NON-TANGENTIAL LIMITS, FINE LIMITS 
AND THE DIRICHLET INTEGRAL 

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Abstract. Let $B$ denote the unit ball in $\mathbb{R}^n$. This paper characterizes the subsets $E$ of $B$ with the property that $\sup_E h = \sup_B h$ for all harmonic functions $h$ on $B$ which have finite Dirichlet integral. It also examines, in the spirit of a celebrated paper of Brelot and Doob, the associated question of the connection between non-tangential and fine cluster sets of functions on $B$ at points of the boundary.

1. Introduction 

Let $B(x, r)$ denote the open ball of centre $x$ and radius $r$ in Euclidean space $\mathbb{R}^n$ ($n \geq 2$), and let $B = B(0, 1)$. If $A$ is a collection of harmonic functions on $B$, then it is natural to ask which non-empty subsets $E$ of $B$ have the property that 

$$\sup_E h = \sup_B h \quad \text{for all } h \in A.$$ 

In the case where $A = h^\infty$, the collection of all bounded harmonic functions on $B$, it is known (cf. [2]) that (1) holds if and only if $E_{NT} = \partial B$, where $\sigma$ denotes surface area measure on $\partial B$ and $E_{NT}$ is the (Borel) set of points of $\partial B$ which can be approached non-tangentially by a sequence in $E$. In the case where $A = h^1$, the collection of differences of positive harmonic functions on $B$, it has been shown (see [10] and [8]) that (1) holds if and only if 

$$\int_{E(1/2)} |x - y|^{-n} \, dx = +\infty \quad \text{for all } y \in \partial B,$$

where $E(1/2) = \bigcup_{x \in E} B(x, (1 - |x|)/2)$. Below we present the corresponding result when $A = \mathcal{D}$, the collection of all harmonic functions $h$ on $B$ which have finite Dirichlet integral; that is, $\int_B |\nabla h(x)|^2 \, dx < +\infty$. We will use $\mathcal{C}(\cdot)$ to denote Newtonian (if $n \geq 3$) or logarithmic (if $n = 2$) capacity on $\mathbb{R}^n$.

Theorem 1. Let $\emptyset \neq E \subseteq B$ and $A = \mathcal{D}$. Then (1) holds if and only if $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$.

When $n = 2$, Theorem 1 is closely related to a recent result of Stray [13] concerning holomorphic functions in the Dirichlet space. However, our methods are completely different. If $\sigma(E_{NT}) = \sigma(\partial B)$, then it follows easily, by considering
Poisson integrals in \( B \) of suitable potentials, that \( C(E_{NT}) = C(\partial B) \). In fact, the capacitary condition is much weaker, as the following example shows.

**Example 1.** Let \( n = 2 \), let \( \mathbb{R}^2 \) be identified with \( \mathbb{C} \) in the usual manner, and let
\[
E = \left\{ (1 - 2^{-2j}) \exp(ik2^{1-j} \pi) : j \in \mathbb{N} \text{ and } k = 0, 1, \ldots, 2^{j-1} \right\}.
\]
Then \( C(E_{NT}) = C(\partial B) \), but \( \sigma(E_{NT}) = 0 \). (See §3.4 for details.)

Brelot and Doob, in their landmark paper [3], were able to relate classical and potential theoretic boundary limit theorems by establishing the relationship between non-tangential and minimal fine cluster sets of functions. Inspired by their work and Theorem 1 we will now provide corresponding results which describe the relationship between non-tangential and fine cluster sets of functions.

