i \setminus B) \cap I_n = \emptyset$ for every $n \in X_i$, and we are done.

We next give a short proof of Mekler's result [7] that a P-point yields a density with $\mathbf{AP}(*)$.

Theorem 2. *If $\mathcal{U}$ is a P-point ultrafilter, then $\nu_{\mathcal{U}}$ is a density having property* $\mathbf{AP}(*)$.

Proof. For an increasing sequence $(A_i)_i$ we take sets X_i belonging to $\mathcal{U}$ and such that

$$\left| \frac{|A_i \cap n|}{n} - \nu(A_i) \right| < \frac{1}{i},$$

whenever $n \in X_i$. As in the proof of Theorem 1, we write $\alpha = \lim_i \nu(A_i)$. Now, since $\mathcal{U}$ is a P-point, there is a set $Y \in \mathcal{U}$ such that $Y \subseteq^* X_i$ for every i. Choose an increasing sequence of numbers $m_i \in \omega$ so that $Y \subseteq X_i \cup m_i$.

We define a set B so that $B \cap [m_i, m_{i+1}) = A_i \cap [m_i, m_{i+1})$ for every i. We have $A_i \subseteq^* B$, since $A_i \setminus B \subseteq m_i$. If $n \in [m_i, m_{i+1})$, then $B \cap n \subseteq A_i \cap n$. If, moreover, $n \in Y$, then $n \in X_i$ so

$$\frac{|B \cap n|}{n} \leq \frac{|A_i \cap n|}{n} < \nu(A_i) + \frac{1}{i} \leq \alpha + \frac{1}{i}.$$

It follows that the inequality above holds for every $n \in Y \setminus m_i$, and therefore $\nu_{\mathcal{U}}(B) \leq \alpha$. On the other hand, the reverse inequality holds because $\nu_{\mathcal{U}}(B) \geq \nu_{\mathcal{U}}(A_i)$ for every i.

Theorem 3. *Suppose that $\mathcal{U}$ is an ultrafilter containing a thin set. Then $\nu_{\mathcal{U}}$ has property $\mathbf{AP}(*)$ if and only if $\mathcal{U}$ is a P-point.*

Proof. One direction is Theorem 2 (and doesn't need thinness). To prove the other direction, fix a thin set X belonging to $\mathcal{U}$; again let I_n stand for I_n^X. Let $(X_m)_m$ be a decreasing sequence in $\mathcal{U}$. For every m write

$$A_m = \bigcup_{k \in X \setminus X_m} I_k.$$

Then $A_m \cap I_n = \emptyset$ for every $n \in X_m$, and Lemma 1 gives $\nu_{\mathcal{U}}(A_m) = 0$.

Since $\nu_{\mathcal{U}}$ has property $\mathbf{AP}(*)$, there is a set A with $\nu_{\mathcal{U}}(A) = 0$, and such that $A_m \subseteq^* A$ for every m. It follows that

$$Y = \left\{ k : \frac{|A \cap I_k|}{|I_k|} < \frac{1}{2} \right\} \in \mathcal{U}.$$

Since A_m is almost contained in A, we have $A_m \subseteq A \cup k_m$ for some $k_m \in X$. Then $Y \subseteq X_m \cup k_m$. Indeed, if $k \in Y \setminus X_m$, then

$$\frac{|A \cap I_k|}{|I_k|} < \frac{1}{2} \quad \text{and} \quad A_m \cap I_k = I_k.$$

Hence $A_m \cap I_k \not\subseteq A \cap I_k$, which means $k \leq k_m$. The proof is complete.

In connection with Theorem 3 it is perhaps worth remarking that, assuming Martin's axiom, one can easily construct a P-point which does not contain any thin set.

Given an arbitrary (infinite) thin set X, it is easy to find a free ultrafilter $\mathcal{U}$ which contains X but is *not* a P-point (take a sequence if distinct ultrafilters containing X and let $\mathcal{U}$ be its cluster point). In view of Theorem 2 and Theorem 3, this remark yields the following.

Corollary 4. *There exists a free ultrafilter $\mathcal{U}$ such that $\nu_{\mathcal{U}}$ has property $\mathbf{AP}(\text{null})$ but not property $\mathbf{AP}(*)$.*

3. Different ultrafilters giving the same density

Extending an idea suggested by Example 1.5 of [7], we shall show that rather dissimilar ultrafilters $\mathcal{U}$ can lead to similar densities or even the same density.