Recall that the fine topology on \( \mathbb{R}^n \) is the coarsest topology for which all superharmonic functions are continuous. A set \( A \) is said to be thin at a point \( x \) if \( x \) is not a fine limit point of \( A \). (For an account of these concepts see Chapter 1.XI of the book by Doob [7].) By Wiener's criterion, this is equivalent to the condition
\[
\sum_k 2^{k(n-2)} C^*(\{y \in A : 2^{-k-1} \leq |x-y| \leq 2^{-k} \}) < +\infty \quad (n \geq 3)
\]
or
\[
\sum_k \log 1/C^*(\{y \in A : 2^{-k-1} \leq |x-y| \leq 2^{-k} \}) < +\infty \quad (n = 2),
\]
where \( C^*(\cdot) \) denotes outer (Newtonian or logarithmic) capacity. If, instead, the weaker condition
\[
2^{k(n-2)} C^*(\{y \in A : 2^{-k-1} \leq |x-y| \leq 2^{-k} \}) \to 0 \quad (n \geq 3)
\]
or
\[
\frac{k}{\log 1/C^*(\{y \in A : 2^{-k-1} \leq |x-y| \leq 2^{-k} \})} \to 0 \quad (n = 2)
\]
holds, then \( A \) is said to be semi-thin at \( x \). Now let \( f : S \to [-\infty, +\infty] \), where \( S \subseteq \mathbb{R}^n \), let \( x \in S \) and \( l \in [-\infty, +\infty] \). We say that \( l \) is a fine (respectively, semi-fine) cluster value of \( f \) at \( x \) if, for every neighbourhood \( N \) of \( l \) in \([-\infty, +\infty] \), the set \( f^{-1}(N) \) is not thin (respectively, not semi-thin) at \( x \). Finally, we define the non-tangential approach region
\[
K(z, \delta, \varepsilon) = \{ x : \varepsilon > 1 - |x| > \delta |x-z| \} \quad (z \in \partial B; 0 < \delta < 1; 0 < \varepsilon < 1).
\]

The following result is straightforward to prove, using ideas from [14].

**Proposition 1.** Let \( h \) be a harmonic function on \( K(z, \delta, \varepsilon) \) such that
\[
\int_{K(z, \delta, \varepsilon)} (1-|x|)^{2-n} |\nabla h(x)|^2 \, dx < +\infty,
\]
and let \( \delta < \delta_1 < 1 \). If there is a sequence \( (x_k) \) of points in \( K(z, \delta_1, \varepsilon) \) such that \( x_k \to z \) and \( h(x_k) \to l \), then \( l \) is a semi-fine (and hence a fine) cluster value of \( h \) at \( z \).
Less obvious is the next result, which goes in the opposite direction. If \( f : B \to [-\infty, +\infty] \), then the non-tangential and fine cluster sets of \( f \) at a point \( z \in \partial B \) are defined respectively by

\[
C_{NT}(f, z) = \{ l \in [-\infty, +\infty] : f(x_k) \to l \text{ for some sequence } (x_k) \text{ of points in } B \text{ which approaches } z \text{ non-tangentially} \}
\]
and

\[
C_{F}(f, z) = \{ l \in [-\infty, +\infty] : l \text{ is a fine cluster value of } f \text{ at } z \}.
\]

**Theorem 3.** Let \( f : B \to [-\infty, +\infty] \). Then there is a Euclidean-\( G_\delta \) set \( A \subseteq \partial B \) such that \( C(A) = C(\partial B) \) and \( C_F(f, z) \subseteq C_{NT}(f, z) \) whenever \( z \in A \).

Theorem 2 can be viewed as a fine topology analogue of a maximality theorem of Collingwood and Lohwater (see Theorem 4.10 in [5]). We note that Mizuta [12] has also considered the relationship between non-tangential, normal and fine cluster sets. His results, which are specific to harmonic functions satisfying a Dirichlet-type integral condition, are of a completely different nature. The sharpness of Theorem 2, even for harmonic functions, is demonstrated by the next result.

**Theorem 3.** Let \( A \subseteq \partial B \) be a Euclidean-\( G_\delta \) set such that \( C(A) = C(\partial B) \). Then there is a harmonic function \( h \) on \( B \) such that \( C_F(h, z) = [-\infty, +\infty] \) and \( C_{NT}(h, z) = \{ 0 \} \) whenever \( z \in \partial B \setminus A \).

Proposition 1 and Theorems 1 and 2 will be proved in §§2 - 4 respectively. Theorem 3 will be proved in §5 using recent results concerning approximation by harmonic functions.

2. Proof of Proposition 1

Let \( h \) be a harmonic function on \( K(z, \delta, \varepsilon) \) such that

\[
\int_{K(z, \delta, \varepsilon)} (1 - |x|)^{2-n} |\nabla h(x)|^2 \, dx < +\infty,
\]

let \( \delta < \delta_0 < \delta_1 < 1 \) and \( 0 < \varepsilon_1 < \varepsilon_0 < \varepsilon \). Then there is a decreasing function \( \phi : (0, +\infty) \to (0, +\infty) \) such that \( \phi(t) \leq 2\phi(2t) \) for all \( t \) and \( \phi(t) \to +\infty \) as \( t \to 0 \), and such that \( L < +\infty \), where

\[
L = \int_{K(z, \delta, \varepsilon)} (1 - |x|)^{2-n} \phi(1 - |x|) |\nabla h(x)|^2 \, dx.
\]

Let \( \eta_0 > 0 \) be small enough so that \( B(x, \eta_0 (1 - |x|)) \subseteq K(z, \delta, \varepsilon) \) whenever \( x \in K(z, \delta_0, \varepsilon_0) \). By the volume mean value inequality, applied to the subharmonic function \( |\nabla h|^2 \) and the ball \( B(x, \eta_0 (1 - |x|)) \), we see that there is a positive constant \( M \), depending only on \( \eta_0 \) and \( n \), such that

\[
(1 - |x|)^{2-n} \phi(1 - |x|) |\nabla h(x)|^2 \leq ML (1 - |x|)^{-n} \quad (x \in K(z, \delta_0, \varepsilon_0)),
\]

and hence

\[
(1 - |x|) |\nabla h(x)| \leq \sqrt{\frac{ML}{\phi(1 - |x|)}} \quad (x \in K(z, \delta_0, \varepsilon_0)).
\]

Let \( \eta_1 > 0 \) be small enough so that \( B(x, \eta_1 (1 - |x|)) \subseteq K(z, \delta_0, \varepsilon_0) \) whenever \( x \in K(z, \delta_1, \varepsilon_1) \). Also, let \( (x_k) \) be a sequence of points in \( K(z, \delta_1, \varepsilon_1) \) such that
$x_k \to z$ and $h(x_k) \to l$, and let $D = \bigcup_k B(x_k, \eta_1(1 - |x_k|))$. It follows from (2) and the mean value theorem of differential calculus that
$$h(x) \to l \quad (x \to z; x \in D),$$
in view of the fact that $\phi(t) \to +\infty$ as $t \to 0$. Since $C(B(x,r))$ is $r^{n-2}$ ($n \geq 3$) or $r$ ($n = 2$), it is clear that $D$ is not semi-thin at $z$. Thus $l$ is a semi-fine (and hence a fine) cluster value of $h$ at $z$.

3. Proofs of Theorem 1 and Example 1

3.1. For the proofs of Theorems 1 - 3 we will assume that $n \geq 3$ and do not the minor modifications required to adapt our arguments to the plane. Before proving Theorem 1 we will assemble some preliminary observations. A function $f : S \to \mathbb{R}$ is said to have a finite limit at a finite limit point $x$ of $S$ if $C_F(f,x)$ consists of only one point. If, further, $C_F(f,x) = \{ f(x) \}$, then $f$ is said to be finely continuous at $x$.

**Lemma A.** Let $f : B \to [0, +\infty]$ be integrable on $B$ and let $0 < \varepsilon < 1$. Then there is a set $Y \subset \partial B$, of zero $(n - 2)$-dimensional Hausdorff measure, such that
$$\int_{K(z,\delta,\varepsilon)} (1 - |x|)^{2-n} f(x) dx < +\infty \quad (z \in \partial B \setminus Y; 0 < \delta < 1).$$

**Theorem A.** If $h \in D$, then there is a finite-valued extension $\overline{h}$ of $h$ to $\overline{B}$, and a polar set $Z \subset \partial B$, such that
$$h(rz) \to \overline{h}(z) \quad (r \to 1-; z \in \partial B \setminus Z)$$
and $\overline{h}$ is finely continuous at each point of $\partial B \setminus Z$.

In proving Lemma A it is clearly enough to show that, for a fixed choice of $\delta$, there is a set $Y_\delta \subset \partial B$, of zero $(n - 2)$-dimensional Hausdorff measure, such that
$$\int_{K(z,\delta,\varepsilon)} (1 - |x|)^{2-n} f(x) dx < +\infty \quad (z \in \partial B \setminus Y_\delta).$$
This can be done by imitating the proof of the analogous result for functions on half-spaces, which may be found in Lemma 5 of [14].