If $\mathcal{U}$ is an ultrafilter on a set X and $g : X \longrightarrow Y$ is any mapping, then $g(\mathcal{U})$ denotes the ultrafilter on Y consisting of those $B \subseteq Y$ for which $g^{-1}(B) \in \mathcal{U}$. Note that

$$g(\mathcal{U})\text{-}\lim_y \alpha(y) = \mathcal{U}\text{-}\lim_x \alpha(g(x)),$$

for every function α from Y into $[0,1]$.

We consider the set $\Delta = \{(n,k) : n < k\}$ and "the standard pairing function" $p : \Delta \longrightarrow \omega$, where

$$p(n,k) = \frac{k(k-1)}{2} + n.$$

Put also $q(k) = p(0,k)$ and denote by $\pi : \Delta \longrightarrow \omega$ the projection onto the second coordinate.

Lemma 2. *If $\mathcal{U}$ is a free ultrafilter on Δ, then $\nu_{p(\mathcal{U})} = \nu_{q\circ\pi(\mathcal{U})}$.*

Proof. (1) Note first that for any $A \subseteq \omega$ and natural numbers Q, P, if $Q \le P \le Q + 2\sqrt{Q}$, then

$$\left| \frac{|A\cap Q|}{Q} - \frac{|A\cap P|}{P} \right| \le \frac{2}{\sqrt{Q}}.$$

Indeed, (1) follows immediately from the following two inequalities:

$$0 \ge \frac{|A\cap Q|}{Q} - \frac{|A\cap P|}{Q} = -\frac{|A\cap (P\setminus Q)|}{Q} \ge -\frac{P-Q}{Q} \ge -\frac{2}{\sqrt{Q}};$$

$$0 \le \frac{|A\cap P|}{Q} - \frac{|A\cap P|}{P} = |A\cap P|(\frac{1}{Q} - \frac{1}{P}) = \frac{|A\cap P|}{P}\cdot\frac{P-Q}{Q} \le \frac{2}{\sqrt{Q}}.$$

(2) For every $n < k$ we have

$$q(k) \le p(n,k) \le q(k) + 2\sqrt{q(k)}.$$

Indeed, the definitions of $p(n,k)$ and $q(k)$ make the first inequality trivial, and they imply that

$$p(n,k) - q(k) = n \le k-1 \le \sqrt{2q(k)} \le 2\sqrt{q(k)},$$

which gives the second inequality.

(3) Using (2) and (1) (where $Q = q(k)$, $P = p(n,k)$), we get for any $A \subseteq \omega$

$$\nu_{p(\mathcal{U})}(A) = p(\mathcal{U})\text{-}\lim_m \frac{|A\cap m|}{m} = \mathcal{U}\text{-}\lim_{(n,k)} \frac{|A\cap p(n,k)|}{p(n,k)}$$

$$= \mathcal{U}\text{-}\lim_{(n,k)} \frac{|A\cap q(k)|}{q(k)} = q\circ\pi(\mathcal{U}) \lim \frac{|A\cap m|}{m} = \nu_{q\circ\pi(\mathcal{U})}(A).$$

Theorem 5. *(a) If there exist P-point ultrafilters on ω, then there is an ultrafilter $\mathcal{V}$ which is not a P-point, and such that $\nu_{\mathcal{V}}$ has property* $\mathbf{AP}(*)$.

(b) There is an ultrafilter $\mathcal{V}$ which does not contain a thin set, and such that $\nu_{\mathcal{V}}$ has property $\mathbf{AP}(\text{null})$ *but not* $\mathbf{AP}(*)$.

Proof. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be ultrafilters on ω and let $\mathcal{U} = \mathcal{U}_1 \otimes \mathcal{U}_2$ denote their product. By definition, $D \in \mathcal{U}_1 \otimes \mathcal{U}_2$ if $\{n : D_{|n} \in \mathcal{U}_2\} \in \mathcal{U}_1$, where $D_{|n}$ is defined to be $\{k : (n,k) \in D\}$. Note that $\Delta \in \mathcal{U}_1 \otimes \mathcal{U}_2$, so we may consider the product ultrafilter as defined on Δ. Note also that $\pi(\mathcal{U}_1 \otimes \mathcal{U}_2) = \mathcal{U}_2$.