Theorem A is taken from Deny ([6], Chap. IV, Théorème 3), who proved the result for the more general class of Beppo Levi functions $h$ on $B$. Actually, Deny’s result asserts only that $\overline{h}$ has a finite limit at each point of $\partial B \setminus Z$, but the proof makes it clear that $\overline{h}$ is actually finely continuous at each point of a set which has this form.

**Lemma 1.** The unit sphere $\partial B$, with the topology induced on it by the fine topology on $\mathbb{R}^n$, is a Baire space.

To prove Lemma 1, we note that the fine topology on $\mathbb{R}^n$ has a base which consists of (Euclidean) compact sets. The same is therefore also true of the topology it induces on $\partial B$. We can now adopt the argument given on pp. 167, 168 of [12] to see that this space is Baire.

If $A$ is a bounded set in $\mathbb{R}^n$, then we use $\overline{R}_1 A$ to denote the capacitary potential of $A$. Thus the Riesz measure $\nu_A$ associated with $\overline{R}_1 A$ satisfies $\nu_A(\mathbb{R}^n) = C^*(A)$.

**Lemma 2.** Let $A \subseteq \partial B$ and $z \in \partial B$. If $A$ is thin at $z$, then $\overline{R}_1 A(z) < 1$. 


To see this we note that, if \( z \in \overline{A} \), then there is a positive superharmonic function \( u \) on \( \mathbb{R}^n \) such that
\[
\liminf_{x \to z, x \notin A} u(x) > u(z).
\]
If we define \( v(x) = u(x) + a \|x + z\|^{2-n} \), where \( a \) is a suitably chosen positive number, then \( v \) is a positive superharmonic function on \( \mathbb{R}^n \) such that
\[
\inf_{A \setminus \{z\}} v > v(z).
\]
It follows easily that \( \widehat{R}_1^A(z) < 1 \), and this inequality is obviously also true when \( z \notin \overline{A} \).

3.2. We now turn to the “if” part of Theorem 1. \( \text{Suppose that } \mathcal{C}(E_{NT}) = \mathcal{C}(\partial B) \) and suppose further, for the sake of contradiction, that \( E_{NT} \) is thin at some point \( z \) of \( \partial B \). Then \( \widehat{R}_1^{E_{NT}}(z) < 1 \), by Lemma 2, and this leads to the contradictory conclusion that \( \mathcal{C}(E_{NT}) = \widehat{R}_1^{E_{NT}}(0) < 1 = \mathcal{C}(\partial B) \). Hence the fine closure of \( E_{NT} \) is all of \( \partial B \).

Now let \( h \in \mathcal{D} \). By Theorem A, there exist a function \( \tilde{h} : B \to \mathbb{R} \) and a polar set \( Z \subset \partial B \) such that \( \tilde{h}|_B = h \) and (3) holds, and such that \( \tilde{h} \) is finely continuous at all points of \( \partial B \setminus Z \). Let \( M = \sup_B \tilde{h} \) and suppose, to avoid triviality, that \( M < +\infty \). By Proposition 1 and Lemma A (with \( f = |\nabla h|^2 \)), there is a polar subset \( Y \) of \( \partial B \) such that \( \overline{h} \leq M \) on \( E_{NT} \setminus (Y \cup Z) \). (Here we are using the fact that a bounded set of finite \((n-2)\)-dimensional Hausdorff measure is polar; see [3], Chap. IV, Theorem 1). The fine closure of \( E_{NT} \setminus (Y \cup Z) \) is also \( \partial B \), so \( \overline{h} \leq M \) on \( \partial B \setminus Z \). We note that the subharmonic function \( h^2 \) has a harmonic majorant on \( B \) because
\[
\int_B (1 - |x|) (\Delta (h^2))(x) dx \leq 2 \int_B |\nabla h(x)|^2 dx < +\infty.
\]
Hence \( h \) is equal to the Poisson integral of a function \( q \) on \( \partial B \) (see [7], 1.II.14). Further, from [3] and Fatou’s boundary limit theorem, \( g = \overline{h}|_{\partial B} \) almost everywhere (\( \sigma \)) on \( \partial B \), and so \( h \leq M \) on \( B \). Thus (1) holds and we have now established the “if” part of Theorem 1.