To check (a) take a P-point $\mathcal{U}_2$ and arbitrary $\mathcal{U}_1$, and consider $\mathcal{U} = \mathcal{U}_1 \otimes \mathcal{U}_2$. Using Lemma 2, we get

$$\nu_{p(\mathcal{U}_1\otimes\mathcal{U}_2)} = \nu_{q\circ\pi(\mathcal{U}_1\otimes\mathcal{U}_2)} = \nu_{q(\mathcal{U}_2)}.$$

Since $q(\mathcal{U}_2)$ is a P-point, $\nu_{p(\mathcal{U})}$ has property $\mathbf{AP}(*)$. On the other hand, $\mathcal{U}$ and its isomorphic copy $p(\mathcal{U})$ are not P-points, for no element of $\mathcal{U}$ is almost contained in every $\Delta_N = \{(n,k) \in \Delta : n \ge N\}$.

We argue for (b) in a similar manner. Take an ultrafilter $\mathcal{U}_2$ which is not a P-point and contains a thin set. Then $q(\mathcal{U}_2)$ contains a thin set, so $\nu_{q(\mathcal{U}_2)}$ has property $\mathbf{AP}(\text{null})$. On the other hand, $q(\mathcal{U}_2)$ is not a P-point, so $\nu_{q(\mathcal{U}_2)}$ does not have property $\mathbf{AP}(*)$ by Theorem 3. Now $p(\mathcal{U})$ defines the same density but it does not contain a thin set. Indeed, if $A \subseteq \omega$ and $p^{-1}(A) \in \mathcal{U}$, then there are $i, j \in A$ with $i < j < 2i$.

4. Densities without additivity properties

Recall that it is relatively consistent that no density has property $\mathbf{AP}(*)$; see Mekler [7], where Shelah's argument from [8] is suitably adapted. Frankiewicz, Shelah, Zbierski [3] obtained a model of set theory, in which there are no *ccc* P-sets in $\beta\omega \setminus \omega$. Let us note that this result improves Mekler's theorem. Indeed, every density ν defines the unique Radon measure $\widehat{\nu}$ on the compact space $\beta\omega \setminus \omega$. If S denotes the support of $\widehat{\nu}$, then S is clearly *ccc*. If, moreover, ν has property $\mathbf{AP}(*)$ then S is easily seen to be a P-set.

Corollary 4 above shows that no extra axioms are needed to find an ultrafilter $\mathcal{U}$ such that $\nu_{\mathcal{U}}$ fails to have $\mathbf{AP}(*)$. We sketch here another, more constructive, argument for this fact.

(1) First note that there is a sequence (A_i) of subsets of ω such that for every k and $\varepsilon > 0$ the system of inequalities

$$\frac{|A_i \cap n|}{n} < \varepsilon, \quad i = 1, 2, \dots, k, \qquad \frac{|A_{k+1} \cap n|}{n} \geq \frac{1}{2},$$

is satisfied for infinitely many n.

This may be proved by a Baire category argument but, as the referee remarked, it suffices to put $A_i = \{[n_k, n_{k+1}) : \ k \in W_i\}$, where $(n_k)_k$ is an enumeration of a thin set, and $(W_i)_i$ is a partition of ω into infinite sets.

(2) Denote by $\mathcal{B}$ the family of all sets B almost including A_i for every i. Put

$$X_{\varepsilon,i} = \left\{ n : \ \frac{|A_i \cap n|}{n} < \varepsilon \right\}; \qquad Y_B = \left\{ n : \ \frac{|B \cap n|}{n} \geq \frac{1}{2} \right\},$$

for every i and $\varepsilon > 0$, and for every $B \in \mathcal{B}$.

(3) Note that the set

$$X_{\varepsilon,1} \cap X_{\varepsilon,2} \cap \ldots \cap X_{\varepsilon,k} \cap Y_B$$

is infinite for any ε, k, and every $B \in \mathcal{B}$. Since $Y_B \cap Y_C \supseteq Y_{B \cap C}$ for $B, C \in \mathcal{B}$, it follows that the family of all $X_{\varepsilon,i}$ and all Y_B has the strong finite intersection property, and therefore is contained in some free ultrafilter $\mathcal{U}$. Now we have $\nu_{\mathcal{U}}(A_i) = 0$ for every i, and $\nu_{\mathcal{U}}(B) \geq 1/2$ for all $B \in \mathcal{B}$, so $\nu_{\mathcal{U}}$ does not have property $\mathbf{AP}(*)$.

Of course, property $\mathbf{AP}(\text{null})$ is much harder to destroy, and we do not know if this can be done by a suitable modification of the argument above. In response to our question whether there is an ultrafilter giving a density without property $\mathbf{AP}(\text{null})$, David Fremlin presented the following result (in a letter of October, 1999).

Theorem 6 (Fremlin)**.** *Suppose that $\mathcal{U}$ is an ultrafilter on ω such that for every $A \in \mathcal{U}$ there is a $k > 0$ with $A + k \in \mathcal{U}$. Write $\mathcal{V} = g(\mathcal{U})$, where $g : \omega \to \omega$, $g(n) = 2^n$. Then the density $\nu_{\mathcal{V}}$ does not have property* $\mathbf{AP}(\text{null})$.

Proof. Clearly $\nu_{\mathcal{V}} = \nu$, where ν is a density defined by the formula

$$\nu(A) = \mathcal{U}\text{-}\lim_n \frac{|A \cap 2^n|}{2^n}.$$

For every $k \in \omega$ we find a set $I_k \in \mathcal{U}$ such that $I_k \cap (k+1) = \emptyset$ and $|i-j| > k$ whenever $i, j \in I_k$, $i \neq j$. For every k we put

$$A_k = \bigcup_{i \in I_k} \left[2^{i-k-1}, 2^{i-k}\right) \cap \omega.$$

Note that $\nu(A_k) \leq 2^{-k}$. Indeed, if $i \in I_k$, then $A_k \cap [2^{i-k}, 2^i) = \emptyset$, so $|A_k \cap 2^i|/2^{-i} \leq 2^{-k}$.

Now we check that, given a set B with $\nu(B) < 1/4$, we have $\nu(A_k \setminus B) > 0$ for infinitely many k. Let

$$J = \{i : \ |B \cap [2^{i-1}, 2^i)| \leq 2^{i-2}\}.$$

Then $J \in \mathcal{U}$, since otherwise we would have $\nu(B) \geq 1/4$.

It follows from the assumption on $\mathcal{U}$ that $J + k \in \mathcal{U}$ for infinitely many k. Fix a number k with this property, put $K = (J+k) \cap I_k$, and consider any $i \in K$. Since $i \in I_k$ and $i - k \in J$, we have

$$A_k \supseteq [2^{i-k-1}, 2^{i-k}) \cap \omega, \qquad |B \cap [2^{i-k-1}, 2^{i-k})| \leq 2^{i-k-2}.$$

Hence

$$|(A_k \setminus B) \cap 2^i| \geq |(A_k \setminus B) \cap [2^{i-k-1}, 2^{i-k})| \geq 2^{i-k} - 2^{i-k-1} - 2^{i-k-2} = 2^{i-k-2};$$

$$\frac{|(A_k \setminus B) \cap 2^i|}{2^i} \geq 2^{-(k+2)},$$

whenever $i \in K \in \mathcal{U}$. Hence $\nu(A_k \setminus B) > 0$. Therefore, the sequence $A_4, A_4 \cup A_5, \ldots$ witnesses that ν fails to have $\mathbf{AP}$(null).

In order to get a density without property $\mathbf{AP}$(null), it is now sufficent to find an ultrafilter $\mathcal{U}$ such that for every $A \in \mathcal{U}$ there is a $k > 0$ with $A + k \in \mathcal{U}$. D. Fremlin pointed out that the existence of an ultrafilter with this property may be quickly derived form Glazer's theorem on idempotent ultrafilters (see e.g. [9], section 15). This suggested to us the following straightforward argument.

The formula $\phi(\mathcal{X}) = \{A : \ A+1 \in \mathcal{X}\}$ defines a continuous mapping $\phi : \beta\omega \setminus \omega \to \beta\omega \setminus \omega$. By compactness, there is a minimal ϕ–invariant closed and nonempty subset S of $\beta\omega \setminus \omega$. We check that any $\mathcal{U} \in S$ has the required property. Indeed, the orbit $\{\phi^k(\mathcal{U}) : k \geq 1\}$ must be dense in S, since its closure is ϕ–invariant. Therefore, given $A \in \mathcal{U}$, there is a $k \geq 1$ such that the ultrafilter $\phi^k(\mathcal{U})$ is in the closed and open set defined by A. Hence $A \in \phi^k(\mathcal{U})$, so $A + k \in \mathcal{U}$, and we are done.

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