3.3. Conversely, suppose that (1) holds and let
\[
F(r) = \partial B \cap \left( \bigcup_{x \in E \setminus B(0,r)} B(x, 2(1 - |x|)) \right) \quad (0 < r < 1).
\]
Clearly (1) implies that \( \overline{E} \cap \partial B \neq \emptyset \), so \( F(r) \neq \emptyset \) for all \( r \). Suppose further, for the sake of contradiction, that there exists \( r \) such that the fine closure of \( F(r) \) is a proper subset of \( \partial B \). Then the function \( h = 1 - \widehat{R}_1^{F(r)} \), which is harmonic on \( B \), is strictly positive there, in view of Lemma 2. We note (see [7], 1.IV.5) that there is an increasing sequence \((u_k)\) of \( C^\infty\) Newtonian potentials, with associated measures \((\mu_k)\), such that \( u_k \uparrow \widehat{R}_1^{F(r)} \) and \( u_k \) is harmonic outside \( \overline{B(0, 1+1/k) \setminus B(0, 1-1/k)} \) and that
\[
\int_{\mathbb{R}^n} |\nabla u_k(x)|^2 dx = a_n \int u_k d\mu_k
\]
by Green’s first identity, where $a_n$ is a positive constant depending only on $n$. Since $|\nabla u_k| \to |\nabla h|$ locally uniformly on $B$, it follows from the reciprocity law that

$$\int_B |\nabla h(x)|^2 \, dx \leq a_n \int B \tilde{R}_1^{F(r)} \, d\nu_{F(r)} \leq a_n C(F(r)) < +\infty;$$

that is, $h|_B \in D$.

Let $M = \sup_B h$ and let $v_r$ denote the harmonic measure of $\partial B \setminus F(r)$ in $B$. The definition of $F(r)$ ensures that there is a constant $c \in (0, 1)$, independent of $E$, such that $v_r \leq c$ on $E \setminus B(0, r)$. Since $h = 1 - \tilde{R}_1^{F(r)} = 0$ on $F(r)$, it follows that $h \leq cM$ on $E \setminus B(0, r)$, and so

$$\sup_E h \leq \max \left\{ cM, \sup_{B(0, r)} h \right\} < M = \sup_B h.$$ 

This contradicts (1). Hence, for every $r \in (0, 1)$, the fine closure of $F(r)$ is all of $\partial B$.

Let

$$F = \bigcap_{k=1}^{\infty} F \left( \frac{k}{k+1} \right).$$

Then, by Lemma 1, the fine closure of $F$ is also all of $\partial B$. Finally, it is clear that $F \subseteq E_{NT}$, so the fine closure of $E_{NT}$ is also $\partial B$. Thus $\tilde{R}_{1^{E_{NT}}} = \tilde{R}_{1^{\partial B}}$, and hence $C(E_{NT}) = C(\partial B)$, as required.

3.4. We now present the details of Example 1. Let $E$ be the set defined there. Then

$$E_{NT} = \bigcup_{l=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{j=m}^{\infty} F_{j,k,l},$$

where

$$F_{j,k,l} = \{ e^{i\theta} : |\theta - k 2^{j-1} \pi| < 2^{2j-2} \}.$$ 

The set $F_{j,k,l}$ is an open arc of the unit circle and $C(F_{j,k,l}) \geq 2^{2j-2}$ (see [11], p. 173, (2.4.4)). It follows easily from Wiener’s criterion that the set

$$A_{l,m} = \bigcup_{j=m}^{\infty} \bigcup_{k=0}^{2^j-1} F_{j,k,l}$$

is finely dense in the circle. Hence, for any $l \in \mathbb{N}$, the set $\bigcap_{m=1}^{\infty} A_{l,m}$ is finely dense in the circle, by Lemma 1. It follows that $C(E_{NT}) = C(\partial B)$. However,

$$\sigma(A_{l,m}) \leq \sum_{j=m}^{\infty} \sum_{k=0}^{2^j-1} l^{2j-2j} = l^{2^j-2} - m,$$

so $\sigma(\bigcap_{m=1}^{\infty} A_{l,m}) = 0$ and hence $\sigma(E_{NT}) = 0$, as claimed.
4. Proof of Theorem 2

Let \( f : B \to [-\infty, +\infty] \) and
\[
D = \{ z \in \partial B : C_F(f, z) \setminus C_K(f, z) \neq \emptyset \},
\]
where, for each \( z \in \partial B \),
\[
C_K(f, z) = \{ l \in [-\infty, +\infty] : f(x_k) \to l \text{ for some sequence } (x_k) \text{ of points in } K(z, 1/2, 1/2) \text{ such that } x_k \to z \}.
\]

Further, let \( \mathcal{I} \) denote the collection of closed intervals of \([-\infty, +\infty]\) with endpoints in \( \mathbb{Q} \cup \{-\infty, +\infty\} \). Suppose that \( z \in D \). Since \( C_K(f, z) \setminus C_K(f, z) \) is compact in \( \mathbb{R} \), we can find \( I \in \mathcal{I} \), a finite union \( J \) of intervals from \( \mathcal{I} \), and \( \varepsilon \in \mathbb{Q} \cap (0, 1) \) such that
\[
(4) \quad I \cap C_F(f, z) \neq \emptyset, \quad I \cap J = \emptyset
\]
and
\[
(5) \quad f \left( K \left( z, \frac{1}{2}, \varepsilon \right) \right) \subseteq J.
\]

If \( I, J \) and \( \varepsilon \) are as above, then we say that \( z \in D(I, J, \varepsilon) \). Thus
\[
(6) \quad D \subseteq \bigcup_{I, J, \varepsilon} D(I, J, \varepsilon),
\]
where the union is over all possible choices of \( I, J, \varepsilon \).

Now suppose that one of the sets in this union, \( D_0 = D(I_0, J_0, \varepsilon_0) \) say, has the property that its (Euclidean) closure has non-empty interior \( U \) with respect to the fine-induced topology on \( \partial B \), let
\[
E = B \setminus \left( \bigcup_{z \in D_0} K \left( z, \frac{1}{2}, \varepsilon_0 \right) \right)
\]
and \( h = 1 - \overline{R}_B \setminus U \). Then \( h = 0 \) on \( \partial B \setminus U \) since the latter set is finely perfect. Also, \( E \cap \partial B \neq \emptyset \) by (4) and (5), and
\[
B(x, 2(1 - |x|)) \cap U = \emptyset \quad (x \in E \setminus B(0, 1 - \varepsilon_0))
\]
by the definition of \( E \). Reasoning as in \S 3.3, we now see that \( h > 0 \) on \( B \), and on \( U \) by Lemma 2, but \( \sup_E h < \sup_B h \). Let \( \sup_E h < M < \sup_B h \) and \( V = \{ x \in \mathbb{R}^n : h(x) > M \} \), and choose \( z_1 \in U \) such that \( h(z_1) > M \). Then \( V \) is a fine neighbourhood of \( z_1 \) and \( V \cap E = \emptyset \). It now follows from (5) and (7) that \( C_F(f, z_1) \subseteq J_0 \), but this contradicts (4). Thus \( U = \emptyset \).

Let
\[
(8) \quad A = \bigcap_{I, J, \varepsilon} \partial B \setminus \overline{D(I, J, \varepsilon)}.
\]

Thus \( A \) is a (Euclidean) \( G_\delta \)-set and each of the sets \( \partial B \setminus \overline{D(I, J, \varepsilon)} \) is open and dense in the fine-induced topology on \( \partial B \). By Lemma 1, the fine closure of \( A \) is \( \partial B \) and hence \( C(A) = C(\partial B) \). Since \( C_K(f, z) \subseteq C_NT(f, z) \), it follows that
\[
C_F(f, z) \subseteq C_NT(f, z) \quad (z \in \partial B \setminus D).
\]

From (6) and (8) we see that \( A \subseteq \partial B \setminus D \), and so Theorem [2] is established.
5. Proof of Theorem

Let $A \subseteq \partial B$ be a Euclidean-G$_d$ set such that $\mathcal{C}(A) = \mathcal{C}(\partial B)$. As we argued at the beginning of §3.2, it follows that $A$ is finely dense in $\partial B$. Thus there is an increasing sequence $(F_k)$ of compact sets such that $\partial B \setminus A = \bigcup_k F_k$ and such that $\partial B \setminus F_k$ is finely dense in $\partial B$ for each $k$. To avoid trivialities we may assume that the sets $\partial B \setminus A$ and $F_1$ are non-empty.

Let $g_k : \partial B \rightarrow [0, 1)$ be defined by

$$g_k(z) = \left\{ \frac{\text{dist}(z, F_k)}{5} \right\} (z \in \partial B; k \in \mathbb{N}).$$

Then the sets

$$E_k = \left\{ rz : z \in \partial B \setminus F_k \text{ and } 1 - r = \frac{g_k(z)}{3k - 1} \right\} (k \in \mathbb{N}),$$
and

$$D_1 = B \cap \left\{ rz : z \in \partial B \text{ and } 1 - r \geq g_1(z) \right\},$$
and

$$D_{k+1} = B \cap \left\{ rz : z \in \partial B \text{ and } \frac{g_k(z)}{3k} \geq 1 - r \geq \frac{g_{k+1}(z)}{3k + 1} \right\} (k \in \mathbb{N})$$
are all closed relative to $B$ and are pairwise disjoint. Let

$$(9) \quad E = \left( \bigcup_k E_k \right) \cup \left( \bigcup_k D_k \right)$$

and let $B^*$ denote the Alexandroff (one-point) compactification of $B$. Since, for a given value of $r$ in $(0, 1)$, only finitely many sets in the union in $\mathbf{(9)}$ meet $B(0, r)$, the set $E$ is also closed relative to $B$.

If we define $u : E \rightarrow \mathbb{R}$ by

$$(10) \quad u(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \in \bigcup_k D_k, \\ q_k & \text{if } x \in E_k; k \geq 1, \end{array} \right.$$ 

where $(q_k)$ is an enumeration of $\mathbb{Q}$, then $u$ extends to a locally constant (and hence harmonic) function on an open set which contains $E$. It is easy to check that $B^* \setminus E$ is connected and locally connected (see §3.2 of [9] for a discussion of local connectedness in this context). We note that, if $z \in \partial B \setminus A$ and $0 < \delta < 1$, then there exists a (smallest) number $k_0$ such that $z \in F_{k_0}$ and a number $\varepsilon_{z, \delta}$ in $(0, 1)$ such that

$$(11) \quad K(z, \delta, \varepsilon_{z, \delta}) \subseteq D_{k_0} \subseteq \bigcup_k D_k \quad (0 < \delta < 1).$$

We also claim that, if $z \in \partial B \setminus A$, then $E_k$ is non-thin at $z$ for all sufficiently large $k$. To see this, we choose $k_0$ such that $z \in F_{k_0}$ and suppose that $E_k$ is thin at $z$ for some $k \geq k_0$. Since the radial projection map from $E_k$ to $\partial B \setminus F_k$ is a Lipschitz map with Lipschitz constant $2$, we have

$$C(\{ y \in \partial B \setminus F_k : 2^{-j-1} \leq |y - z| \leq 2^{-j} \})$$

$$\leq 2^{n-2} C(\{ y \in E_k : 2^{-j-2} \leq |y - z| \leq 2^{-j+1} \})$$

for each $j \in \mathbb{N}$, by standard contraction and dilation properties of Newtonian capacity (see [11], Chap. 2, §3). Hence, by Wiener’s criterion, we obtain the contradictory conclusion that $\partial B \setminus F_k$ is also thin at $z$. Thus our claim is verified.
We now apply a recent harmonic approximation result (see [1], or Corollary 3.10 in [9]) to observe that there is a harmonic function \( h \) on \( B \) such that
\[
|h(x) - u(x)| < 1 - |x| \quad (x \in E).
\]
It follows from (10) and (11) that \( C_{NT}(h, z) = \{0\} \) whenever \( z \in \partial B \setminus A \). For such \( z \), it also follows from (10) and the claim verified in the previous paragraph that \( C_{F}(h, z) \) contains all but a finite number of the rationals, and so \( C_{F}(h, z) = [-\infty, +\infty] \). The proof of Theorem 3 is now complete.

References


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